An Evidence Logic Perspective on Schotch-Jennings Forcing

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Abstract. Traditional epistemic and doxastic logics cannot deal with inconsistent beliefs nor do they represent the evidence an agent possesses. So-called 'evidence logics' have been introduced to deal with both of those issues. The semantics of these logics are based on neighbourhood or hypergraph frames. The neighbourhoods of a world represent the basic evidence available to an agent. On one view, beliefs supported by evidence are propositions derived from all maximally consistent collections evidence. An alternative concept of beliefs takes them to be propositions derivable from consistent partitions of one's inconsistent evidence; this is known as Schotch-Jennings Forcing. This paper develops a modal logic based on the hypergraph semantics to represent Schotch-Jennings Forcing. The modal language includes an operator $U(\varphi_1, \ldots, \varphi_n; \psi)$ which is similar to one introduced in Instantial Neighbourhood Logic. It is of variable arity and the input formulas are enjoy distinct roles. The U operator expresses that all evidence at a particular world that supports ψ can be supported by at least one of the φ_is. U can then be used to express that all the evidence available can be unified by the finite set of formulas $\varphi_1, \ldots, \varphi_n$ if ψ is taken to be $\top$. Future developments will then use that semantics as the basis for a doxastic logic akin to evidence logics.

Keywords: Evidence Logic · Epistemic Logic · Paraconsistent Logic · Schotch-Jennings Forcing · Pointed Operators

1 Introduction

In [4] and [2] the authors proffer modal logics for reasoning about beliefs which are based on *evidence*. Traditionally, epistemic and doxastic logics are about how an agent reasons from propositions they know or believe. How the agent arrives at those propositions they reason from is not part of the model. However, these new "Evidence Logics" include an explicit representation of what evidence an agent has. They then can go on to define conditions for belief on the basis of what evidence the agent possesses.

One of the challenges of doxastic and epistemic logic has been that agents often possess inconsistent evidence. Traditional modal logics based on (binary) relational semantics cannot tolerate inconsistency; everything is believed when beliefs are inconsistent. These evidence models suggest a different approach. They allow the evidence one accumulates to be inconsistent, while restraining beliefs based on that evidence in ways that ensure consistency of resulting belief—at least in the case of [2]. Filtering beliefs from evidence requires novel ways of combining the evidence and deriving conclusions from it that will avoid—if not eliminate—inconsistencies.

The approaches to evidence based belief in [2] and [4] relate to the method of dealing with inconsistent data/premises proposed by [12] in which one reasons from maximally consistent subsets of one's data. We take a different starting point, namely, the preservationist approach to paraconsistency in [8]. The preservationist method of reasoning from inconsistent data is to reason from special partitions of one's data; when something follows from one of these partitions, that conclusion is *forced*, and this inference method is called *forcing*. The relationship between these two approaches to paraconsistent propositional inference has been studied in [11]. Before any application of this preservationist approach can be made in the present context—to the modal logic representation of evidence—it must first be given a semantic representation (§ 3) that facilitates comparison between the two starting points.

The meeting point of the two views is the use of neighbourhood models to represent the evidence of an agent: the collection of neighbourhoods is the set of basic evidence available at that world. The preservationist approach to paraconsistency was inspired by modal logics which use n-ary relation frames, cf. [7], [1]. It came to be understood that those frames corresponded to neighbourhood frames for modal logic [9]. The paraconsistent n-ary modal logics could be interpreted on those neighbourhood frames when a variation on the truth condition for the $\Box$ operator is used: in order for $\Box\varphi$ to be true at x, there must be a neighbourhood of x where that φ is true throughout. This differs from the usual truth condition, in which all the worlds where φ is true must *be* a neighbourhood of x. A thorough study of the relations between n-ary modal logics and n-relation modal logics has been conducted in [6] which explores these connections via neighbourhood semantics.

Here, we offer a way to use neighbourhood models to represent a preservationist approach of deriving belief from evidence. The goal of this paper is to capture the general forcing relation in a neighbourhood semantics. To do this, we introduce an operator, similar to that found in [3], which takes two arguments: a non-empty list of formulas and a formula. This operator expresses the sufficiency of the formulas in the list of the first argument for implying the formula in the second argument. What we show is that a semantics can be given which represents Schotch-Jennings forcing on classical propositional logic, and provide a logic which is sound and complete for that semantics.

1.1 Evidence Models

Evidence models are built on the standard set up from modal logic where we have a non-empty set of ways the world might be W, i.e., possible worlds, and propositions or facts that might be true in those worlds represented by subsets of W. An agent's evidence will be represented by a so-called 'evidence frame'

$$\mathfrak{F} = \langle W, \mathcal{E} \rangle$$

consisting of W and a function $\mathcal{E} : W \to \mathcal{P}(\mathcal{P}(W))$. For each $x \in W$, $\mathcal{E}(x)$ represents the evidence the agent has collected at x; the agent's *basic* evidence at x. The only conditions that we will impose on $\mathcal{E}(x)$—at the moment—are that $\varnothing \notin \mathcal{E}(x) \neq \varnothing$. Thus, the agent can never collect a contradiction as evidence. There are no conditions at this point on whether $\mathcal{E}(x)$ must be closed under various set-theoretic operations like supersets or intersection. We will consider an agent to *have evidence* that $X \subseteq W$, when there is $Y \in \mathcal{E}(x)$ such that $Y \subseteq X$. That is, agents have all evidence that their basic evidence, taken individually, implies. That makes the requirement of closure under supersets unnecessary.

Although the evidence an agent has is simply what that agent's evidence individually implies, what an agent's evidence *supports* is an holistic matter. Intuitively, evidential support should be computed by combining the basic evidence somehow, but it is not clear how that should be done. We have not assumed that $\mathcal{E}(x)$ is factual or even consistent: the actual world may not be in $\bigcap\mathcal{E}(x)$ nor is it guaranteed that $\bigcap\mathcal{E}(x) \neq \varnothing$, respectively. So a simple combining of one's basic evidence via taking what is common between all of it may result in "supporting" everything since all propositions are implied by an inconsistent set: when $\bigcap\mathcal{E}(x) = \varnothing$, $\bigcap\mathcal{E}(x) \subseteq X$ for any $X \subseteq W$. The authors van Benthem et al. and Baltag et al. have suggested two fruitful ways of combining evidence. Inspired by their ideas, we here offer a method of combining evidence by using a representation of Schotch-Jennings Forcing in modal logic.

Schotch-Jennings Forcing offers a way to disentangle any inconsistency, and then to infer from the disentangled collection. In the following section we will review the syntactic account of this method, survey the extant connections between modal logic and forcing, and then develop a semantic analog of forcing in neighbourhood models, suitable as a basis for a modal logic.

1.2 Forcing and Level

In a series of papers, [13], [7], and [14] Jennings and Schotch developed a method of drawing inferences from inconsistent sets which they refer to as 'forcing'. The set up is to find the minimal way to partition the premises so that each element, or 'cell', of the partition is consistent. Then one reasons from those consistent cells. Taking the smallest or minimal partitions of a set, if some conclusion follows from at least one cell in *every* such partition, then the set forces that conclusion.

More precisely, lets say that a partition Π is a *cover* of a set of formulas Γ iff, $\bigcup\Pi = \Gamma$ and for all $\pi \in \Pi$, $\pi \nvdash \bot$ where $\vdash$ is simply the consequence relation of classical logic. We will also refer to the cardinality of Π as its width. There is another definition of a syntactic cover as follows: a collection of consistent sets of sentences Π (not necessarily a partition of Γ) such that for each $\gamma \in \Gamma$, there is $\pi \in \Pi$ such that $\pi \vdash \gamma$. If we introduce $\mathbb{C}(\Gamma) = \{\alpha : \Gamma \vdash \alpha\}$ to refer to the deductive closure of Γ, then we can say that Π is a cover of Γ when $\Gamma \subseteq \bigcup_{\pi\in\Pi} \mathbb{C}(\pi)$ and each π is consistent. Partitions are a special case of this more general kind of cover and are thus referred to as 'partition covers'.

The *level* of Γ, $\ell(\Gamma)$, is a kind of measure of how inconsistent Γ is, and it is determined by the minimum width a set of sets must have in order to be a cover of Γ, but if there is no such minimum, its level is ∞. Thus:

$$\ell(\Gamma) = \begin{cases} 0 & \Gamma \subseteq \mathbb{C}(\varnothing) \\ \min\{\,|\Pi| : \Pi \text{ is a cover of } \Gamma\,\} & \text{if it exists } \& \ \Gamma \nsubseteq \mathbb{C}(\varnothing) \\ \infty & \text{otherwise} \end{cases}$$

We assign the level of 0 to the special case where Γ is a set of theorems. We can then say that Γ forces α, $\Gamma \Vdash \alpha$ iff in any cover of Γ (partition or otherwise), Π, such that $|\Pi| = \ell(\Gamma)$, there is $\pi \in \Pi$ such that $\pi \vdash \alpha$. However, it can be shown that forcing is determined by the collection of partition covers since we can always generate a partition cover from a cover.

Most conceptions of consequence are based on considering what is true across all the ways things could be and forcing incorporates this 'all the ways things could be' kind of thinking by consulting all covers of Γ to determine the forcing consequences. It is interesting to note that this does not simply mean looking at all $\ell(\Gamma)$-tuples of distinct maximally consistent subsets of Γ. This could seem odd since obviously each cell in a (partition) cover of Γ can be extended to a maximally consistent subset (i.e., Γ' is a maximally consistent subset of Γ iff $\Gamma' \subseteq \Gamma$, $\Gamma' \nvdash \bot$ and for any $\alpha \in \Gamma \smallsetminus \Gamma'$, $\Gamma' \cup \{\,\alpha\,\} \vdash \bot$). The issue is that some maximally consistent subsets may not be reachable by such extensions. For example, consider the following set from classical logic:

$$\Phi = \{\,\neg q \wedge p, q \rightarrow r, \neg r, q, \neg p\,\}$$

This set gives rise to maximally consistent subsets. We will not list all of them, but for instructional purposes here are two of them:

(A) $\{\,q \rightarrow r, \neg r, \neg p\,\}$, and
(B) $\{\,q \rightarrow r, \neg r, \neg q \wedge p\,\}$, and

It is easy to see that $\ell(\Phi) = 2$ since it is inconsistent but we only need a cover of width 2:

$$\{\,\{\,q \rightarrow r, \neg r, \neg q \wedge p\,\}, \{\,q, \neg p\,\}\,\}.$$

What this means is that the covers that would be used to determine the forcing consequences of Φ would all have width 2. This gives rise to a curious situation when we consider set A above. Set A would never appear in a cover of Φ that was used to calculate forcing consequences. The reason is that, if set A is removed from Φ, the set that is left over has level 2 as well. That means no cover of Φ with width 2 could be constructed with set A as a cell.

If one ends up with inconsistent evidence, another way to make inferences from it, or another way to calculate what the evidence supports, is by what the evidence forces. Of course, if one's evidence is *consistent*, then the conclusions one can draw are simply all those which follow, classically speaking. We now consider a semantics for forcing which relates it to the semantics of prior evidence logics.

2 Forcing and Modal Logic

Although we are interested in representing forcing in an evidence logic manner, there already exist some connections between modal logics and forcing. In fact, these modal logics are non-normal and have natural semantics in terms of evidence logic-like semantics. First we will define a language which we will add to as we encounter problems. The basic semantic set up is just like that for evidence logics; we start with a frame and then a model:

Definition 1. A structure $\mathfrak{F} = \langle W, \mathcal{E} \rangle$ is a hypergraph/evidence **frame** iff:

1. $W \neq \varnothing$, and
2. $\mathcal{E} : W \rightarrow \mathcal{P}(\mathcal{P}(W))$ such that for all $x \in W$
   (a) $\varnothing \notin \mathcal{E}(x)$, and
   (b) $\mathcal{E}(x) \neq \varnothing$.

A hypergraph/evidence **model** is a structure $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ where $\mathfrak{F}$ is a hypergraph/evidence frame and $V : \mathbf{At} \to \mathcal{P}(W)$ where $\mathbf{At}$ is the set of atomic formulas of a propositional language.

