Validity in Choice Logics
A Game-theoretic Investigation

Robert Freiman[1⋆] and Michael Bernreiter[1]

Institute of Logic and Computation, TU Wien
robert@logic.at, michael.bernreiter@tuwien.ac.at

Abstract. Qualitative Choice Logic (QCL) is a framework for jointly dealing with truth and preferences. We develop the concept of degree-based validity by lifting a Hintikka-style semantic game [10] to a provability game. Strategies in the provability game are translated into proofs in a novel labeled sequent calculus where proofs come in degrees. Furthermore, we show that preferred models can be extracted from proofs.

Keywords: choice logics · game semantics · sequent calculus.

1 Introduction

Preferences are important in many research areas, including computer science and artificial intelligence [14]. A formalism for preference representation that has gained considerable attention is Qualitative Choice Logic (QCL) [6], which extends classical propositional logic with a connective $\vec{\times}$ called ordered disjunction. $F \vec{\times} G$ expresses that F or G should be satisfied, but satisfying F is preferable to satisfying only G. QCL and its variations [2,4,5] have been studied with regards to applications [1,7,12,15,16], computational properties [4], and proof systems [3].

Recently, QCL has been reexamined through the lens of game theoretic semantics (GTS). Specifically, Game-induced Choice Logic (GCL) [10] was introduced as an extension of Hintikka's semantic game for classical logic [11]. In this semantic game, two players – *Me* and *You*– play over a fixed formula F and an interpretation $\mathcal{I}$. Hintikka's modeling of truth as a win for *Me* and falsity as a loss is refined by more fine-grained outcomes. The more preferences I am able to satisfy during the game, the higher the payoff for *Me*. Besides providing a new understanding of ordered disjunction, GCL addresses some contentious behavior of negation in QCL, where a formula F is not necessarily semantically equivalent to the double negation $\neg\neg F$. GCL redefines negation using game-theoretic methods and thus provides semantics where F is equivalent to $\neg\neg F$ and negation behaves more similarly to classical negation in general.

A natural question not yet addressed in existing work on GCL is whether there is an algorithm that finds strategies for *Me* which guarantee a fixed payoff for the game over a fixed formula F and *all* interpretations. Reduced to winning strategies, this corresponds to the question of the validity of F. We answer this

⋆ Research supported by FWF project P32684

question by lifting the GTS to a dialogue game (we prefer the term *provability game*). Intuitively, in this game, the players play the semantic game over all interpretations simultaneously but I am allowed to create backup copies of game states. This technique has been demonstrated to lead to adequate proof systems for a variety of logics [8,13,9]. Our approach is the first to interpret non-classical truth values with non-binary outcomes in both the semantic and the disjunctive game. Our main result states that from a strategy σ for *Me* in the disjunctive game one can extract strategies for the semantic game over every interpretation yielding a payoff at least as good as σ's. Furthermore, from *Your* strategy one can extract an interpretation $\mathcal{I}$ and a strategy for the semantic game over $\mathcal{I}$ yielding at least the same payoff for *You*. In logical terminology, this corresponds to counter-model extraction; in the realm of preference handling, this corresponds to the construction of a preferred model.

While the exposition in this paper is mostly game-theoretic, we demonstrate that strategies for *Me* in the disjunctive game can be formulated as proofs in a labeled sequent calculus. Unlike the system for QCL [3], in our proof system $\mathbf{GS}^*$, proofs have degrees where positive degrees represent proofs of validity, while negative degrees represent refutations of a formula.

This paper is structured as follows: In Section 2, we recall game-theoretic notions and GCL. In Section 3, we lift the semantic game for GCL to a provability game.

In Section 4, we reformulate *My* strategies as a proof system.

2 Preliminaries

In this section, we recall the game-induced choice logic GCL and its two semantics – game-theoretic and degree-based. The language of GCL is the same as QCL's, i.e., it extends the usual propositional language by the choice connective $\vec{\times}$. We assume an infinite countable set of propositional variables $a, b, \dots$. Compound formulas are built according to the following grammar:

$$F ::= a \mid \neg F \mid F \wedge F \mid F \vee F \mid F \vec{\times} F.$$

An *interpretation* $\mathcal{I}$ is a set of propositional variables, with $\mathcal{I} \models a$ iff $a \in \mathcal{I}$.

2.1 Game-theoretic semantics

We start by recalling Hintikka's game [11] over a formula F in the language restricted to the connectives $\vee, \wedge, \neg$ and over a classical interpretation $\mathcal{I}$. The game is played between two players, *Me* and *You*, both of which can act either in the role of Proponent ($\mathbf{P}$) or Opponent ($\mathbf{O}$). At formulas of the form $G_1 \vee G_2$, $\mathbf{P}$ chooses a formula G_i that the game continues with. At formulas of the form $G_1 \wedge G_2$ it is $\mathbf{O}$'s choice. At negations $\neg G$, the game continues with G and a role switch. Every outcome (final state of the game) is an occurrence of a propositional variable a. The player currently in the role of $\mathbf{P}$ wins the game (and **O** loses) iff $a \in \mathcal{I}$. The central result is that I have a winning strategy for the game starting in $\mathbf{P} : F$ iff $\mathcal{I} \models F$.

To deal with ordered disjunction ($\vec{\times}$), Hintikka's game is extended as follows [10]: at $G_1 \vec{\times} G_2$ it is **P**'s choice whether to continue with G_1 or with G_2, *but this player prefers* G_1. The preferences of **O** are the exact opposite of **P**. For both players, the aim in the game is now not only to win the game but to do so with as little compromise to their preferences as possible. Thus, it is natural to express **P**'s preference of G_1-outcomes O_1 over G_2-outcomes O_2 via the relation $O_1 \gg O_2$. We leave the formal treatment of this game for the next section and proceed with some standard game-theoretic definitions.

Definition 1 *A* game *is a pair* $\mathbf{G} = (T, d)$*, where*

1. $T = (V, E, l)$ *is a tree with set of nodes* V *(called* (game) states*) and edges* E*. The leaves of* T *are called* outcomes *and are denoted* $\mathcal{O}(\mathbf{G})$*. The* labeling function l *maps nodes of* T *to the set* $\{I, Y\}$*.*
2. d *is a payoff-function mapping outcomes to elements of a linear order* $(\Lambda, \preceq)$*.*

We write $x \approx y$ *if* $x \preceq y$ *and* $y \preceq x$*.* Λ *is partitioned into two sets,* W *and* L*, where* W *is upward-closed and* $L = \Lambda \setminus W$*. Outcomes* O *are called* winning *if* $d(O) \in W$ *and* losing *if* $d(O) \in L$*. A* run *of the game is a maximal path in* T *starting at the root.*

Hintikka's game can be seen as a game in the sense of this definition: the game tree is the formula tree of F where each occurrence of a subformula G of F is decorated with either **P** or **O**, we write $\mathbf{P} : G$ and $\mathbf{O} : G$, respectively. Let F be decorated with $\mathbf{Q}_0 \in \{\mathbf{P}, \mathbf{O}\}$. If $G = G_1 \vee G_2$, or $G = G_1 \wedge G_2$ then the children of $\mathbf{Q} : G$ are decorated the same. If $G = \neg G'$, then the child of $\mathbf{Q} : G$ is $\bar{\mathbf{Q}} : G'$, where $\bar{\mathbf{Q}}$ is **O** if $\mathbf{Q} = \mathbf{P}$, and **P** otherwise. As for the labeling function, game states of the form $\mathbf{P} : G_1 \vee G_2$, $\mathbf{O} : G_1 \wedge G_2$ are I-states and all other states are Y-states.

As for payoffs, we write $\mathcal{I} \models \mathbf{P} : a$ iff $\mathcal{I} \models a$ and $\mathcal{I} \models \mathbf{O} : a$ iff $\mathcal{I} \not\models a$. The payoff functions maps outcomes to $P = \{0, 1\}$, where $d(o) = 1$ iff $\mathcal{I} \models o$. P carries the usual ordering $0 < 1$ and $W = \{1\}$.

A strategy σ for *Me* in a game can be understood as *My* complete game plan. For every node of the underlying game tree labeled "I", σ tells *Me* to which node I have to move. Here is a formal definition:

Definition 2 *A* strategy σ *for* Me *for the game* $\mathbf{G}$ *is a subtree of the underlying tree such that (1) the root of* T *is in* σ *and for all* v *in* σ*, (2) if* $l(v) = I$*, then at least one successor of* v *is in* σ *and (3) if* $l(v) = Y$*, then all successors of* v *are in* σ*. A strategy for* You *is defined symmetrically. We denote by* Σ_I *and* Σ_Y *the set of all strategies for* Me *and* You*, respectively.*

Conditions (1) and (3) make sure that all possible moves by the other player are taken care of by the game plan. Each pair of strategies $\sigma_I \in \Sigma_I$, $\sigma_Y \in \Sigma_Y$ defines a *unique* outcome of $\mathbf{G}$, denoted by $O(\sigma_I, \sigma_Y)$. We abbreviate $d(O(\sigma_I, \sigma_Y))$

by $d(\sigma_I, \sigma_Y)$. A strategy σ_I^* for *Me* is called *winning* if, playing according to this strategy, I win the game, no matter how *You* move, i.e. for all $\sigma_Y \in \Sigma_Y$, $d(\sigma_I^*, \sigma_Y) \in W$. Let $k \in \Lambda$. A strategy σ_I^k for *Me* guaranteeing a payoff of at least k, i.e. $\min^{\preceq}_{\sigma_Y}(\sigma_I^k, \sigma_Y) \succeq k$ is called a *k-strategy for Me.* A strategy for *You* guaranteeing a payoff of at most k is called a *k-strategy for You.* An outcome O that maximizes *My* pay-off in light of *Your* best strategy is called *maxmin-outcome.* Formally, O is a maxmin-outcome iff $d(O) = \max^{\preceq}_{\sigma_I} \min^{\preceq}_{\sigma_Y} d(\sigma_I, \sigma_Y)$ and $d(O)$ is called the *maxmin-value* of the game. A strategy σ_I^* for *Me* is a *maxmin-strategy* for $\mathbf{G}$ if $\sigma_I^* \in \arg\max^{\preceq}_{\sigma_I} \min^{\preceq}_{\sigma_Y} d(\sigma_I, \sigma_Y)$, i.e, the maximum is reached at σ_I^*. *Minmax* values and strategies for *You* are defined symmetrically.

The class of games that we have defined falls into the category of *zero-sum games of perfect information* in game theory. They are characterized by the fact that the players have strictly opposing interests. In these games, the minimax and maximin values always coincide and are referred to as the *value of the game.*

2.2 Game Choice Logics GCL

We now define the game semantics for GCL [10]. Let $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$. The game over the interpretation $\mathcal{I}$ starting with *Me* in the role $\mathbf{Q}$ of formula F is denoted by $\mathbf{NG}(\mathbf{Q} : F, \mathcal{I})$. The game tree for the *semantic game* $\mathbf{NG}(\mathbf{Q} : F, \mathcal{I})$ is the same as in Hintikka's game, where $\vec{\times}$ is treated like $\vee$.

The main difference is that we now wish to deal with preferences induced by $\vec{\times}$. *My* preferences are expressed via the strict partial order $\ll$ on outcomes of the game tree: If $\mathbf{P} : G_1 \vec{\times} G_2$ appears in the tree, then outcomes reachable from $\mathbf{P} : G_1$ are in $\gg$-relation with outcomes reachable from $\mathbf{P} : G_2$. Similarly, for $\mathbf{O} : G_1 \vec{\times} G_2$, outcomes reachable from $\mathbf{O} : G_1$ are in $\ll$-relation with outcomes reachable from $\mathbf{O} : G_2$.