For simplicity we will refer to hypergraph frames as 'hyperframes'. We can then define the semantics for a language on such models which we will sometimes refer to simply as 'models'. The language consists of the boolean operators and the unary operator '$E\varphi$' which is meant to be interpreted, intuitively, as that there is evidence supporting the proposition φ among one's basic evidence. Its dual is denoted as $\langle E \rangle$. Let $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ be a model, the semantics is:

– $\mathcal{M}, x \vDash p$ iff $x \in V(p)$ for all $p \in \mathbf{At}$
– Boolean cases as usual,
– $\mathcal{M}, x \vDash E\varphi$ iff there is $X \in \mathcal{E}(x)$ such that $X \subseteq [\![\varphi]\!]$,
– $\mathcal{M}, x \vDash \langle E \rangle \varphi$ iff for all $X \in \mathcal{E}(x)$, $X \cap [\![\varphi]\!] \neq \varnothing$.

Of course $\mathcal{M}$ satisfies φ iff there is $x \in W$ such that $\mathcal{M}, x \vDash \varphi$ and satisfies a set of sentences Γ iff $\mathcal{M}$ satisfies all members of Γ at some world $x \in W$. As is also standard, $\Gamma \vDash_E \varphi$ iff for all $\mathcal{M}$ and x, if $\mathcal{M}, x \vDash \Gamma$, then $\mathcal{M}, x \vDash \varphi$. As is well known [5], this logic can be axiomatized as follows:

CL All theorems of classical propositional logic.
D $\vdash_E \neg E\bot$
N $\vdash_E E\top$
M $\dfrac{\vdash_E p \to q}{\vdash_E Ep \to Eq}$
With rules
MP Modus Ponens, and
US Uniform Substitution.

This is the basic logic of hypergraphs as we have defined them above. But as one might expect it is nowhere near expressive enough to capture forcing. But there are near-by logics based on hyperframes that connect to forcing and are fairly well understood. First, there are the K_n modal logics which *sometimes* represent the forcing consequences of a set of formulas.

The modal logics K_n are non-normal modal logics which are axiomatized in the following way:[3]

CL All theorems of classical propositional logic.
N $\vdash_{K_n} \langle E \rangle \top$
$K_n^{\Diamond}$ $\vdash_{K_n} (\langle E \rangle p_1 \wedge \ldots \wedge \langle E \rangle p_{n+1}) \to \langle E \rangle \bigvee_{1 \le i < j \le n+1} (p_i \wedge p_j)$
With rules
M $\dfrac{\vdash_{K_n} p \to q}{\vdash_{K_n} \langle E \rangle p \to \langle E \rangle q}$
MP Modus Ponens, and
US Uniform Substitution.

What is unique about these modal logics is the axiom $K_n^{\Diamond}$[4] which weakens the adjunctive properties of the logic and keeps inconsistent formulas from interacting. The modal logic K_n axiomatizes the logic valid on the class of all n-bounded hyperframes. A hyperframe is n-bounded when for all $X \in \mathcal{E}(x)$ and $x \in W$, $|X| \le n$. This doesn't mean that an n-bounded hyperframe is finite, just that each edge in each hypergraph is finite.

What can be shown is that if the level of a set Γ is n, then

$$\Gamma \Vdash \alpha \text{ iff } \langle E \rangle [\Gamma] \vdash_{K_n} \langle E \rangle \alpha$$

where $\langle E \rangle [\Gamma] = \{ \langle E \rangle \gamma : \gamma \in \Gamma \}$. These logics, however, are not suitable for forcing in general. They capture what is called 'fixed-level forcing' which is when one consults all of the covers of Γ which have a fixed width, say, n.[5] The problem with K_n is two fold. If Γ's level is less than n, then one will lose many forcing consequences. And if the level of Γ is larger than n, then Γ is treated as inconsistent, so it 'fixed-level forces' everything. The source of the issue is that the K_n logics cannot discern what level a set of sentences has before determining its consequences.

There are also the P_n logics studied in [6]. These logics are axiomatized as follows:

[3] Usually, they are presented with Es ($\Box$s) in the place of all the $\langle E \rangle$ ($\Diamond$s) which makes the connection to the modal logic K clearer in which $K_1 = K$. But we are choosing to remain consistent with the notation in the literature on evidence logic.
[4] This is the name of the axiom as presented in [6].
[5] A more appropriate name would be 'fixed-width forcing'.

CL All theorems of classical propositional logic.
N $\vdash_{P^n} E\top$
P^n $\vdash_{P^n} (Ep_1 \wedge \ldots \wedge Ep_{n+1}) \to \bigvee_{1\leq i<j\leq n+1} E(p_i \wedge p_j)$
With rules

M $\dfrac{\vdash_{P^n} p \to q}{\vdash_{P^n} Ep \to Eq}$
MP Modus Ponens, and
US Uniform Substitution.

The P^n logics are determined by the class of all consistent and n-bounded in degree hyperframes. A hyperframe is n-bounded in degree iff for all $x \in W$, $|\mathcal{E}(x)| \leq n$ and consistent iff for all $X \in \mathcal{E}(x)$, $|X| \geq 1$ for all $x \in W$.

These logics have the resources to determine what level a set of sentences has. If Γ has level n and $m < n$, then $E[\Gamma] \vdash_{P^m} \bot$. That is because if $\mathcal{M}, x \models E[\Gamma]$, then for all $\gamma \in \Gamma$ there is $X \in \mathcal{E}(x)$ such that $X \subseteq [\![\gamma]\!]$. Now, if $|\mathcal{E}(x)| < \ell(\Gamma)$, then by a pigeon hole argument we could create a syntactic cover of Γ whose width is less than $\ell(\Gamma)$; but that should be impossible when $\ell(\Gamma) = n$. So, when $E[\Gamma] \vdash_{P^m} \bot$, $\ell(\Gamma) \geq m$. Similarly, if $E[\Gamma] \nvdash_{P^m} \bot$ then $\ell(\Gamma) \leq m$. If Γ is finite, then $\ell(\Gamma) \leq |\Gamma|$. So, for finite sets Γ, $\ell(\Gamma) = n$ iff $E[\Gamma] \vdash_{P^{n-1}} \bot$ and $E[\Gamma] \nvdash_{P^n} \bot$.

But just because the P^n logics can determine the level of a set, that doesn't mean that it can determine the forcing consequences. Indeed, it doesn't. The logic K_n determines the forcing consequences of sets which have level n.

The fundamental issue is that forcing is a dynamic, global and contextual conception of consequence. Generally, the logical consequences of a set of sentences are dependent on what the set contains but are not influenced by global properties of that set. Forcing, on the other hand, contextually adapts to a particular, and important, global property of the set, namely the set's level. Typically, logics do *not* change their behaviour from context to context; that is kind of the point of them. But forcing must, since it depends on preserving the overall coherence of the set of premises, not just interactions between some individual premises. So to develop a semantics for forcing we have to find a way to overcome that narrow focus. We need a logic that can both determine the level of a set and its forcing consequences.

3 Covers: Syntactic vs. Semantic

Although there are various logics that can represent certain kinds of forcing, none captures forcing in general. The goal is to represent forcing using an evidence logic style semantics. The first thing which is needed is a semantic analog of a cover in order to represent the level of a set via a semantic object, i.e., an evidence set.

Given a set $\mathcal{X}$ of subsets of a set W, we can define the level of this set in much the same way as we defined the level of a set of formulas since, after all, subsets of W are supposed to represent propositions. We start with a *cover*.[6] A *cover* of $\mathcal{X}$ is a set $\mathcal{Y} \subseteq \mathcal{P}(W) \setminus \{\varnothing\}$ such that for each $X \in \mathcal{X}$, there is $Y \in \mathcal{Y}$ and $Y \subseteq X$. Again,

$$\ell(\mathcal{X}) = \begin{cases} 0 & \text{when } \mathcal{X} = \{W\} \\ \min\{|\Pi| : \Pi \text{ is a cover of } \mathcal{X}\} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

Like in the syntactic case, $\ell(\mathcal{X}) = \infty$ iff there is a self-inconsistent proposition in $\mathcal{X}$, i.e., $\varnothing \in \mathcal{X}$. The conditions on evidence frames will rule out $\varnothing$ ever being in an $\mathcal{E}(x)$, so no evidence set will have level ∞. A major difference is that since $\mathcal{E}(x)$ could be uncountable, $\ell(\mathcal{E}(x))$ could be an uncountable cardinal, which cannot happen in the syntactic case when one is only working with countable languages. But even in the syntactic case one could have an evidence set of level ω. An evidence set like that would have covers where the extension of each formula is in its own cell. However, given an evidence set whose narrowest cover is of size ω, its forcing consequences boil down to only what follows from the individual pieces of evidence on their own.

[6] We could define a cover of $\mathcal{X}$ as a subset of $\mathcal{P}(\mathcal{P}(W))$, Π such that for each $\pi \in \Pi$, $\cap\pi \neq \varnothing$ and for each $X \in \mathcal{X}$ there is $\pi \in \Pi$ such that $\cap\pi \subseteq X$ and if Π is a partition of $\mathcal{X}$ we say that Π would be a *partition* cover. However, the definition on offer is slightly more economical.

Another fact which is easy to see is that if $\mathcal{Y}$ is a cover of $\mathcal{X}$, then $\ell(\mathcal{Y}) \geq \ell(\mathcal{X})$. For suppose that $\mathcal{Y}'$ is a cover of minimal width of $\mathcal{Y}$. Then $|\mathcal{Y}'| = \ell(\mathcal{Y})$. But the transitivity of $\subseteq$ means that $\mathcal{Y}'$ is also a cover of $\mathcal{X}$. Thus, $\ell(\mathcal{X}) \leq |\mathcal{Y}'| = \ell(\mathcal{Y})$.

We now introduce some closely related concepts to connect semantic covers to syntactic covers via the evidence models. These concepts help us discuss the various ways that sets of sentences may relate to sets of basic evidence, given a model and point within it. Note that if $\mathcal{M}$ is a model, $[\![\Gamma]\!]_{\mathcal{M}} = \{ [\![\gamma]\!] : \gamma \in \Gamma \}$ rather than the more common understanding of that notation as $\bigcap \{ [\![\gamma]\!] : \gamma \in \Gamma \}$. We will usually omit the subscript $\mathcal{M}$.

Definition 2. Let $\mathcal{M}$ be a model, $x \in W$, Γ a set of sentences, and $X \in \mathcal{E}(x)$. We will say,

- $\mathcal{M}$ **covers** Γ at x iff $\forall \gamma \in \Gamma, \exists X \in \mathcal{E}(x), X \subseteq [\![\gamma]\!]$.
- $\mathcal{M}$ **strongly covers** Γ at x iff $[\![\Gamma]\!] \subseteq \mathcal{E}(x)$.
- $\mathcal{M}$ is **unified** by Γ at x iff $\forall X \in \mathcal{E}(x), \exists \gamma \in \Gamma, [\![\gamma]\!] \subseteq X$.
- $\mathcal{M}$ is **strongly unified** by Γ at x iff $[\![\Gamma]\!] \subseteq \mathcal{E}(x)$ and $\mathcal{M}$ is unified by Γ at x.

In the vocabulary of evidence models from section 1, $\mathcal{M}$ covers Γ at x iff there *is evidence that* γ at x for each $\gamma \in \Gamma$, and strong covering is, intuitively, the claim that Γ is among the basic evidence at x. For unification, $\mathcal{M}$ unifies Γ at x when every piece of basic evidence is evidenced by something in Γ. Finally, strong unification is when the evidence at x is unified by a subset of the evidence at x. These concepts (and those that can be defined in terms of them) exhausts the ways in which we will need to refer to the relationships between theories and evidence sets, in order to establish a correspondence between syntactic covers of Γ and semantic covers of $\mathcal{E}(x)$. Moreover, note that covering is stable under subsets of Γ and it is easy to see that $\mathcal{M}$ covers Γ at x iff $\mathcal{M}, x \vDash E[\Gamma]$. Also, when $\mathcal{M}$ is unified by Γ at x, then $[\![\Gamma]\!]$ is a cover of $\mathcal{E}(x)$.

The natural epistemic interpretation of unification is that the evidence at x can be theoretically unified by taking Γ as a set of hypotheses, e.g., each piece of evidence can be predicted by the propositions in Γ. When we have Γ in hand, this is clearly an epistemic virtue often sought after in scientific theories: good theories should imply our evidence.[7] While philosophically important, we neglect further discussion of the intuitive philosophical interpretation of these concepts. Instead, we show that unification provides a relationships between evidence sets $\mathcal{E}(x)$ and theories Γ that suffices for a preservationist approach to evidence, by ensuring that syntactic level and semantic level coincide.