A sensible payoff function must respect both truth (winning conditions) and preferences (the relation $\ll$). Our payoff function $\delta_{\mathcal{I}}$ takes values in the domain $Z := (\mathbb{Z} \setminus \{0\}, \trianglelefteq)$. The ordering $\trianglelefteq$ is the inverse ordering on $\mathbb{Z}^+$ and on $\mathbb{Z}^-$, for $a \in \mathbb{Z}^+, b \in \mathbb{Z}^-$ we set $b \triangleleft a$, i.e. $-1 \triangleleft -2 \triangleleft \ldots 2 \triangleleft 1$. For each outcome o, let $\pi_{\ll}(o)$ be the longest $\ll$-chain starting in o, i.e. pairwise different outcomes $o_1, \ldots, o_n$ such that $o = o_1 \ll \cdots \ll o_n$. Let $|\pi_{\ll}(o)| = n$ denote its length. For an interpretation $\mathcal{I}$, and an outcome $\mathbf{Q} : a$, we define[1]

$$\delta_{\mathcal{I}}(\mathbf{Q} : a) = \begin{cases} |\pi_{\ll}(\mathbf{Q} : a)|, & \text{if } \mathcal{I} \models \mathbf{Q} : a, \\ -|\pi_{\gg}(\mathbf{Q} : a)|, & \text{if } \mathcal{I} \not\models \mathbf{Q} : a. \end{cases}$$

By design, $\delta_{\mathcal{I}}$ maps true outcomes to $\mathbb{Z}^+$ and false outcomes to $\mathbb{Z}^-$. We, therefore, declare all outcomes with a payoff in $\mathbb{Z}^+$ as winning, and all other outcomes as losing for *Me*. The game can be thus seen as a refined extension of Hintikka's game. Indeed, let F^* be F with all $\vec{\times}$s replaced by $\vee$s. Then I have a winning strategy for $\mathbf{NG}(\mathbf{P} : F, \mathcal{I})$ iff I have a winning strategy for F^* in Hintikka's game over $\mathcal{I}$. Furthermore, $\delta_{\mathcal{I}}$ respects the relation $\ll$: if $o_1 \ll o_2$ and both are winning (or both are losing) for *Me*, then $\delta_{\mathcal{I}}(o_1) \triangleleft \delta_{\mathcal{I}}(o_2)$.

[1] Notice the flipped $\ll$-sign in the second case.

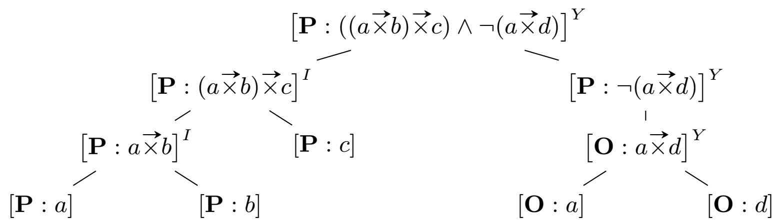

Fig. 1: The game tree for $\mathbf{NG}(\mathbf{P} : ((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d))$.

Example 1. Consider the formula $((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)$. The game tree, where *I* am initially the Proponent can be found in Figure 1. The order on outcomes is $\mathbf{P} : c \ll \mathbf{P} : b \ll \mathbf{P} : a$ and $\mathbf{O} : a \ll \mathbf{O} : d$.

Let $\mathcal{I} = \{b\}$. If *You* go to the left at the root node, *I* will move to reach the outcome $\mathbf{P} : b$, winning the game with payoff 2. Therefore, *You* might choose to go right at the root to reach $\mathbf{O} : a$ or $\mathbf{O} : d$ with payoff 2 and 1 respectively. It is better for *You* to reach $\mathbf{O} : a$ with payoff 2. Thus, the value of the game is 2.

Now consider the game starting in $\mathbf{O} : ((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)$, again with $\mathcal{I} = \{b\}$. The game tree is the same, except that $\mathbf{P}$ and $\mathbf{O}$ are flipped everywhere, as are the labels I, Y and the order over outcomes. *You* can now win the game: if *I* go left at the root, *You* will move to $\mathbf{O} : b$ with payoff -2. The alternative is not better for *Me*: if *I* go right, *I* can choose between $\mathbf{P} : a$ and $\mathbf{P} : d$ with payoffs -1 and -2 respectively. Thus, the value of this game is -2.

2.3 Degree-based semantics for GCL

Although the motivation for GCL is game-theoretic, it also admits a degree semantics that is more common in choice logics. We first need the following notion of optionality:

Definition 3 *The optionality of GCL-formulas is defined inductively as follows: (i)* $\text{opt}(a) = 1$ *for variables* a*, (ii)* $\text{opt}(\neg F) = \text{opt}(F)$*, (iii)* $\text{opt}(F \circ G) = \max(\text{opt}(F), \text{opt}(G))$ *for* $\circ \in \{\vee, \wedge\}$*, and (iv)* $\text{opt}(F \overrightarrow{\times} G) = \text{opt}(F) + \text{opt}(G)$.

In [10], we show that $\text{opt}(F)$ computes the length of the longest $\ll$-chain in the outcomes reachable from $\mathbf{P} : F$ in the semantic game. The degree function of GCL is denoted by $\deg^{\mathcal{G}}_{\mathcal{I}}$.[2] It assigns to each formula a degree relative to an interpretation $\mathcal{I}$ and is defined inductively as follows:

$$\deg^{\mathcal{G}}_{\mathcal{I}}(a) = 1 \text{ if } a \in \mathcal{I}, -1 \text{ otherwise}$$
$$\deg^{\mathcal{G}}_{\mathcal{I}}(\neg F) = -\deg^{\mathcal{G}}_{\mathcal{I}}(F)$$

[2] The superscript $\mathcal{G}$ is used to differentiate from the standard degree function $\deg_{\mathcal{I}}$ of QCL used in the literature.

$$\deg^{\mathcal{G}}_{\mathcal{I}}(F \wedge G) = \min(\deg^{\mathcal{G}}_{\mathcal{I}}(F), \deg^{\mathcal{G}}_{\mathcal{I}}(G))$$
$$\deg^{\mathcal{G}}_{\mathcal{I}}(F \vee G) = \max(\deg^{\mathcal{G}}_{\mathcal{I}}(F), \deg^{\mathcal{G}}_{\mathcal{I}}(G))$$
$$\deg^{\mathcal{G}}_{\mathcal{I}}(F \overrightarrow{\times} G) = \begin{cases} \deg^{\mathcal{G}}_{\mathcal{I}}(F) & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^+ \\ \mathrm{opt}(F) + \deg^{\mathcal{G}}_{\mathcal{I}}(G) & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^-, \\ & \quad \deg^{\mathcal{G}}_{\mathcal{I}}(G) \in \mathbb{Z}^+ \\ \deg^{\mathcal{G}}_{\mathcal{I}}(F) - \mathrm{opt}(G) & \text{otherwise} \end{cases}$$

Here min and max are relative to $\trianglelefteq$. If $\deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^+$ then we say that $\mathcal{I}$ classically satisfies F, or that $\mathcal{I}$ is a model of F. A model $\mathcal{I}$ of F is *preferred*, if for every other model $\mathcal{I}'$ of F we have $\deg^{\mathcal{G}}_{\mathcal{I}}(F) \trianglerighteq \deg^{\mathcal{G}}_{\mathcal{I}'}(F)$.

Theorem 4 (Theorem 4.7 in [10]) *The value of* $\mathbf{NG}(\mathbf{P} : F, \mathcal{I})$ *is* $\deg^{\mathcal{G}}_{\mathcal{I}}(F)$. *The value of* $\mathbf{NG}(\mathbf{O} : F, \mathcal{I})$ *is* $-\deg^{\mathcal{G}}_{\mathcal{I}}(F)$.

3 A Provability Game

Usually, in a semantic view of a logic, validity of a formula F is defined as truth of F in all interpretations. In our context of graded truth, however, we can refine this notion to *graded validity*. Thus, we define the *degree (of validity)* of F to be the least possible degree of F in an interpretation:

$$\deg^{\mathcal{G}}(F) := \min_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F)$$

In this section, we give a game-theoretic characterization of this degree. To this end, we lift the semantic game **NG** to a provability game that adequately characterizes validity in GCL. Our framework will be able to deal with the following central notion in preference handling:

Definition 5 *An interpretation* $\mathcal{I}$ *is a* preferred model of F *iff* $\deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^+$ *and for all other interpretations* $\mathcal{J}$, $\deg^{\mathcal{G}}_{\mathcal{J}}(F) \trianglelefteq \deg^{\mathcal{G}}_{\mathcal{I}}(F)$.

We now describe the lifting of **NG** to the provability game **DG** (we call our game *disjunctive game*). We want a winning strategy for *Me* for the provability game starting at g to imply the existence of winning strategies in *all* semantic games starting at g. Note that the game trees of g over different interpretations $\mathcal{I}$ are identical, except for the payoff at outcomes. Therefore, a simultaneous play can be modeled by changing the pay-off at outcomes o to be the worst possible pay-off of o under all interpretations: $\delta(o) = \min_{\mathcal{I}}(o)$.

However, this variant does not capture validity yet, as I do not have winning strategies for this game even for simple cases, like $\mathbf{P} : a \vee \neg a$. This variant of the game is too restrictive, as it would require the existence of a *uniform strategy* – a single strategy that works in all semantic games. To remedy this shortcoming, we allow *Me* to create "backup copies" of game states during the provability game. If the game is unfavorable for *Me* in one copy, *I* can always come back to have another shot at the other copy. *My* goal is to win at least one of these copies. The game states of this game can be thus read as disjunctions, and are therefore called *disjunctive game states*[3], (hence the name of the game).

Formally, game states of the disjunctive game are finite multisets of game states of the game **NG**. We prefer to write $g_1 \bigvee ... \bigvee g_n$ for the disjunctive game state $D = \{g_1, ..., g_n\}$, but keep the convenient notation $g \in D$ if g belongs to the multiset D. A disjunctive state is called *elementary* if all its game states are leaf-states of **NG**. Following the intuition of backup states, the payoff at an elementary disjunctive state D is the maximum of the payoffs of its game states:

$$\delta(D) = \min_{\mathcal{I}} \max_{1 \leq i \leq n} \delta_{\mathcal{I}}(g_i).$$

Additionally, I take the rule of a scheduler who decides which of the copies is played upon next.

At the disjunctive state $D \bigvee g$, I can point at a non-leaf state g, codified by underlining: $D \bigvee \underline{g}$. After the corresponding player takes their turn in $\mathbf{NG}(g_i, \mathcal{I})$ and moves to a state, say g', the game continues with $D \bigvee g'$.

As mentioned, instead of pointing to a game state of the disjunctive state, I can *duplicate* any of its states, i.e. create a backup copy. If I decide to duplicate g, the game continues with $D \bigvee g \bigvee g$. Due to this rule, it is now possible to have infinite runs of the game. In these runs, I repeatedly create backup copies. To prevent such behavior, we punish the "delaying" of the game by declaring infinite runs losing for *Me* with the worst possible pay-off -1.

Formally, we define the game tree of the disjunctive game $\mathbf{DG}(D, \mathcal{I})$ recursively as follows. We say that $D' \bigvee \underline{g}$ is obtained from $D = D' \bigvee g$ by *underlining* a game state and $D \bigvee g \bigvee g$ is obtained by *duplicating* a game state. If no states in D are underlined, it is an "I"-disjunctive state and its successor nodes are all disjunctive states obtainable by underlining, or duplicating a game state. If a game state is underlined, say we are in $D = D' \bigvee \underline{g}$, then this disjunctive state is labeled the same as g in the semantic game. The children of D are all $D' \bigvee g'$, where g' ranges over the children of g in the semantic game. For example, if $D = D' \bigvee \underline{\mathbf{P} : G_1 \vee G_2}$, then it is an "I"-disjunctive state and its children are $D' \bigvee \mathbf{P} : G_1$ and $D' \bigvee \mathbf{P} : G_2$.