Notice first that covering does not suffice. When $\mathcal{M}$ covers Γ at x, the level of $\mathcal{E}(x)$ is not guaranteed to be the same as the level of Γ. Take $\Gamma = \{ p, q, r, \neg p, r \to \neg q \}$. This set has level 2 since

$$\Pi = \{ \pi_1 = \{ p, q, r \}, \pi_2 = \{ \neg p, r \to \neg q \} \}$$

is a partition cover. Then take any model $\mathcal{M}$ in which $\cap [\![\pi_1]\!] \neq \varnothing$ and $\cap [\![\pi_2]\!] \neq \varnothing$ such that there are a, b, c for which $a \subset [\![p]\!] \setminus [\![q]\!] \cup [\![r]\!]$ and $b \subset [\![q]\!] \setminus [\![p]\!] \cup [\![r]\!]$ and $c \subset [\![r]\!] \setminus [\![q]\!] \cup [\![p]\!]$. Let $\mathcal{E}(x) = \{ a, b, c \}$. Then $\mathcal{M}$ covers Γ at x, since for each $\gamma \in \Gamma$ one of a, b, c is a subset of its extension. (Obvious for p, q and r). Consider $[\![\neg q]\!]$, e.g. $a \subset [\![\neg q]\!]$, and likewise $a \subset [\![r]\!]^c \cup [\![q]\!]^c$.[8] But now we have an $\mathcal{M}$ that covers Γ at x but where $\ell(\mathcal{E}(x)) > \ell(\Gamma)$, since a, b, c all pairwise disjoint, $\ell(\mathcal{E}(x)) = 3$. We will also notice that in this model $\ell([\![\Gamma]\!]) > \ell(\Gamma)$. In general, by a similar pigeon hole argument as above, it will always be the case that $\ell([\![\Gamma]\!]) \geq \ell(\Gamma)$ for any Γ.

However, although $\mathcal{M}$ covers Γ does not ensure that the semantic cover has the same level as Γ, if $\mathcal{M}$ is also *unified* by Γ then the evidence set will have the same level as the extensions of all of the sentences in Γ.

Observation 1. *Let $\mathcal{M}$ be a model and $x \in W$. If $\mathcal{M}$ is unified by Γ at x, then $\ell([\![\Gamma]\!]_{\mathcal{M}}) \geq \ell(\mathcal{E}(x))$. If, in addition, $\mathcal{M}$ covers Γ at x, then $\ell([\![\Gamma]\!]_{\mathcal{M}}) = \ell(\mathcal{E}(x))$.*

Proof. $\ell([\![\Gamma]\!]_{\mathcal{M}}) \geq \ell(\mathcal{E}(x))$ is immediate since $[\![\Gamma]\!]_{\mathcal{M}}$ is a cover of $\mathcal{E}(x)$ when $\mathcal{M}$ is unified by Γ at x.

Suppose also that for all $[\![\gamma]\!] \in [\![\Gamma]\!]_{\mathcal{M}}$, there is $X \in \mathcal{E}(x)$ such that $X \subseteq [\![\gamma]\!]$, i.e., $\mathcal{M}$ covers Γ at x. That means $\mathcal{E}(x)$ is a cover of $[\![\Gamma]\!]_{\mathcal{M}}$ and as we have observed, then, $\ell(\mathcal{E}(x)) \geq \ell([\![\Gamma]\!]_{\mathcal{M}})$. Therefore, $\ell(\mathcal{E}(x)) = \ell([\![\Gamma]\!]_{\mathcal{M}})$

[7] Of course, it is also a property that can be trivially satisfied by taking Γ to be large enough—assuming that each $X \in \mathcal{E}(x)$ can be represented by a formula. Of course, if Γ has other properties, e.g., finiteness, that makes a better case for a non-trivial unification.

[8] X^c is the relative complement of X with respect to W.

In the modal language introduced so far, we can express covering, but not unification. The above result thus gives us reason to introduce an operator which allows us to express in the object language that a set of sentences unifies one's evidence. This operator, having variable arity, will be somewhat unorthodox. However, a similar operator has been introduced by [3] in the development of Instantial Neighbourhood Logic (INL).[9] The operator is constructed as follows:

If $\varphi_1, \dots, \varphi_n, \psi$ are formulas, so is $U(\varphi_1, \dots, \varphi_n; \psi)$.[10]

The inclusion of the formula at the end '$\dots;\psi)$' is an effort to build a logic that is parallel with INL. In future work we intend to investigate classes of operators—which we call 'pointed operators', with ψ is the point—that all have the same syntactic form and whose truth conditions have a similar shape. Having said that, the pointedness of the formula provides some very useful, and perhaps required, expressive power. The semantics of this operator is as one might expect given the discussion above:

$$\mathcal{M}, x \vDash U(\varphi_1, \dots, \varphi_n; \psi) \iff \text{ for all } X \in \mathcal{E}(x), \text{ if } X \subseteq [\![\psi]\!] \text{ then there is } i \leq n \text{ s.t. } [\![\varphi_i]\!] \subseteq X.$$

We have yet to bring syntactic and semantic conceptions of level together and a major stumbling block is that the syntactic consistency of a set of formulas requires looking at *all* the models whereas semantic level is determined merely by the model at hand. In some cases this gap can be bridged. Let $\mathbf{At}(\Gamma) = \{ p \in \mathbf{At} : p \text{ is mentioned in } \Gamma \}$ where $\mathbf{At}$ is the set of atomic sentences. Note that in the following observation we will just be using the conceptions of syntactic level derived from classical consequence. Let's call a model $\mathcal{M}$ **consistency comprehensive** for Γ when for all $X \subseteq \mathbf{At}(\Gamma)$, there is $x \in W$ such that for all $p \in \mathbf{At}(\Gamma)$, $\mathcal{M}, x \vDash p$ iff $p \in X$.

Observation 2. *Suppose Γ is a set of pure Boolean formulas. If $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ is a consistency comprehensive model, then $\ell(\Gamma) = \ell([\![\Gamma]\!]_{\mathcal{M}})$.*

Proof. First notice that if Γ contains some formula equivalent to $\bot$, then $\varnothing \in [\![\Gamma]\!]_{\mathcal{M}}$, and $\ell(\Gamma) = \infty = \ell([\![\Gamma]\!]_{\mathcal{M}})$.

Next, notice that if $\Gamma' \subseteq \Gamma$ and is propositionally consistent, there is a truth value assignment v to the atoms that are mentioned in Γ' such that $\vDash_v \Gamma'$. Let $X = \{ p : v(p) = T \} \cap \mathbf{At}(\Gamma)$. Then, by hypothesis there is a $x \in W$ such that $\mathcal{M}, x \vDash p$ iff $p \in X$ for all $p \in \mathbf{At}(\Gamma)$, so $\mathcal{M}, x \vDash \Gamma'$.

Let Π be a (syntatic) cover of Γ of width $\ell(\Gamma)$. Without loss of generality, we can assume that all logically equivalent formulas are in the same cells of the partition. Now form the following partition of $[\![\Gamma]\!]_{\mathcal{M}}$ by

$$\Pi' = \{ \{ [\![\gamma]\!] : \gamma \in \pi \} : \pi \in \Pi \}.$$

Claim: If $\pi' \in \Pi'$, then $\cap \pi' \neq \varnothing$. Since $\pi \subseteq \Gamma$ is a consistent subset of Γ (it is a cell in a cover of Γ), by the observation above there is $x \in W$ such that $\mathcal{M}, x \vDash \pi$. Thus, $x \in \cap \pi'$. Hence $\Pi'' = \{ \cap \pi' : \pi' \in \Pi' \}$ is a cover of $[\![\Gamma]\!]_{\mathcal{M}}$ and its width is $\ell(\Gamma)$ by construction. Thus $\ell([\![\Gamma]\!]_{\mathcal{M}}) \leq \ell(\Gamma)$. Since $\ell([\![\Gamma]\!]_{\mathcal{M}})$ cannot be less than $\ell(\Gamma)$, $\ell([\![\Gamma]\!]_{\mathcal{M}}) = \ell(\Gamma)$.

For a finite and purely Boolean Γ, consistency comprehensiveness can be expressible if we include a standard modal operator: $\Diamond\varphi$ meaning that φ is true at some "related" world. Although a relation could be added to interpret $\Diamond$, we will simply interpret $\Diamond$ as a global modality:

$$\mathcal{M}, x \vDash \Diamond\varphi \iff \text{ there is } w \in W \text{ s.t. } \mathcal{M}, w \vDash \varphi.$$

[9] The operator in [3] is $\Box(\varphi_1, \dots, \varphi_n; \psi)$ which is true at $\mathcal{M}, x$ iff there is $X \in \mathcal{E}(x)$ such that $X \subseteq [\![\psi]\!]$ and $X \cap [\![\varphi_i]\!] \neq \varnothing$ for each $i \leq n$. Its dual would be true, then, iff for all $X \in \mathcal{E}(x)$ if $X \subseteq [\![\psi]\!]$, then $X \subseteq [\![\varphi_i]\!]$ for some $i \leq n$. Whereas that operator says that all of the evidence is sufficient for at least one of φ_is—when it is sufficient for ψ, our U operator says that any piece of evidence is necessary for at least one of the φ_is, when it is sufficient for ψ.

[10] As abbreviations, we will write $\vec{\varphi}$ to mean $\varphi_1, \dots, \varphi_n$, and $(\vec{\varphi}/\psi)_i$ to mean

$$\varphi_1, \dots, \varphi_{i-1}, \psi, \varphi_{i+1}, \dots, \varphi_n$$

for $i \leq n$.

Now we can express consistency comprehensiveness. When Γ is finite and purely Boolean, let $\Diamond\mathbf{At}(\Gamma)$ be the formula:

$$\bigwedge_{\Gamma' \subseteq \mathbf{At}(\Gamma)} \Diamond((\bigwedge_{p \in \Gamma'} p) \wedge \bigwedge_{q \in \mathbf{At}(\Gamma) \setminus \Gamma'} (\neg q)).$$

So, for example, if $\mathbf{At}(\Gamma) = \{p, q, r\}$, then $\Diamond\mathbf{At}(\Gamma)$ is

$$\begin{aligned}&\Diamond(p \wedge q \wedge r) \wedge \Diamond(p \wedge q \wedge \neg r) \wedge \Diamond(p \wedge \neg q \wedge r) \wedge \\ &\Diamond(\neg p \wedge q \wedge r) \wedge \Diamond(p \wedge \neg q \wedge \neg r) \wedge \Diamond(\neg p \wedge q \wedge \neg r) \wedge \\ &\Diamond(\neg p \wedge \neg q \wedge r) \wedge \Diamond(\neg p \wedge \neg q \wedge \neg r)\end{aligned}$$

As is easily verified, $\mathcal{M}$ satisfies $\Diamond\mathbf{At}(\Gamma)$ iff $\mathcal{M}$ is consistency comprehensive for Γ.

While the results above assumed a particular proof theoretic relation to define the syntactic covers, none of its specifics beyond being an extension of classical propositional logic were used. It can be replaced by any extension of CPL even one that is merely determined by a semantics. In the latter case we replace consistency with satisfiability and consequence with entailment, both relative to whatever semantics is being used. As long as the resulting (semantic) consequence relation is reflexive, transitive and monotonic, the syntactic covers and thus the level function will have all the necessary properties. This is fortunate since the subsequent extensions we have made to the language have not been given any sort of axiomatization so far. Nonetheless, the results above still hold for our new language which includes the U operator and $\Diamond/\Box$. With this observation in mind we can then show the following:

Lemma 1. *Suppose that* $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. *If* $\mathcal{M}, x \vDash (E\gamma_1 \wedge \ldots \wedge E\gamma_n) \wedge U(\gamma_1, \ldots, \gamma_n; \top)$, *then* $\ell(\mathcal{E}(x)) = \ell([\![\Gamma]\!]_{\mathcal{M}})$. *If in addition* Γ *is purely boolean and* $\mathcal{M}$ *is consistency comprehensive for* Γ, $\ell(\Gamma) = \ell(\mathcal{E}(x))$.

Proof. From the previous observations and the definition of U.

Just as a reminder of the goal, we are trying to find a representation for forcing in terms of evidence logic in the same way that the K_n logics represent 'fixed-level forcing'. We have so far been able to find a way to express the level of a set of formulas, at least in the Boolean case (which is all we need).

To express that a formula is a forcing consequence we also need a way to canvas all the relevant covers of a set of sentences. While we will have to add an operator to the language to express the relevant relationship, it is expressible by a relation definable on the *frames* rather than on the models given. To define this relation we first need the idea of the *core* of $\mathcal{E}(x)$, denoted '$cor(\mathcal{E}(x))$' which is the set of any $\subseteq$-minimal elements of $\mathcal{E}(x)$. More precisely, $cor(\mathcal{E}(x)) = \{X \in \mathcal{E}(x) : \nexists\, Y \in \mathcal{E}(x), Y \subseteq X\}$, i.e., the set of elements of $\mathcal{E}(x)$ for which there is no proper subset also in $\mathcal{E}(x)$. A frame will be said to be *core complete* iff the core represents all the sets in $\mathcal{E}(x)$ in the sense that if $Y \in \mathcal{E}(x)$ there is some set $X \in cor(\mathcal{E}(x))$ such that $X \subseteq Y$.

It is fairly easy to see that $\ell(cor(\mathcal{X})) = \ell(\mathcal{X})$. What is also fairly easy to see is that if all the elements in the core are mutually exclusive, then the size of the core is the level of the set, i.e., $|cor(\mathcal{X})| = \ell(\mathcal{X})$, if for all distinct $X, Y \in cor(\mathcal{X})$, $X \cap Y = \varnothing$.

Now we define a relation $\mathrm{cov}_{\mathfrak{F}} \subseteq W \times W$ as follows:

Definition 3. Let $\mathfrak{F} = \langle W, \mathcal{E}\rangle$ be a hyperframe. For all $x, y \in W$, $\mathrm{cov}_{\mathfrak{F}}(x, y)$ holds iff

1. for all $X \in \mathcal{E}(x)$ there is $Y \in \mathcal{E}(y)$ such that $Y \subseteq X$,
2. for all $Y \in cor(\mathcal{E}(y))$ there is $X \in \mathcal{E}(x)$ such that $Y \subseteq X$, and
3. $|cor(\mathcal{E}(y))| = \ell(\mathcal{E}(x))$.

The idea is to have $\mathrm{cov}_{\mathfrak{F}}(x, y)$ iff the "evidence set" at y forms a cover of minimal width of the evidence at x relative to $\mathfrak{F}$. So, if we were to look at all models on all frames we would be able to find all possible covers of Γ of width $\ell(\Gamma)$. We can now extend the language to include a new operator F to interpret the $\mathrm{cov}_{\mathfrak{F}}$ relation on the frames, but the relation which interprets F needn't be all of $\mathrm{cov}_{\mathfrak{F}}$. In fact, it needn't be a subset of $\mathrm{cov}_{\mathfrak{F}}$ for the application that we have in mind. All that matters is that R_F and $\mathrm{cov}_{\mathfrak{F}}$ agree when $\mathcal{E}(x)$ is finitely unifiable, but we will discuss this in more detail in section 5. So, we can simply use a relation R_F on W which we will assume agrees with $\mathrm{cov}_{\mathfrak{F}}$ on the relevant pairs (x, y).

$$\mathcal{M}, x \vDash F\varphi \Longleftrightarrow \forall w \in W, R_F(x, w), \mathcal{M}, w \vDash \varphi.$$

Define $\Gamma \vDash_{\mathbf{F}} \varphi$ iff for all hypermodels over the language defined so far with the semantics developed so far, if $\mathcal{M}, x \vDash \Gamma$, then $\mathcal{M}, x \vDash \varphi$. $\vDash_{\mathbf{F}} \varphi$ when φ is true at all worlds in all hypermodels. As discussed above, we can use $\vDash_{\mathbf{F}}$ to define syntactic covers and observations 1 and 2 will carry over to the current context. Just to be explicit about how that is done: Π is a syntactic cover of Γ relative to $\mathbf{F}$ iff for each $\pi \in \Pi$, π is satisfiable and for each $\gamma \in \Gamma$ there is $\pi \in \Pi$ such that $\pi \vDash_{\mathbf{F}} \gamma$.

Lemma 2. *Let $\mathcal{M}$ be a hypergraph model. If $\mathcal{M}$ covers and is unified by Γ at x and $\mathrm{cov}_{\mathfrak{F}}(x,y)$, then $\Pi_{\mathcal{E}(y)} = \{\{\varphi : Y \subseteq [\![\varphi]\!]\} : Y \in cor(\mathcal{E}(y))\}$ is a syntactic cover of Γ (not necessarily a partition cover). If, in addition, Γ is pure boolean and $\mathcal{M}$ is consistency comprehensive for Γ, then the width of $\Pi_{\mathcal{E}(y)}$ is $\ell(\Gamma)$.*

Proof. Consider $\Pi_{\mathcal{E}(y)} = \{\{\varphi : Y \subseteq [\![\varphi]\!]\} : Y \in cor(\mathcal{E}(y))\}$. Since $\mathcal{M}$ covers Γ at x, for each $\gamma \in \Gamma$, there is $X \in \mathcal{E}(x)$ such that $X \subseteq [\![\gamma]\!]$. By condition 1 in the definition of cov, there is $Y \in \mathcal{E}(y)$ such that $Y \subseteq X$ and, by definition of the core, there is $Y' \in cor(\mathcal{E}(y))$ such that $Y' \subseteq Y$. Thus, for any $\gamma \in \Gamma$, there is a $Y' \in cor(\mathcal{E}(y))$ such that $Y' \subseteq [\![\gamma]\!]$. That means, for each $\gamma \in \Gamma$, there is $\pi \in \Pi_{\mathcal{E}(y)}$ such that $\gamma \in \pi$, hence $\pi \vDash_F \gamma$. Furthermore, by definition of $\Pi_{\mathcal{E}(y)}$, for each $\pi \in \Pi_{\mathcal{E}(y)}$ there is a $Y \in cor(\mathcal{E}(y))$ such that $Y \subseteq \cap[\![\pi]\!]$ and of course $Y \neq \varnothing$ since $\mathcal{M}$ is a hypermodel and so $\varnothing \notin \mathcal{E}(y)$. Hence each π is satisfiable. Thus $\Pi_{\mathcal{E}(y)}$ is a syntactic cover of Γ. Also notice that the width of $\Pi_{\mathcal{E}(y)}$ is $|cor(\mathcal{E}(y))|$.

From observation 1 we know that $\ell(\mathcal{E}(x)) = \ell([\![\Gamma]\!])$ since $\mathcal{M}$ covers and is unified by Γ at x. So, $|cor(\mathcal{E}(y))| = \ell([\![\Gamma]\!])$ by condition 3 in the definition of cov. If we also assume that Γ is pure boolean and that $\mathcal{M}$ is consistency comprehensive for Γ, then by observation 2, $\ell([\![\Gamma]\!]) = \ell(\Gamma)$. Hence the width of $\Pi_{\mathcal{E}(y)}$ is $\ell(\Gamma)$.

Now we can ask the relevant question: is this logic one that allows us to capture classical forcing in at least the finite cases? The answer, fortunately, is 'yes'.

Theorem 1. *Suppose $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ and φ are purely Boolean.*

$$\Gamma \Vdash \varphi \iff \vDash_{\mathbf{F}} [(E\gamma_1 \wedge \ldots \wedge E\gamma_m) \wedge U(\gamma_1, \ldots, \gamma_m; \top) \wedge \Diamond \mathbf{At}(\Gamma)] \to FE\varphi$$

Proof. The only if direction follows by proving the contrapositive using lemma 2. If $\nvDash_{\mathbf{F}} [(E\gamma_1 \wedge \ldots \wedge E\gamma_m) \wedge U(\gamma_1, \ldots, \gamma_m; \top) \wedge \Diamond \mathbf{At}(\Gamma)] \to FE\varphi$, then there is a model and world $\mathcal{M}, x$ such that $\mathcal{M}, x \vDash (E\gamma_1 \wedge \ldots \wedge E\gamma_m) \wedge U(\gamma_1, \ldots, \gamma_m; \top) \wedge \Diamond \mathbf{At}(\Gamma)$, but $\mathcal{M}, x \nvDash FE\varphi$. Since $\Diamond \mathbf{At}(\Gamma)$ is true at x, $\mathcal{M}$ is consistency comprehensive for Γ. And since $\mathcal{M}, x \nvDash FE\varphi$, there is a $y \in W$ such that $\mathrm{cov}(x,y)$ such that $\mathcal{M}, y \nvDash E\varphi$. Now we can apply lemma 2 to $\mathcal{E}(y)$ and given that $\mathrm{cov}(x,y)$, $|cor(\mathcal{E}(y))| = \ell(\Gamma)$. Given that $|\Pi_{\mathcal{E}(y)}| = |cor(\mathcal{E}(y))|$, we get a cover of Γ such that each cell does not entail φ.

For the if direction again we will argue contrapositively. Assume that $\Gamma \nVdash \varphi$. Note that in this case, $\Vdash$ is forcing based on classical propositional logic.

Notice that since Γ is finite $\ell(\Gamma) = n \leq |\Gamma|$. Since Γ does not force φ there is a syntactic partition cover $\Pi = \{\pi_i : 1 \leq i \leq n\}$ of Γ (of width n) such that for all $\pi \in \Pi$, $\pi \nvdash \varphi$. Hence there are n truth value assignments $v_1, \ldots, v_n$ such that for each $\pi \in \Pi$, there is $i \leq n$ such that $\vDash_{v_i} \pi$ and $\nvDash_{v_i} \varphi$ by the completeness of CPL with respect to two-valued truth value assignments.

Define $W_\Pi = \{v : \exists X \subseteq \mathbf{At}(\Gamma), \forall p \in \mathbf{At}, p \in X \text{ only if } v(p) = T\}$; and $V_\Pi : \mathbf{At} \to \mathcal{P}(W)$ such that $V_\Pi(p) = \{v \in W_\Pi : v(p) = T\}$. Finally, define $\mathcal{E}_\Pi(v)$ as

$$\mathcal{E}_\Pi(v) = \begin{cases} \{[\![\gamma]\!] : \gamma \in \Gamma\} & v = v_1 \\ \{\cap[\![\pi]\!] : \pi \in \Pi\} & v \neq v_1 \end{cases}$$

Let $\mathcal{M}_\Pi = \langle W_\Pi, \mathcal{E}_\Pi, V_\Pi \rangle$. In the case where Γ is consistent $\mathcal{E}(v) = \{\cap[\![\Gamma]\!]\}$ for all $v \neq v_1$. Since all the v_i above are in W_Π, $W_\Pi \neq \varnothing$. Similarly, for all $v \in W$, and $X \in \mathcal{E}_\Pi(v)$, there is v_i from above such that $v_i \in X$. So, $\varnothing \notin \mathcal{E}_\Pi(v)$. As we have defined $\mathcal{M}_\Pi$, it is consistency comprehensive for Γ, and since $\mathcal{E}_\Pi(v_1) = \{[\![\gamma]\!] : \gamma \in \Gamma\}$, $\mathcal{M}_\Pi, v_1 \vDash [(E\gamma_1 \wedge \ldots \wedge E\gamma_m) \wedge U(\gamma_1, \ldots, \gamma_m; \top) \wedge \Diamond \mathbf{At}(\Gamma)]$.

Let $\mathfrak{F}_\Pi = \langle W_\Pi, \mathcal{E}_\Pi \rangle$. Now we must exhibit at least one world that v_1 relates to by $\mathrm{cov}_{\mathfrak{F}_\Pi}$ at which φ is false. Again since $\mathcal{M}_\Pi$ is consistency comprehensive for Γ, by lemma 2,

$$\ell(\mathcal{E}_\Pi(v_1)) = \ell(cor(\mathcal{E}_\Pi(v_1))) = \ell([\![\Gamma]\!]) = \ell(\Gamma) = |\mathcal{E}_\Pi(v)|$$

for all $v \neq v_1$. Since for each $\gamma \in \Gamma$, there is $\pi \in \Pi$ such that $\pi \models \gamma$, $\cap[\![\pi]\!] \subseteq [\![\gamma]\!]$. But also, since each π is a consistent non-empty subset of Γ, there is $\gamma \in \Gamma$ such that $\cap[\![\pi]\!] \subseteq [\![\gamma]\!]$, since $\gamma \in \pi$. Thus, $\text{cov}_{\mathfrak{F}_\Pi}(v_1, v)$ for all $v \neq v_1$. So, in particular $\text{cov}_{\mathfrak{F}_\Pi}(v_1, v_2)$ and we can set $R_F = \text{cov}_{\mathfrak{F}_\Pi}$.