Example 2. Let F be $((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} b)$. Figure 2 shows a compact representation of a strategy for *Me* for the game $\mathbf{DG}(\mathbf{O} : F)$. Underlining moves are clear from context and are therefore hidden. First, I duplicate $\mathbf{O} : F$ and move to $\mathbf{P} : ((a \overrightarrow{\times} b) \overrightarrow{\times} c)$ in one copy and to $\mathbf{O} : \neg(a \overrightarrow{\times} d)$ in the other. The latter is immediately converted to $\mathbf{P} : a \overrightarrow{\times} d$, for which I repeat the strategy of duplicating and moving into both options. Finally, I point to $\mathbf{O} : (a \overrightarrow{\times} b) \overrightarrow{\times} c$, where it is *Your* turn. All *Your* possible choices are shown in the strategy. The payoffs are

$$\delta(\mathbf{O} : a \bigvee \mathbf{P} : a \bigvee \mathbf{P} : d) = \min_{\mathcal{I}} \max\{\delta_{\mathcal{I}}(\mathbf{O} : a), \delta_{\mathcal{I}}(\mathbf{P} : a), \delta_{\mathcal{I}}(\mathbf{P} : d)\}$$

[3] To avoid confusion, we always refer to game states of the disjunctive game **DG** as "*disjunctive (game) states*". "*(Game) states*" is reserved for the semantic game **NG**.

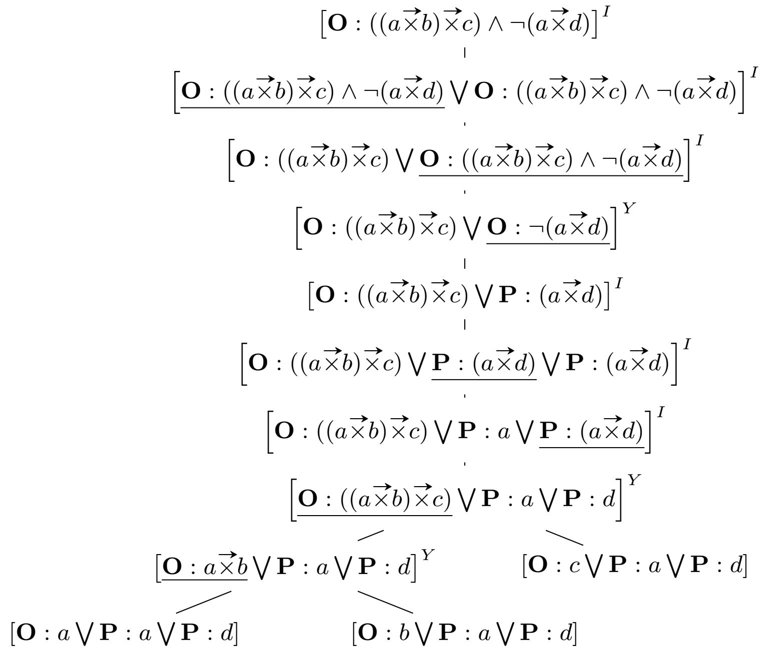

Fig. 2: A compact representation of the strategy for *Me* for an instance of **DG**

$$
\begin{aligned}
&= \max\{\delta_\emptyset(\mathbf{O} : a), \delta_\emptyset(\mathbf{P} : a), \delta_\emptyset(\mathbf{P} : d)\} = \max\{3, -2, -1\} = 3,\\
\delta(\mathbf{O} : b \bigvee \mathbf{P} : a \bigvee \mathbf{P} : d) &= \min_{\mathcal{I}} \max\{\delta_{\mathcal{I}}(\mathbf{O} : b), \delta_{\mathcal{I}}(\mathbf{P} : a), \delta_{\mathcal{I}}(\mathbf{P} : d)\}\\
&= \max\{\delta_{\{b\}}(\mathbf{O} : b), \delta_{\{b\}}(\mathbf{P} : a), \delta_{\{b\}}(\mathbf{P} : d)\} = \max\{-2, -2, -1\} = -2,\\
\delta(\mathbf{O} : c \bigvee \mathbf{P} : a \bigvee \mathbf{P} : d) &= \min_{\mathcal{I}} \max\{\delta_{\mathcal{I}}(\mathbf{O} : c), \delta_{\mathcal{I}}(\mathbf{P} : a), \delta_{\mathcal{I}}(\mathbf{P} : d)\}\\
&= \max\{\delta_{\{c\}}(\mathbf{O} : c), \delta_{\{c\}}(\mathbf{P} : a), \delta_{\{c\}}(\mathbf{P} : d)\} = \max\{-3, -2, -1\} = -3.
\end{aligned}
$$

Given these payoffs, *You* prefer the second outcome, giving *Me* a payoff of -2. We note two things. First, *I* cannot do better by playing another strategy. If the outcomes do not contain game states resulting from $\mathbf{O} : (a \overrightarrow{\times} b) \overrightarrow{\times} c$, then their pay-offs are the same, or even less. The strategy of first duplicating, then exploiting all possible moves is therefore – in a way – optimal for *Me*. Hence, we can conclude that the value of the game is -2.

The remainder of this section is devoted to proving the adequacy of **DG**.

Theorem 6 I *have a k-strategy in* $\mathbf{DG}(D)$ *iff for every interpretation* $\mathcal{I}$*, there is some* $g \in D$ *such that* I *have a k-strategy in* $\mathbf{NG}(g, \mathcal{I})$. You *have a k-strategy in* $\mathbf{DG}(D)$ *iff there is an interpretation* $\mathcal{I}$ *such that* You *have k-strategies in* $\mathbf{NG}(g, \mathcal{I})$*, for all* $g \in D$.

We prove the above theorem with the help of two lemmas.