Finally, each of the v_i from above are such that $v_i \in \cap[\![\pi_i]\!]$ but $v_i \notin [\![\varphi]\!]$ for $i \leq n$; hence $\cap[\![\pi_i]\!] \nsubseteq [\![\varphi]\!]$. So, by definition, $\mathcal{M}_\Pi, v_2 \nvDash E\varphi$ and so $\mathcal{M}_\Pi, v_1 \nvDash FE\varphi$. Therefore, $\nvDash_F [(E\gamma_1 \wedge \ldots \wedge E\gamma_m) \wedge U(\gamma_1, \ldots, \gamma_m; \top) \wedge \Diamond\mathbf{At}(\Gamma)] \to FE\varphi$.

Thus, this logic allows one to represent classical forcing via a modal evidence logic. The next step, is to axiomatize the system. We will first give an axiomatization for a logic with the operators $E, \Box, F$, and U relative to all hypergraph models for which $\varnothing \notin \mathcal{E}(x) \neq \varnothing$. The logic **F** required by Theorem 1 is obtained by adding axioms to the logic **U** and is discussed in section 5.

4 Semantics and Axiomatization for U

We start with the language $\mathcal{L}_\mathbf{U}$. It is defined by the following BNF:

$$\varphi \coloneqq \bot \mid p \mid \neg\varphi \mid F\varphi \mid E\varphi \mid \Box\varphi \mid \varphi \to \varphi \mid U(\underbrace{\varphi, \ldots, \varphi}_{n-\text{times}}; \varphi) \; n \in Z^+$$

Where $p \in \mathbf{At}$ the set of atoms. The operators $\Diamond$, $\langle F \rangle$, and $\langle E \rangle$ are defined via their duals $\neg \blacksquare \neg\varphi$ for $\blacksquare \in \{\Box, F, E\}$, and the other Boolean connectives are defined in the usual way. In the interest of limiting the number of operators to keep it in line with the literature on evidence logics we won't introduce additional neighbourhood operators like: for all $X \in \mathcal{E}(x), X \subseteq [\![\varphi]\!]$ which have been discussed elsewhere [10]. Next we have a frame and then a model:

Definition 4. A structure $\mathfrak{F} = \langle W, \mathcal{E} \rangle$ is a hypergraph frame iff:

1. $W \neq \varnothing$, and
2. $\mathcal{E} : W \to \mathcal{P}(\mathcal{P}(W))$ such that for all $x \in W$
   (a) $\varnothing \notin \mathcal{E}(x)$, and
   (b) $\mathcal{E}(x) \neq \varnothing$
3. R_F is a relation on W
4. The frame is **augmented** when there is an equivalence relation $R_\Box \subseteq W \times W$ added to the frame.

A hypergraph[11] **model** is a structure $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ where $\mathfrak{F}$ is a hypergraph frame and $V : \mathbf{At} \to \mathcal{P}(W)$.

Let $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ be a hypermodel. The semantics for the logic **U** for hypermodels is:

- $\mathcal{M}, x \models p$ iff $x \in V(p)$ for all $p \in \mathbf{At}$
- Boolean cases as usual,
- $\mathcal{M}, x \models E\varphi$ iff there is $X \in \mathcal{E}(x)$ such that $X \subseteq [\![\varphi]\!]$,
- $\mathcal{M}, x \models \langle E \rangle\varphi$ iff for all $X \in \mathcal{E}(x)$, $X \cap [\![\varphi]\!] \neq \varnothing$,
- $\mathcal{M}, x \models \Box\varphi$ iff $[\![\varphi]\!] = W$,
- $\mathcal{M}, x \models \Diamond\varphi$ iff $[\![\varphi]\!] \neq \varnothing$,
- $\mathcal{M}, x \models F\varphi$ iff $R_F(x) \subseteq [\![\varphi]\!]$,
- $\mathcal{M}, x \models U(\varphi_1, \ldots, \varphi_n; \psi)$ iff for all $X \in \mathcal{E}(x)$, $X \subseteq [\![\psi]\!]$ only if for some $i \leq n$, $[\![\varphi_i]\!] \subseteq X$

This semantics gives rise to a semantic consequence relation $\models_\mathbf{U}$, defined in the usual way. This system is complete with respect to the following axioms, which will give rise to the syntactic system $\vdash_\mathbf{U}$. In the following axioms $\vec{\varphi}$ refers to a tuple of formulas $\varphi_1, \ldots, \varphi_n$ as before, but in cases where it is not the only argument on the left of the ';' in a U operator it can be empty. $n!$ refers to all permutations of $\{1, 2, \ldots, n\}$ and σ will be a specific permutation in $n!$ where $\sigma(k)$ is the number that k is permuted to by the permutation σ. Let p, q, r, s, p_i be in **At**.

[11] We are using a 'hypergraph model' in the sense found in [9] rather than in [6]. Our hypergraph models are what they call neighbourhood models and what [5] calls 'Minimal Models'. Topologically speaking, it would make more sense to call neighbourhood models those minimal models $\langle W, \mathcal{E} \rangle$ in which for each $x \in W$, $x \in \bigcap \mathcal{E}(x)$ since a neighbourhood of x would usually contain x.

CL All theorems of classical propositional logic.
S5 The axioms of S5 for $\Box$.
KF $(Fp \wedge Fq) \longleftrightarrow F(p \wedge q)$
$\Box$F $\Box p \to Fp$
D $\neg E\bot$
N $E\top$
E$\Box$ $\Box(p \to q) \to (Ep \to Eq)$
MergeE $(Ep \wedge \Box q) \to E(p \wedge q)$
U$\bot$ $U(\bot; q)$
U! $U(p_1, \ldots, p_n; \psi) \to (\bigwedge_{\sigma \in n!} U(p_{\sigma(1)}, \ldots, p_{\sigma(n)}; q))$
UE $\neg U(\vec{p}; q) \to Eq$
U+ $U(\vec{p}; q) \to U(\vec{p}, r; q)$
U- $U(\vec{p}, r, r; q) \to U(\vec{p}, r; q)$
UV $(U(\vec{p}; q) \wedge Eq) \to \bigvee_{i=1}^{n} \Box(p_i \to q)$
U$\Box$R $\Box(q \to r) \to (U(\vec{p}; r) \to U(\vec{p}; q))$
U$\Box$L $\Box(q \to r) \to (U((\vec{p}/r)_i; s) \to U((\vec{p}/q)_i; s))$
With rules
US Uniform Substitution,
MP Modus Ponens,
Nec $\vdash \varphi$ only if $\vdash \Box\varphi$
UInf

$$\frac{\vdash \theta \to (\Box(p \to \psi) \to (\bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p) \to \neg Ep))}{\vdash \theta \to U(\varphi_1, \ldots, \varphi_n; \psi)} \; p \text{ foreign to } \varphi_1, \ldots, \varphi_n, \psi, \theta$$

The usual definitions for Hilbert-style proof theory are used: $\Gamma \vdash_{\mathbf{U}} \varphi$ iff there are $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that $\vdash_{\mathbf{U}} (\gamma_1 \wedge \ldots \wedge \gamma_n) \to \varphi$. As will be shown in section 6:

Theorem 2. *The system $\vdash_{\mathbf{U}}$ is sound and complete with respect to $\vDash_{\mathbf{U}}$.*

A few comments about the system are in order. The axiomatization is obviously not finite, but it is recursive. We can also treat the tuple of formulas before the semicolon in the U operators as a set given axioms U! and U-. The contrapositive of UV is equivalent to $\bigwedge_{i=1}^{n} \Diamond(\varphi_i \wedge \neg\psi) \to (E\psi \to \neg U(\vec{\varphi}; \psi))$, and given UE, $\bigwedge_{i=1}^{n} \Diamond(\varphi_i \wedge \neg\psi) \to (E\psi \longleftrightarrow \neg U(\vec{\varphi}; \psi))$ is derivable for any formulas $\vec{\varphi}, \psi$. That also indicates how to interpret the UInf rule. UInf formalizes the idea that if no proposition that both implies ψ and is not implied by any of the φ_is in $U(\varphi_1, \ldots, \varphi_n; \psi)$, can be in an evidence set at a world when θ is also true, then $U(\varphi_1, \ldots, \varphi_n; \psi)$ must be true.

5 Definability and the Logic F

The first thing we will point out is that we know the logic **U** is distinct from Instantial Neighbourhood Logic (INL) of [3]. The reason for this is that using the U and E operators we can define $\Box$ in the context of the $E\Box$ axiom[12]:

$$(U(\neg\varphi; \varphi) \wedge E\varphi) \longleftrightarrow \Box\varphi$$

While the E operator can be defined in INL—it is a special case of it—the authors show that $\Box$ is not definable in INL. Although this means that $\Box$ isn't needed in **U**, it is convenient to treat it as separate.

The system U is complete with respect to the class of all hypermodels. But the system needed to meet the requirements for the proof of Theorem 1 asks more of the relation R_F which interprets the F operator. The condition that is sufficient for Theorem 1 is the following: If $\mathcal{M}$ is a hypergraph model based on the frame $\mathfrak{F} = \langle W, \mathcal{E} \rangle$, then for all $x \in W$, $\mathcal{E}(x)$ is finitely unifiable only if for all $y \in W$ such that $R_F(x, y)$, $\mathrm{cov}_{\mathfrak{F}}(x, y)$. I.e., when $\mathcal{E}(x)$ is finitely unifiable, all the R_F-realted worlds are minimal covers of $\mathcal{E}(x)$.

[12] A syntactic derivation of this equivalence proceeds as follows: Suppose $U(\neg\varphi; \varphi) \wedge E\varphi$. An instance of UV is $(U(\neg\varphi; \varphi) \wedge E\varphi) \to \Box(\neg\varphi \to \varphi)$, so we can infer $\Box(\neg\varphi \to \varphi)$ which is equivalent to $\Box\varphi$ in any normal modal logic. Conversely, suppose $\Box\varphi$. Thus, in any normal modal logic $\Box\varphi \to \Box(\neg\varphi \to \bot)$ is a theorem. By U$\bot$, $U(\bot; \varphi)$ is a theorem of U, and by U$\Box$L, $\Box(\neg\varphi \to \bot) \to (U(\bot; \varphi) \to U(\neg\varphi; \varphi))$ is a theorem and thus, $U(\neg\varphi; \varphi)$ follows. Since $\Box\varphi \to \Box(\top \to \varphi)$ is a theorem of any normal modal logic, using N, and E$\Box$, we can derive $E\varphi$.

The task is to find axioms which guarantee that the conditions in the definition of $\mathrm{cov}_{\mathfrak{F}}(x,y)$ are met. Thus, we need to show that if, $R_F(x,y)$ and $\mathcal{E}(x)$ is finitely unifiable, then 1) for all $X \in \mathcal{E}(x)$ there is $Y \in \mathcal{E}(y)$ such that $Y \subseteq X$, 2) for all $Y \in cor(\mathcal{E}(y))$ there is $X \in \mathcal{E}(x)$ such that $Y \subseteq X$, and 3) $|cor(\mathcal{E}(y))| = \ell(\mathcal{E}(x))$.

These requirements can be achieved by imposing axioms which define certain properties of the frames, since, after all, the properties that are required depend on the frames rather than the models. As per usual, a formula α is valid on a frame $\mathfrak{F} = \langle W, \mathcal{E}, R_F \rangle$ iff for all models $\mathcal{M}$ based on $\mathfrak{F}$, and all $x \in W$, $\mathcal{M}, x \vDash \alpha$, and we will denote that α is valid on $\mathfrak{F}$ by $\mathfrak{F} \vDash \alpha$.

Ensuring that condition 1 is met requires a fairly simple axiom which we refer to as EF: $Ep \to FEp$.

Proposition 1. *Let $\mathfrak{F}$ be a hyperframe. $\mathfrak{F} \vDash Ep \to FEp$ iff for all $x, y \in W$, if $R_F(x,y)$, then for all $X \in \mathcal{E}(x)$ there is $Y \in \mathcal{E}(y)$ such that $Y \subseteq X$. The proof is standard and uncomplicated, so we will omit it.*

To capture the other conditions we will first define some operators as abbreviations to simplify the expression of the axioms. One of the first things that we can notice is the one can express that a (finite) set of formulas forms of cover of $\mathcal{E}(x)$. We define $\mathrm{cov}(\vec{p})$:

$$\mathrm{cov}(p_1, \ldots, p_n) \coloneqq \bigwedge_{i=1}^{n} \Diamond p_i \wedge U(p_1, \ldots, p_n; \top)$$

When $\mathrm{cov}(\vec{\varphi})$ is true at $x \in W$, then $\vec{\varphi}$ unifies $\mathcal{E}(x)$ so $\{ [\![\varphi]\!] : \varphi \in \vec{\varphi} \}$ could serve as a semantic cover for $\mathcal{E}(x)$ since none of the $[\![\varphi]\!]$ is empty, but not necessarily a partition cover. But $\vec{\varphi}$ may not strongly unify $\mathcal{M}$ at x when $\mathrm{cov}(\vec{\varphi})$ is true.