Lemma 7 *Let π be a finite run of the game $\mathbf{DG}(D)$ such that for every I-state g in π, all of its children appear in π, too. Let k be the payoff of π. Then there is a model $\mathcal{I}_0$ such that* You *have a k-strategy for $\mathbf{NG}(g, \mathcal{I}_0)$, for each $g \in D$.*

Proof (of Lemma 7). Let D_{fin} be the outcome of π and let $\mathcal{I}_0$ be such that $\delta_{\mathcal{I}_0}(D_{fin}) \trianglelefteq k$. For $g_0 \in D$, let σ_{g_0} be the set of successors of g appearing in π. Note that σ_{g_0} carries the structure of a subtree of $\mathbf{NG}(g_0, \mathcal{I})$. Indeed, it is a strategy for *You*: by assumption, for every I-state in σ_{g_0}, all successors appear in π, and thus in σ_{g_0}. Every Y-state g in σ_{g_0} comes from a disjunctive state $D' \bigvee g$ appearing in π. At some point, $D' \bigvee \underline{g}$ is in π. The next disjunctive state in π is $D' \bigvee g'$, so g' is the unique successor of the Y-state g in σ_{g_0}.

To verify that σ_{g_0} is a k-strategy, it is enough to notice that all outcomes o in σ_{g_0} appear in D_{fin}. Thus, $\delta_{\mathcal{I}_I}(o) \trianglelefteq \max_{h \in D} \delta_{\mathcal{I}_0}(h) \trianglelefteq \delta_{\mathcal{I}_0}(D_{fin}) \trianglelefteq k$. □

Lemma 8 *Let σ be a strategy for* Me *for $\mathbf{DG}(D_0)$ and let S be a set containing exactly one game state of each outcome of σ. Then for every interpretation $\mathcal{I}$, there is a strategy for* Me *for $\mathbf{NG}(g_0, \mathcal{I})$ with $g_0 \in D_0$ and outcomes in S.*

Proof (of Lemma 8). We define recursively for each $D \in \sigma$ a strategy σ^D for $\mathbf{NG}(g, \mathcal{I})$, where $g \in D$ and outcomes are in S. In the base case, D is an outcome of σ, so we set σ^D to be the singleton $S \cap D$.

If D is an I-state, and its unique child $H \in \sigma$ is obtained by duplicating or underlining a game state, we use the inductive hypothesis and set $\sigma^D = \sigma^H$. If $D = D' \bigvee \underline{g}$, where g is an I-state, then $H = D' \bigvee g'$, where g' is a child of g. If σ^H is a strategy for $\mathbf{NG}(h, \mathcal{I})$ with $h \in D'$, we can simply set $\sigma^D = \sigma^H$. Otherwise, σ^H is a strategy for $\mathbf{NG}(g', \mathcal{I})$. We can thus set $\sigma^D = \{g\} \cup \sigma^H$.

If D is a Y-state, then it is of the form $D = D' \bigvee \underline{g}$, where g is a Y-state. The children of D are of the form $D' \bigvee g'$, where g' ranges over the children of g. Since σ is a strategy for *Me*, all these children appear in σ. If for some g', $\sigma^{D' \bigvee g'}$ is a strategy for $\mathbf{NG}(h, \mathcal{I})$ and $h \in D'$, we can set $\sigma^D = \sigma^{D' \bigvee g'}$. Otherwise, all $\sigma^{D' \bigvee g'}$ are strategies for $\mathbf{DG}(g', \mathcal{I})$, and we can set $\sigma^D = \{g\} \cup \bigcup \sigma^{D' \bigvee g'}$.

In all the inductive steps it is clear that σ^D contains only outcomes from S. The claim follows for $D = D_0$. □

Proof (of Theorem 6). We prove the left-to-right directions (ltr) of both statements. The right-to-left directions (rtl) then follow easily: for example, suppose, for every $\mathcal{I}$, there is a $g \in D$, such that *I* have a k-strategy in $\mathbf{NG}(g, \mathcal{I})$. Let $l \triangleleft k$ be maximal. We infer that for every $\mathcal{I}$, there is some $g \in D$ such that *You* do not have a k-strategy in $\mathbf{NG}(g, \mathcal{I})$. By ltr of Statement 2, *You* do not have an l-strategy for $\mathbf{DG}(g, \mathcal{I})$. Since *You* cannot enforce the payoff to be below k, *I* have a k-strategy. The rtl of the other statement is similar.

Let us prove the ltr of Statement 1. Fix a k-strategy σ for *Me* in $\mathbf{DG}(D)$ and an interpretation $\mathcal{I}$. By assumption, for every outcome of the disjunctive game O in σ, there is a game state $o \in O$ such that $\delta_{\mathcal{I}}(o) \trianglerighteq k$. Collect for each outcome such an o into a set S. We apply Lemma 8 to obtain a strategy μ for

Me for $\mathbf{NG}(g, \mathcal{I})$, for some $g \in D$ with outcomes in S. These outcomes have a payoff of at least k, i.e., μ is a k-strategy.

Ltr of Statement 2: suppose *You* have a k-strategy for $\mathbf{DG}(D)$. Let π be the run of the game where *I* play according to the following strategy: if the current disjunctive state is H, *I* underline an arbitrary $h \in H$. If h is an I-state and has only one child h', *I* go to that child in the corresponding copy. If h has two children h_1 and h_2, *I* first duplicate h, then go to h_1 in the first and to h_2 in the second copy. Let L be the outcome of π. By assumption, $\delta(L) \trianglelefteq k$. By Lemma 7, there is $\mathcal{I}$ such that *You* have k-strategies for $\mathbf{NG}(g, \mathcal{I})$, for each $g \in D$. □

Corollary 9 *The values of the games* $\mathbf{DG}(\mathbf{P} : F)$ *and* $\mathbf{DG}(\mathbf{O} : F)$ *are given by* $\deg^{\mathcal{G}}_{\mathcal{I}}(F) = \min_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F)$ *and* $-\max_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F)$*, respectively.*

Proof. For each interpretation $\mathcal{I}$, let $v_{\mathcal{I}}$ be the value of $\mathbf{DG}(D, \mathcal{I})$. It follows from the theorem that the value of $\mathbf{DG}(D)$ is $\min_{\mathcal{I}} v_{\mathcal{I}}$. Thus, by Theorem 4, the values of $\mathbf{DG}(\mathbf{P} : F)$ and $\mathbf{DG}(\mathbf{O} : F)$ are $\min_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F)$ and $\min_{\mathcal{I}} - \deg^{\mathcal{G}}_{\mathcal{I}}(F)) = -\max_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F)$, respectively. □

Corollary 10 *Let* $\mathcal{I}$ *be a preferred model of* F *and let* k *be the value of* $\mathbf{DG}(\mathbf{O} : F)$*. Then* $k = -\deg^{\mathcal{G}}_{\mathcal{I}}(F)$ *and a preferred model of* F *can be extracted from* Your *k-strategy for* $\mathbf{DG}(\mathbf{O} : F)$.

Proof. The first statement immediately follows from Corollary 9. Let σ be *Your* k-strategy for $\mathbf{DG}(\mathbf{O} : F)$. Since there is an interpretation making F true, k must be negative and thus winning for *You*. By the proof of Theorem 6, all the information for a preferred model is contained in the outcome of the run of the game, where *I* play according to the strategy sketched in that proof and *You* play according to σ. Let L be the outcome of that run. L must be winning for *You*. We, therefore, set $\mathcal{I}^{\pi} = \{a \mid \mathbf{O} : a \in L\}$ and obtain a k-strategy for *You* for $\mathbf{DG}(\mathbf{O} : F, \mathcal{I}^{\pi})$. Let v be the value of that game. We have that $v \trianglelefteq k$, by the existence of *Your* k-strategy and $v \trianglerighteq k$, since by Theorem 4 and Corollary 9, $v = -\deg^{\mathcal{G}}_{\mathcal{I}^{\pi}}(F) \trianglerighteq -\max_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F) = k$.

This shows $\deg^{\mathcal{G}}_{\mathcal{I}^{\pi}}(F) = \max_{\mathcal{I}} \deg^{\mathcal{G}}_{\mathcal{I}}(F)$, i.e., $\mathcal{I}^{\pi}$ is a preferred model of F. □

4 Proof systems

In this section, we study the proof-theoretic content of the provability game by reinterpreting strategies as proofs in three different labeled sequent calculi. Essentially, proofs in these systems are nothing but representations of *My* strategies for the disjunctive game. Sequents in these calculi consist of labeled formulas: each formula is decorated with two numbers $k, l \geq 1$ and we write ${}^{k}_{l}F$. The intuitive reading is that all winning outcomes of $\mathbf{Q} : F$ have a longest $\ll$-chain of at least l, and thus their payoff is at most l. Losing outcomes have a longest $\gg$-chain of at least k and thus their payoff is at least $-k$.

Sequents are of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are multisets of labeled formulas. There is a direct translation from disjunctive states of the game $\mathbf{DG}(D_0)$ into sequents: Each disjunctive state D is translated into the sequent

$$\{{}^k_l F \mid \mathbf{O} : F \in D\} \Rightarrow \{{}^k_l F \mid \mathbf{P} : F \in D\},$$

where $k = \min\{|\pi_{\gg}(o)| : o \in \mathcal{O}(\mathbf{Q} : F)\}$ and $l = \min\{|\pi_{\ll}(o)| : o \in \mathcal{O}(\mathbf{Q} : F)\}$. In particular, we have $k = l = 1$ if the game starts at $\mathbf{Q} : F$. We assign degrees to sequents $\Gamma \Rightarrow \Delta$ as follows: for each interpretation $\mathcal{I}$,

$$\text{if } {}^k_l F \in \Delta, \text{ we set } \deg^{\mathcal{G}}_{\mathcal{I}}({}^k_l F) = \begin{cases} l + \deg^{\mathcal{G}}_{\mathcal{I}}(F) - 1, & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^+, \\ -k + \deg^{\mathcal{G}}_{\mathcal{I}}(F) + 1, & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^-, \end{cases}$$

$$\text{if } {}^k_l F \in \Gamma, \text{ we set } \deg^{\mathcal{G}}_{\mathcal{I}}({}^k_l F) = \begin{cases} l - \deg^{\mathcal{G}}_{\mathcal{I}}(F) - 1 & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^-, \\ -k - \deg^{\mathcal{G}}_{\mathcal{I}}(F) + 1 & \text{if } \deg^{\mathcal{G}}_{\mathcal{I}}(F) \in \mathbb{Z}^+. \end{cases}$$

We then set

$$\deg^{\mathcal{G}}(\Gamma \Rightarrow \Delta) = \min_{\mathcal{I}} \max_{{}^k_l F \in \Gamma \cup \Delta} \deg^{\mathcal{G}}_{\mathcal{I}}({}^k_l F).$$

In the simplest case, $\deg^{\mathcal{G}}(\Rightarrow {}^1_1 F)$ coincides with $\deg^{\mathcal{G}}(F)$. We now have all ingredients to present our proof systems.

The first proof system, **GS** in Figure 1, is closer to the game-theoretic view. Proofs are (bottom-up) representations of *My* strategies for the disjunctive game. What is unusual is that *all* sequents consisting of labeled propositional variables are allowed as initial sequents. A proof with all initial sequents of degree $\trianglerighteq k$, therefore, represents a k-strategy for *Me*. Hence, in this case, we speak of a k-*proof*. Note that in accordance with a k-strategy, k-proofs are not per se optimal: they merely witness that the degree of the proved sequent is at least k. In particular, every k proof is also an l-proof, if $k \trianglerighteq l$.

Table 1: Proof systems **GS** and $\mathbf{S}^k$.