The next operator indicates that the extensions of the formulas to which it applies are found in the core of $\mathcal{E}(x)$:

$$\mathrm{core}(p_1, \ldots, p_n) \coloneqq \bigwedge_{i=1}^{n} (Ep_i \wedge U(p_i; p_i))$$

The ability to express that the extension of a formula is in the core of an evidence set is a great side-effect of making the U operator parallel with those found in the Instantial Neighbourhood Logic of [3]. Without the operator's "point"—the formula after the semicolon—we could not guarantee that, when Ep is also true, $[\![p]\!]$ is the only element of $\mathcal{E}(x)$ which is contained in $[\![p]\!]$. If we add $U(p_1, \ldots, p_n; \top)$ to $\mathrm{core}(p_1, \ldots, p_n)$, we get a formula that expresses that $\mathcal{E}(x)$ contains a cover of itself as its core, i.e., $cor(\mathcal{E}(x)) = \{ [\![p_i]\!] : i \leq n \}$. This operator expresses that the sequence of formulas constitute the entire core of $\mathcal{E}(x)$:

$$\mathrm{totalcore}(p_1, \ldots, p_n) \coloneqq \bigwedge_{i=1}^{n} (Ep_i \wedge U(p_i; p_i)) \wedge U(p_1, \ldots, p_n; \top).$$

To capture conditions 2 and 3 in the definition of $\mathrm{cov}_{\mathfrak{F}}(x,y)$ we use recursive sets of formulas. While EF provided condition 1 without the assumption that $\mathcal{E}(x)$ is finitely unifiable, our next "axioms" make that assumption explicit.

While we usually work with individual axioms or collections of various axioms to define frame conditions, the following "axioms" are actually recursive sets of formulas. Define the set of formulas Cor by

$$\mathrm{Cor} \coloneqq \left\{ \mathrm{totalcore}(p_1, \ldots, p_n) \to (\langle F \rangle\, \mathrm{core}(q) \to \bigvee_{i=1}^{n} \Box(q \to p_i)) : n > 0 \,\&\, p_i, q \in \mathbf{At} \right\}$$

Proposition 2. *Let $\mathfrak{F}$ be a hyperframe. $\mathfrak{F} \vDash Cor$ iff for all $x, y \in W$ if $R_F(x,y)$ and $\mathcal{E}(x)$ has a finite and non-empty core, then for all $Y \in cor(\mathcal{E}(y))$ there is $X \in cor(\mathcal{E}(x))$ such that $Y \subseteq X$.*

The condition on frames above is stronger than what condition 2 requires since it says that for each set in the core of any world that x R_F-relates to will imply all the elements of the core of $\mathcal{E}(x)$, provided there is a core. That could pose a problem since condition 2 doesn't require that all evidence sets have cores. However, as we discuss at the end of section 6.2, **U** is complete with respect to the class of core-complete hyperframes, so we can restrict attention to only core-complete hyperframes in this context as well. In addition, the proof of Theorem 1 only use a model which is core-complete, so the assumption of core-completeness leaves all results intact.

Now we can home in on finding conditions for $|cor(\mathcal{E}(y))| = \ell(\mathcal{E}(x))$. The first thing to notice is that in section three the results were limited to finite cases, so while there can be evidence sets which have infinite levels, we are setting those to the side for the moment. We shall define another set of formulas UpLev as follows:

$$\text{UpLev} = \{ \operatorname{cov}(q_1,\dots,q_k) \to (\langle F\rangle \operatorname{totalcore}(p_1,\dots,p_n) \to U(p_1,\dots,p_n;\top)) : n,k \in \mathbb{N} \ \&\ p_i, q_j \in \mathbf{At} \}$$

Proposition 3. *Let $\mathfrak{F}$ be a hyperframe. $\mathfrak{F} \vDash$ UpLev iff for all $x,y \in W$, if $cor(\mathcal{E}(y))$ and $\ell(\mathcal{E}(x))$ are finite, then $R_F(x,y)$ only if $cor(\mathcal{E}(y))$ is a cover of $\mathcal{E}(x)$.*

The effect of this result is to enforce an upper bound on $\ell(\mathcal{E}(x))$ when it is finite; hence the name. Notice that if $|cor(\mathcal{E}(y))|$ is finite and $R_F(x,y)$ in an UpLev-frame, i.e., a frame $\mathfrak{F}$ where all formulas in UpLev are valid on $\mathfrak{F}$, then $\ell(\mathcal{E}(x)) \leq |cor(\mathcal{E}(y))|$. That follows since if $cor(\mathcal{E}(y))$ is a cover of $\mathcal{E}(x)$, then the level of $\ell(\mathcal{E}(x))$ can't be any larger than the size of that cover. What is needed, then, is a lower bound. For that we define:

$$\text{LowLev} = \left\{ \operatorname{cov}(r_1,\dots,r_n) \to (\langle F\rangle \operatorname{core}(p_1,\dots,p_k) \to (U(q_1,\dots,q_m;\top) \to \bigvee_{i=1}^{m} \neg\Diamond q_i)) : n,k,m \in \mathbb{N} \ \&\ m < k \right\}$$

Proposition 4. *Let $\mathfrak{F}$ be an hyperframe. $\mathfrak{F} \vDash$ LowLev iff for all $x,y \in W$, if $R_F(x,y)$, then $\ell(\mathcal{E}(x)) \geq |cor(\mathcal{E}(y))|$ when $\ell(\mathcal{E}(x))$ is finite.*

Proof. Suppose that $\mathfrak{F}$ is a hyperframe such that for all $x,y \in W$, if $R_F(x,y)$, then $\ell(\mathcal{E}(x)) \in \mathbb{N}$ only if $\ell(\mathcal{E}(x)) \geq |cor(\mathcal{E}(y))|$. Now suppose that $\mathcal{M}$ is a model based on $\mathfrak{F}$ and that $x \in W$ such that $\mathcal{M}, x \vDash \operatorname{cov}(r_1,\dots,r_n) \wedge \langle F\rangle \operatorname{core}(p_1,\dots,p_k) \wedge U(q_1,\dots,q_m;\top)$ where $m < k$. From $\mathcal{M}, x \vDash \operatorname{cov}(r_1,\dots,r_n)$, we can infer that $\ell(\mathcal{E}(x))$ is finite and from $\mathcal{M}, x \vDash \langle F\rangle \operatorname{core}(p_1,\dots,p_k)$ we can infer that there is $y \in W$ such that $R_F(x,y)$ (and that $\mathcal{M}, y \vDash \operatorname{core}(p_1,\dots,p_k)$). Thus, by our assumption about $\mathfrak{F}$, $\ell(\mathcal{E}(x)) \geq |cor(\mathcal{E}(y))|$. Since $\mathcal{M}, y \vDash \operatorname{core}(p_1,\dots,p_k)$, $\{ [\![p_i]\!] : i \leq k \} \subseteq cor(\mathcal{E}(y))$, hence, $|cor(\mathcal{E}(y))| \geq k$. Now suppose for reductio that for each $i \leq m$, $[\![q_i]\!] \neq \varnothing$. Since $\mathcal{M}, x \vDash U(q_1,\dots,q_m;\top)$, $\{ [\![q_i]\!] : i \leq m \}$ is a cover of $\mathcal{E}(x)$. In general, if $\mathcal{X}$ is a cover of $\mathcal{Y}$, then $\ell(\mathcal{Y}) \leq \ell(\mathcal{X})$, and $\ell(\mathcal{X}) \leq |\mathcal{X}|$. Thus, $\ell(\mathcal{E}(x)) \leq \ell(\{ [\![q_i]\!] : i \leq m \}) \leq |\{ [\![q_i]\!] : i \leq m \}| = m$. Thus,

$$\ell(\mathcal{E}(x)) \geq |cor(\mathcal{E}(y))| \geq k > m \geq \ell(\{ [\![q_i]\!] : i \leq m \}) \geq \ell(\mathcal{E}(x)),$$

a contradiction. So, some $[\![q_i]\!] = \varnothing$. Therefore, $\mathcal{M}, x \vDash \bigvee_{i=1}^{m} \neg\Diamond q_i$. Since n, m, and k were arbitrary as was the model $\mathcal{M}$ based on $\mathfrak{F}$, $\mathfrak{F} \vDash$ LowLev.

Conversely, suppose that $\mathfrak{F}$ is such that there are $x,y \in W$ such that $R_F(x,y)$ and $\ell(\mathcal{E}(x))$ is finite, but that $\ell(\mathcal{E}(x)) < |cor(\mathcal{E}(y))|$. Since $\ell(\mathcal{E}(x))$ is finite suppose it is n and then suppose that $\{ X_1,\dots,X_n \}$ is a cover of minimal width of $\mathcal{E}(x)$. Suppose that $\{ Y_1,\dots,Y_{n+1} \} \subseteq cor(\mathcal{E}(y))$ which must exist since $|cor(\mathcal{E}(y))| > n$. Define $\mathcal{M}$ in which $V(r_i) = X_i = V(q_i)$ for $i \leq n$ and $V(p_j) = Y_j$ for $j \leq n+1$. Since $n < n+1$, the formula:

$$\operatorname{cov}(r_1,\dots,r_n) \to (\langle F\rangle \operatorname{core}(p_1,\dots,p_{n+1}) \to (U(q_1,\dots,q_m;\top) \to \bigvee_{i=1}^{n} \neg\Diamond q_i))$$

is in LowLev. Furthermore, since the X_i's form a cover of $\mathcal{E}(x)$ none of them is empty nor are any of the Y_js since they are from the core of $\mathcal{E}(y)$. As we have assumed that $R_F(x,y)$, $\mathcal{M}, y \vDash \operatorname{core}(p_1,\dots,p_{n+1})$ and so $\mathcal{M}, x \vDash \langle F\rangle \operatorname{core}(p_1,\dots,p_{n+1})$. And as we have assumed the X_is are a cover of $\mathcal{E}(x)$, $\mathcal{M}, x \vDash \operatorname{cov}(r_1,\dots,r_n)$, but also as part of that $\mathcal{M}, x \vDash U(q_1,\dots,q_n)$. However, since none of the X_is is empty $\mathcal{M}, x \nvDash \bigvee_{i=1}^{n} \neg\Diamond q_i$. Thus

$$\mathcal{M}, x \nvDash \operatorname{cov}(r_1,\dots,r_n) \to (\langle F\rangle \operatorname{core}(p_1,\dots,p_{n+1}) \to (U(q_1,\dots,q_m;\top) \to \bigvee_{i=1}^{n} \neg\Diamond q_i))$$

and so $\mathfrak{F} \nvDash$ LowLev.

Suppose now that $\mathfrak{F}$ is a (core-complete) frame on which EF, UpLev, LowLev, and Cor are all valid. If $x \in W$ and $\ell(\mathcal{E}(x))$ is finite, then for any $y \in W$ such that $R_F(x,y)$, $|cor(\mathcal{E}(y))|$ must also be finite by LowLev. Hence, by UpLev, if $R_F(x,y)$, $cor(\mathcal{E}(y))$ must be a cover of $\mathcal{E}(x)$. Thus, $|cor(\mathcal{E}(y))| = \ell(\mathcal{E}(x))$.

We will refer to a hyperframe $\mathfrak{F}$ which is an EF, LowLev, UpLev, and Cor frame as a **forcing**-frame. If $\mathfrak{F}$ is a forcing-frame, then if $x \in W$ and $\ell(\mathcal{E}(x))$ is finite, then $R_F(x,y)$ only if $\operatorname{cov}_{\mathfrak{F}}(x,y)$. Thus, the relation $\vDash_{\mathbf{F}}$ and its underlying semantics needed to prove Theorem 1 is the class of forcing frames. As an example of a forcing frame, one can consider the frame constructed in the proof of theorem 1.

We can then get the proof theory of the logic **F** by adding to the logic **U** the additional axioms:

EF $Ep \to FEp$
Cor $\mathrm{totalcore}(p_1,\ldots,p_n) \to (\langle F\rangle \mathrm{core}(q) \to \bigvee_{i=1}^{n} \Box(q \to p_i))$ where $n > 0$
UpLev $\mathrm{cov}(q_1,\ldots,q_k) \to (\langle F\rangle \mathrm{totalcore}(p_1,\ldots,p_n) \to U(p_1,\ldots,p_n;\top))$ where $n > 0$
LowLev $\mathrm{cov}(r_1,\ldots,r_n) \to (\langle F\rangle \mathrm{core}(p_1,\ldots,p_k) \to (U(q_1,\ldots,q_m;\top) \to \bigvee_{i=1}^{m} \neg\Diamond q_i))$ where $m < k$ and $n > 0$

6 Soundness and Completeness of U

6.1 Soundness

The validity of most of the axioms is straightforward. The $\Box$ operator is supposed to be a global necessity, and F is, at this point, just a normal modal operator. D ensures that $\varnothing \notin \mathcal{E}(x)$ and N ensures that $\mathcal{E}(x) \neq \varnothing$. The E operator is a classical modal operator in Segerberg's sense, hence E$\Box$. The other things to notice is that since $\Box$ is global necessity, the truth of $\Box(\varphi \to \psi)$ anywhere in a model translates to $[\![\varphi]\!] \subseteq [\![\psi]\!]$.

U$\bot$ is valid becuase $\varnothing$ is a subset of any set and U! is valid because the disjunction used to give the truth condition of U is communative. Similarly, U+ and U- are valid because of properties of disjunction. The U$\Box$ axioms show that the operator is anti-monotonic on both the left and right side of ';' and follows because of the transitivity of the subset relation. The validity of UE can be seen by inspecting the truth condition for U and noticing that it is a conditional with $X \subseteq [\![\psi]\!]$ as its antecedent. Finally, UV is valid again because of the transitivity of the subset relation.

The only really interesting inference rule/axiom is UInf, and to prove that it is sound we need the following standard fact. Say that $\mathcal{M} = \langle W, R_F, \mathcal{E}, V\rangle$ and $\mathcal{M}' = \langle W', R'_F, \mathcal{E}', V'\rangle$ differ at most on $p \in \mathbf{At}$ iff $W = W'$, $\mathcal{E} = \mathcal{E}'$, $R_F = R'_F$ and $V(q) = V'(q)$ for all $q \neq p$ from $\mathbf{At}$. Then we have that:

Lemma 3. *If $\mathcal{M}$ and $\mathcal{M}'$ differ at most on p, then $[\![\varphi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}'}$ for all φ which do not mention p.*

Proof. The usual induction on the complexity of φ.

Proposition 5. UInf *is sound.*

Proof. Suppose that p is foreign to all of $\varphi_1,\ldots,\varphi_n,\psi,\theta$ and $\nvDash \theta \to U(\varphi_1,\ldots,\varphi_n;\psi)$. So there is a model $\mathcal{M} = \langle W, R_F, \mathcal{E}, V\rangle$ and $x \in W$ such that $\mathcal{M}, x \vDash \theta$, but $\mathcal{M}, x \nvDash U(\vec{\varphi};\psi)$. By definition there is $X \in \mathcal{E}(x)$ such that $X \subseteq [\![\psi]\!]_{\mathcal{M}}$ and $[\![\varphi_j]\!] \nsubseteq X$ for all $j \leq n$. The last fact means that $[\![\varphi_j]\!]_{\mathcal{M}} \cap X^c \neq \varnothing$ for all $j \leq n$. Define $\mathcal{M}'$ to be just like $\mathcal{M}$ other than $V'(p) = X$. Then, since $\mathcal{M}$ and $\mathcal{M}'$ differ at most on p, by the lemma above, $[\![\theta]\!]_{\mathcal{M}} = [\![\theta]\!]_{\mathcal{M}'}$, $[\![\psi]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}'}$, and $[\![\varphi_j]\!]_{\mathcal{M}} = [\![\varphi_j]\!]_{\mathcal{M}'}$ for all $j \leq n$. Immediately we have $\mathcal{M}', x \vDash \theta$. Furthermore, $[\![\varphi_j]\!]_{\mathcal{M}'} \cap [\![p]\!]^c_{\mathcal{M}'} \neq \varnothing$ for all $j \leq n$, so $[\![\varphi_j]\!]_{\mathcal{M}'} \cap [\![\neg p]\!]_{\mathcal{M}'} \neq \varnothing$ for all $j \leq n$. Hence, $\mathcal{M}', x \vDash \bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p)$. Since

$$[\![p]\!]_{\mathcal{M}'} = X \subseteq [\![\psi]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}'},$$

$\mathcal{M}', x \vDash \Box(p \to \psi)$. Since $[\![p]\!]_{\mathcal{M}'} = X \in \mathcal{E}(x) = \mathcal{E}'(x)$, there is an $X \in \mathcal{E}'(x)$ such that $X \subseteq [\![p]\!]_{\mathcal{M}'}$, thus $\mathcal{M}', x \vDash Ep$, i.e. $\mathcal{M}', x \nvDash \neg Ep$. Therefore, $\nvDash \theta \to (\Box(p \to \psi) \to (\bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p) \to \neg Ep))$.

6.2 Completeness

The completeness proof resembles the Henkin-style completeness proofs for first-order and hybrid logics in that the domain of the canonical model isn't simply the collection of all maximally consistent sets of formulas. The sets need to have an additional property since $\neg U(\vec{\varphi};\psi)$ can be true, while there is no formula which witnesses that fact, i.e., no formula θ such that $E\theta$, $\Box(\theta \to \psi)$ and $\bigvee_{j=1}^{n} \Diamond(\varphi_j \wedge \neg\theta)$ are all in the set. Naturally, the fix is to choose maximally consistent subsets Γ which are "filled-up" with enough formulas to witness each case where $\neg U(\vec{\varphi},\psi) \in \Gamma$. We will call sets with this property U-**saturated**. Let $\Phi(U(\vec{\varphi};\psi), p) = \{\, Ep, \Box(p \to \psi), \bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p) \,\}$.

Proposition 6. *Each U-consistent set of sentences Γ can be extended to a maximally consistent and U-saturated set of sentences Γ^+.*

Proof. Suppose Γ is an U-consistent set of sentences. Then let $\{\, p_i : i \in \mathbb{N} \,\}$ be a set of atoms not mentioned in Γ. Define a new language which includes the language of Γ and the new atoms and let $\{\, \psi_i : i \in \mathbb{Z}^+ \,\}$ be an enumeration of that language. Then we define the following sequence of sets. Let $\Gamma_0 = \Gamma$ and

$$\Gamma_n = \begin{cases} \Gamma_{n-1} \cup \{\, \neg\psi_n \,\} & \Gamma_{n-1} \cup \{\, \psi_n \,\} \vdash \bot \\ \Gamma_{n-1} \cup \{\, \psi_n \,\} & \Gamma_{n-1} \cup \{\, \psi_n \,\} \nvdash \bot \ \&\ \psi_n \neq \neg U(\vec{\varphi},\psi), \text{ or} \\ \Gamma_{n-1} \cup \{\, \psi_n \,\} \cup \Phi(\psi_n, p_i) & \text{where } i \text{ is the least } i \text{ such that } p_i \text{ is not mentioned in } \Gamma_{n-1} \cup \{\, \psi_n \,\} \end{cases}$$

We can see that each Γ_n is consistent by induction on n. The only case that is non-standard to see this in the inductive step is the third clause in the definition of Γ_n.

Suppose for reductio that $\Gamma_n = \Gamma_{n-1} \cup \{ \psi_n \} \cup \Phi(\psi_n, p) \vdash \bot$ where p is the first p_i not mentioned in $\Gamma_{n-1} \cup \{ \psi_n \}$. That can happen only when $\Gamma_{n-1} \cup \{ \psi_n \} \nvdash \bot$. By definition of ψ_n and $\Phi(\psi_n, p)$, then,

$$\Gamma_{n-1}, \neg U(\vec{\varphi}; \psi), Ep, \Box(p \to \psi), \bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p) \vdash \bot.$$

By the definition of provability, there is a finite subset of Γ_{n-1}, Γ' such that $\Gamma', \neg U(\vec{\varphi}; \psi), Ep, \Box(p \to \psi), \bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p) \vdash \bot$. It then follows by classical logic that $\Gamma', Ep, \Box(p \to \psi), \bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p) \vdash U(\vec{\varphi}; \psi)$ and by U□R, that $\Gamma', Ep, \Box(p \to \psi), \bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p) \vdash U(\vec{\varphi}; p)$. By the contrapositive of UV, $\vdash \bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p) \to (Ep \to \neg U(\vec{\varphi}; p))$ so by MP, and the transitivity and monotonicity of $\vdash$, $\Gamma', Ep, \Box(p \to \psi), \bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p) \vdash \neg U(\vec{\varphi}; p)$ hence $\Gamma', Ep, \Box(p \to \psi), \bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p)$ is inconsistent. But that means, by classical logic, that

$$\Gamma', \Box(p \to \psi) \vdash \bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p) \to \neg Ep$$

and so by UInf $\Gamma' \vdash U(\vec{\varphi}; \psi)$. But that implies that $\Gamma_{n-1}, \neg U(\vec{\varphi}; \psi) \vdash \bot$ contrary to assumption. Naturally, let $\Gamma^+ = \bigcup_{i \in \mathbb{N}} \Gamma_n$.

To define the canonical model, we will start with the set of all maximally U-consistent and U-saturated sets of sentences, but we will always select only all the $R_\Box$-related worlds in order for the $\Box$ operator to represent global necessity. The canonical model $\mathcal{M}^* = \langle W^*, R^*_\Box, R^*_F, \mathcal{E}^*, V^* \rangle$ is augmented and defined in the following way:

- W^* is the set of maximally U-consistent and saturated sets of formulas,
- $R^*_\Box(x) = \{ y \in W^* : \forall \psi, \Box\psi \in x \Rightarrow \psi \in y \}$,
- $R^*_F(x) = \{ y \in W^* : \forall \psi, F\psi \in x \Rightarrow \psi \in y \}$,
- $V^*(p) = \{ x \in W^* : p \in x \}$, and
- $\mathcal{E}^*$ is defined by: $X \in E^*(x)$ iff there is $\{ \theta_i : i \in I \} \subseteq \{ \theta : E\theta \in x \}$ such that
  (a) $\bigcap_{i \in I} |\theta_i| = X$, and
  (b) For all δ, if $\bigcap_{i \in I} |\theta_i| \subseteq |\delta|$, then $E\delta \in x$

Observation 3. *If $E\theta \in x$ then $|\theta| \in \mathcal{E}^*(x)$.*

Proof. This follows since if $|\theta| \subseteq |\delta|$, then $\vdash \theta \to \delta$, so $\vdash \Box(\theta \to \delta)$, hence if $E\theta \in x$, $E\delta \in x$. Thus $\{ |\theta| \}$ satisfies conditions (a) and (b).

From the canonical model we can define the model which will be used for counterexamples. For each $y \in W^*$ define $\mathcal{M}^{*,y}$ as follows:

- $W^{*,y} = R^*_\Box(y)$,
- $R^{*,y}_F(x) = R^*_F(x) \cap W^{*,y}$,
- $\mathcal{E}^{*,y}(x) = \{ X \cap W^{*,y} : X \in \mathcal{E}^*(x) \}$, and
- $V^{*,y}(p) = V^*(p) \cap W^{*,y}$.

It is possible to give alternative representations of $W^{*,y}$. For example, $W^{*,y} = \{ z \in W^* : \Box(y) \subseteq z \}$, where $\Box(y) = \{ \varphi : \Box\varphi \in y \}$. Since $\Box$ is an S5 operator it follows that if $z \in W^{*,y}$, then $\Box(z) = \Box(y)$. So we can also represent $W^{*,y}$ as $\{ z \in W^* : \Box(z) = \Box(y) \}$. In fact, since $\Box$ is an S5 operator, for all $z \in W^{*,y}$, $\Box\varphi \in z$ iff $\Box\varphi \in y$, i.e., all elements of $W^{*,y}$ agree on $\Box$ed formulas. We can also show the following:

Lemma 4. *If $x \in W^{*,y}$ for some $y \in W^*$, then all $X \in \mathcal{E}^{*,y}(x)$ are non-empty.*

Proof. Let $X \in \mathcal{E}^{*,y}(x)$. Suppose that $X = \varnothing$. By definition, there is $\{ \theta_i \}_{i \in I} \subseteq \{ \theta : E\theta \in x \}$ such that $\bigcap_{i \in I} |\theta_i| \in \mathcal{E}^*(x)$ and $\bigcap_{i \in I} |\theta_i| \cap W^{*,y} = X = \varnothing$. Thus, $\bigcap_{i \in I} |\theta_i| \cap W^{*,y} = \varnothing$ iff $\{ \theta_i : i \in I \} \cup \Box(y) \vdash \bot$. By the compactness of $\vdash$, then there are $\{ \varphi_1, \ldots, \varphi_k \} \subseteq \Box(y)$ and $\{ \theta_1, \ldots, \theta_n \} \subseteq \{ \theta_i : i \in I \}$ such that $\bigwedge_{i=1}^{k} \varphi_i \wedge \bigwedge_{j=1}^{n} \theta_j \vdash \bot$. Since each θ_j is one of the θ_is for some $i \in I$, $\bigcap_{i \in I} |\theta_i| \subseteq |\bigwedge_{j=1}^{n} \theta_j|$. So by condition b in the definition of $\mathcal{E}^*$, $E(\bigwedge_{j=1}^{n} \theta_j) \in x$. Due to the fact that $x \in W^{*,y}$, $\Box(x) = \Box(y)$, thus $\Box(\bigwedge_{i=1}^{k} \varphi_i) \in x$. Then, by MergeE, $E(\bigwedge_{i=1}^{k} \varphi_i \wedge \bigwedge_{j=1}^{n} \theta_j)$. But then, by E□, $E\bot \in x$ which is impossible since $\neg E\bot \in x$ and x is consistent.

Now we can show that the truth lemma for $\mathcal{M}^{*,y}$ is true for any $y \in W^*$.

Lemma 5 (Truth Lemma). *For all φ, and $x \in W^{*,y}$, $\mathcal{M}^{*,y}, x \vDash \varphi$ iff $\varphi \in x$, i.e., $[\![\varphi]\!]_{\mathcal{M}^{*,y}} = |\varphi| \cap W^{*,y}$.*

Proof. Let $x \in W^{*,y}$. The proof is by induction on the complexity of φ. The atomic case follows by the definition of $V^{*,y}$. The induction hypothesis (IH) is that for all δ of less complexity than φ, $[\![\delta]\!]_{\mathcal{M}^{*,y}} = |\delta| \cap W^{*,y}$. We will omit the subscript '$\mathcal{M}^{*,y}$' and the $\cap W^{*,y}$ from here on, unless it is important. The Boolean cases are standard and the case for $\Box$ follows since all members of $W^{*,y}$ agree on $\Box$ed formulas. The F and E cases are also fairly straightforward, so we will just do the U case.

Suppose $\varphi = U(\varphi_1, \dots, \varphi_n; \delta)$. Assume that $\mathcal{M}^{*,y}, x \vDash U(\varphi_1, \dots, \varphi_n; \delta)$. By the truth condition for U, $\forall X \in E^{*,y}(x)$, $X \subseteq [\![\delta]\!]$ only if $[\![\varphi_j]\!] \subseteq X$ for some $j \leq n$. Then, by the IH, $\forall X \in E^{*,y}(x)$, $X \subseteq |\delta| \cap W^{*,y}$ only if $|\varphi_j| \cap W^{*,y} \subseteq X$ for some $j \leq n$.

Now suppose for reductio that $U(\varphi_1, \dots, \varphi_n; \delta) \notin x$, by x's maximality, $\neg U(\varphi_1, \dots, \varphi_n; \delta) \in x$. Since x is U-saturated there is θ such that $\varPhi(U(\vec{\varphi}; \delta), \theta) \subseteq x$. From observation 4 above, then, $|\theta| \cap W^{*,y} \in \mathcal{E}^{*,y}(x)$ since $E\theta \in x$. It also follows that, since $\Box(\theta \to \delta) \in x$, $|\theta| \cap W^{*,y} \subseteq |\delta| \cap W^{*,y}$ because $\Box(\theta \to \delta) \in z$ for all $z \in W^{*,y}$. By IH, $|\delta| \cap W^{*,y} = [\![\delta]\!]$, so there is $X \in \mathcal{E}^{*,y}(x)$ such that $X \subseteq [\![\delta]\!]$. But we also have that $\bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg\theta) \in x$, thus for each $j \leq n$, $\Diamond(\varphi_j \wedge \neg\theta) \in x$ which implies $|\varphi_j| \cap W^{*,y} \nsubseteq |\theta| \cap W^{*,y}$. Since $|\theta| \cap W^{*,y} = X \in \mathcal{E}^{*,y}(x)$, that should be impossible according to our first assumption. Thus, $U(\varphi_1, \dots, \varphi_n; \delta) \in x$.

Conversely, suppose $U(\varphi_1, \dots, \varphi_n; \delta) \in x$. Further, suppose that $X \in \mathcal{E}^{*,y}(x)$ and that $X \subseteq [\![\delta]\!]$; if not the conclusion follows vacuously. We need to show that $[\![\varphi_j]\!] \subseteq X$ for some $j \leq n$. By the IH, we get that $X \subseteq |\delta| \cap W^{*,y}$ and by the definition of $\mathcal{E}^{*,y}$ we get $X = \bigcap_{i\in I} |\theta_i| \cap W^{*,y}$ for some $\bigcap_{i\in I} |\theta_i| \in \mathcal{E}^*(x)$.

Suppose for reductio that $[\![\varphi_j]\!] \nsubseteq \bigcap_{i\in I} |\theta_i| \cap W^{*,y}$ for all $j \leq n$. By IH $|\varphi_j| \cap W^{*,y} \nsubseteq \bigcap_{i\in I} |\theta_i| \cap W^{*,y}$ for all $j \leq n$. For each j there is at least one $x_j \in |\varphi_j| \cap W^{*,y}$ and $x_j \notin \bigcap_{i\in I} |\theta_i| \cap W^{*,y}$. Thus, there is θ_{i_j} such that $x_j \notin |\theta_{i_j}| \cap W^{*,y}$ which means that for each j, $|\varphi_j| \cap W^{*,y} \nsubseteq \bigcap_{k=1}^{n} |\theta_{i_k}| \cap W^{*,y}$. Hence, for each j, $|\varphi_j| \cap |\neg(\bigwedge_{k=1}^{n} \theta_{i_k})| \cap W^{*,y} \neq \varnothing$, which implies that $\varphi_j \wedge \neg(\bigwedge_{k=1}^{n} \theta_{i_k}) \in z$ for some $z \in W^{*,y}$ for each $j \leq n$. But that means $\Diamond(\varphi_j \wedge \neg(\bigwedge_{k=1}^{n} \theta_{i_k})) \in x$ for each j and thus, due to x's maximal consistency, $\bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg(\bigwedge_{k=1}^{n} \theta_{i_k})) \in x$.

Since $\bigcap_{i\in I} |\theta_i| \cap W^{*,y} \subseteq |\delta| \cap W^{*,y} \subseteq |\delta|$, $\bigcap_{i\in I} |\theta_i| \cap \bigcap \{ |\psi| : \Box\psi \in y \} \subseteq |\delta|$. By standard facts about proof sets, then, $\{ \theta_i : i \in I \} \cup \{ \psi : \Box\psi \in y \} \vdash \delta$. So there are finite sets $\varTheta \subseteq \{ \theta_i : i \in I \}$ and $\varPsi \subseteq \{ \psi : \Box\psi \in y \}$ such that $\varTheta \cup \varPsi \vdash \delta$. Hence, by monotonicity and classical logic $\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k} \vdash \delta$. Thus,

$$\Box[(\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k}) \to \delta] \in x.$$

Also, $\vdash \varphi_j \wedge \neg(\bigwedge_{k=1}^{n} \theta_{i_k}) \to (\varphi_j \wedge \neg(\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k}))$, which implies $\vdash \Diamond(\varphi_j \wedge \neg(\bigwedge_{k=1}^{n} \theta_{i_k})) \to \Diamond(\varphi_j \wedge \neg(\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k}))$ since $\Box$ is normal, so $\Diamond(\varphi_j \wedge \neg(\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k})) \in x$ for each $j \leq n$ and thus,

$$\bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg(\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k})) \in x.$$

By the contrapositive of UV, then,

$$\vdash [\bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg(\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k}))] \to [E(\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k}) \to \neg U(\varphi_1, \dots, \varphi_n; (\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k}))]$$

is a theorem of U and so in x. But that means, since x is closed under modus ponens, that

$$E(\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k}) \to \neg U(\varphi_1, \dots, \varphi_n; (\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k})) \in x.$$

Notice, however, that since $\bigcap_{i\in I} |\theta_i| \subseteq \bigcap \{ |\theta| : \theta \in \varTheta \} \cap \bigcap_{k=1}^{n} |\theta_{i_k}|$, by standard facts about proof sets we have $\bigcap \{ |\theta| : \theta \in \varTheta \} \cap \bigcap_{k=1}^{n} |\theta_{i_k}| = |\bigwedge\varTheta \wedge \bigwedge_{k=1}^{n} \theta_{i_k}|$, so by condition (b) on $\mathcal{E}^*(x)$, $E(\bigwedge\varTheta \wedge \bigwedge_{k=1}^{n} \theta_{i_k}) \in x$. But since $\varPsi \subseteq \Box(x) = \Box(y)$, and $\Box$ is a normal operator, $\Box(\bigwedge\varPsi) \in x$. But then, by MergeE, $E(\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k}) \in x$. Thus,

$$\neg U(\varphi_1, \dots, \varphi_n; (\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k})) \in x.$$

Since we have already established that $\Box[(\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k}) \to \delta] \in x$, it follows from U$\Box$R, that $U(\varphi_1, \dots, \varphi_n; (\bigwedge\varTheta \wedge \bigwedge\varPsi \wedge \bigwedge_{k=1}^{n} \theta_{i_k})) \in x$. That means x is inconsistent. But x is maximally consistent, so $[\![\varphi_j]\!] \subseteq X$ for some $j \leq n$. Therefore, $\mathcal{M}^{*,y}, x \vDash U(\varphi_1, \dots, \varphi_n; \delta)$.

If we then notice that assuming $E\top \in x$ for each $x \in W^{*,y}$, we get that $[\![\top]\!] \in \mathcal{E}^{*,y}(x)$ by the truth lemma and the definition of $\mathcal{E}^{*,y}$, thus, $\mathcal{E}^{*,y}(x) \neq \varnothing$. Also, if we were to assume $\varnothing \in \mathcal{E}^{*,y}(x)$, then $[\![\bot]\!] \in \mathcal{E}^{*,y}(x)$. However, given the definition of $\mathcal{E}^{*,y}(x)$ and the truth lemma, we would have $E\bot \in x$ which is impossible since x must be consistent. Therefore $\mathcal{M}^{*,y}$ is a hypergraph model as given in definition 4. The standard argument shows that the logic **U** is complete with respect to the arguments validated in all hypergraph models. In fact, it shows something stronger.

The proof of completeness from the previous section shows that the logic **U** is actually complete with respect to the class of all core-complete models. Thus, we may assume that all frames considered are core-complete: i.e., $cor(\mathcal{E}(x)) \neq \varnothing$ and for all $X \in \mathcal{E}(x)$ there is $X' \in cor(\mathcal{E}(x))$ such that $X' \subseteq X$. This observation was made in [6]. Their construction of a canonical model resulted in a core-reduced model rather than just one that is core-complete because they only kept the cores of each $\mathcal{E}^*(x)$ they defined. The core of $\mathcal{E}^*(x)$ consists of all sets $\bigcap\{\,|\theta_i| : i \in I\,\}$ that are maximal subsets of $\{\,|\theta| : E\theta \in x\,\}$ which also satisfy the second condition in the definition of $\mathcal{E}^*(x)$. Since $\mathcal{E}^{*,y}$ is defined from $\mathcal{E}^*(x)$, $\mathcal{E}^{*,y}(x)$ is also core-complete.

Thus, we have given a modal evidence logic for general Schotch-Jennings forcing, not simply the fixed-level versions. Now that we have this logic, in future work we can extend this semantics to a doxastic logic in the style of the evidence logics of van Benthem et al. and Baltag et al. We will also explore generalizations of the U operator which we have called ‘pointed operators’.

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