Initial Sequents for GS

$\Gamma \Rightarrow \Delta$, where Γ and Δ consist of labeled variables

Axioms for $\mathbf{S}^k$

$\Gamma \Rightarrow \Delta$, where $\deg^G(\Gamma \Rightarrow \Delta) \trianglerighteq k$, and Γ and Δ consist of labeled variables

Structural Rules

$$\frac{\Gamma, {}^k_l F, {}^k_l F \Rightarrow \Delta}{\Gamma, {}^k_l F \Rightarrow \Delta}\ (L_c) \qquad \frac{\Gamma \Rightarrow {}^k_l F, {}^k_l F, \Delta}{\Gamma \Rightarrow {}^k_l F, \Delta}\ (R_c)$$

Propositional rules

$$\frac{\Gamma, {}^k_l F \Rightarrow \Delta \qquad \Gamma, {}^k_l G \Rightarrow \Delta}{\Gamma, {}^k_l (F \vee G) \Rightarrow \Delta}\ (L_\vee) \qquad \frac{\Gamma \Rightarrow {}^k_l F, \Delta}{\Gamma \Rightarrow {}^k_l (F \vee G), \Delta}\ (R^1_\vee)$$

$$\frac{\Gamma, {}^k_l F \Rightarrow \Delta}{\Gamma, {}^k_l (F \wedge G) \Rightarrow \Delta}\ (L^1_\wedge) \qquad \frac{\Gamma \Rightarrow {}^k_l G, \Delta}{\Gamma \Rightarrow {}^k_l (F \vee G), \Delta}\ (R^2_\vee)$$

$$\frac{\Gamma, {}^k_l G \Rightarrow \Delta}{\Gamma, {}^k_l (F \wedge G) \Rightarrow \Delta}\ (L^2_\wedge) \qquad \frac{\Gamma \Rightarrow {}^k_l F, \Delta \quad \Gamma \Rightarrow {}^k_l G, \Delta}{\Gamma \Rightarrow {}^k_l (F \wedge G), \Delta}\ (R_\wedge)$$

$$\frac{\Gamma \Rightarrow {}^k_l F, \Delta}{\Gamma, {}^k_l \neg F \Rightarrow \Delta}\ (L_\neg) \qquad \frac{\Gamma, {}^k_l F \Rightarrow \Delta}{\Gamma \Rightarrow {}^k_l \neg F, \Delta}\ (R_\neg)$$

Choice rules

$$\frac{\Gamma, {}^{k}_{l+\mathrm{opt}(G)} F \Rightarrow \Delta \quad \Gamma, {}^{k+\mathrm{opt}(F)}_{l} G \Rightarrow \Delta}{\Gamma, {}^k_l (F \overrightarrow{\times} G) \Rightarrow \Delta}\ (L_{\overrightarrow{\times}}) \qquad \frac{\Gamma \Rightarrow {}^{k+\mathrm{opt}(G)}_{l} F, \Delta}{\Gamma \Rightarrow {}^k_l (F \overrightarrow{\times} G), \Delta}\ (R^1_{\overrightarrow{\times}})$$

$$\frac{\Gamma \Rightarrow {}^{k}_{l+\mathrm{opt}(G)} G, \Delta}{\Gamma \Rightarrow {}^k_l (F \overrightarrow{\times} G), \Delta}\ (R^2_{\overrightarrow{\times}})$$

The second proof system is a proof-theoretically more orthodox system. In fact, it is actually a family of proof systems: for each $k \in Z$, the system $\mathbf{S}^k$ is defined in Figure 1. These proof systems share all the rules with **GS**, but initial sequents are valid iff their degree is at least k. Such initial sequents are axioms in the usual sense.

The conceptual difference between the two approaches is as follows: in **GS**, the value k can be *computed* from the initial sequents. In the second approach, k is *guessed* (implicitly, by picking the proof system $\mathbf{S}^k$, for a concrete k).

Example 3. Figure 3 shows a derivation of $((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg (a \overrightarrow{\times} d) \Rightarrow$ in **GS**. Essentially, it is *My* strategy from Example 2 bottom-up. Degrees of initial sequents:

$$\begin{aligned}
\deg^{\mathcal{G}}({}^3_1 a \Rightarrow {}^1_2 a, {}^2_1 d) &= \deg^{\mathcal{G}}_{\{a\}}({}^3_1 a \Rightarrow {}^1_2 a, {}^2_1 d) = 2, \\
\deg^{\mathcal{G}}({}^2_2 b \Rightarrow {}^1_2 a, {}^2_1 d) &= \deg^{\mathcal{G}}_{\{b\}}({}^2_2 b \Rightarrow {}^1_2 a, {}^2_1 d) = -2, \\
\deg^{\mathcal{G}}({}^1_3 c \Rightarrow {}^1_2 a, {}^2_1 d) &= \deg^{\mathcal{G}}_{\{c\}}({}^1_3 c \Rightarrow {}^1_2 a, {}^2_1 d) = -3.
\end{aligned}$$

Therefore, the derivation is a -2-proof and thus a proof in $\mathbf{S}^{-2}$.

It follows directly from the translation of *My* strategies into proofs:

Theorem 11 *The following are equivalent:*

1. I *have a k-strategy for* $\mathbf{DG}(\mathbf{O} : F_1 \bigvee ... \bigvee \mathbf{O} : F_n \bigvee \mathbf{P} : G_1 \bigvee ... \bigvee \mathbf{P} : G_m)$.
2. $\deg^{\mathcal{G}}({}^1_1 F_1, ..., {}^1_1 F_n \Rightarrow {}^1_1 G_1, ..., {}^1_1 G_m) \trianglerighteq k$.
3. *There is a k-proof of* ${}^1_1 F_1, ..., {}^1_1 F_n \Rightarrow {}^1_1 G_1, ..., {}^1_1 G_m$ *in* **GS**.
4. *There is a proof of* ${}^1_1 F_1, ..., {}^1_1 F_n \Rightarrow {}^1_1 G_1, ..., {}^1_1 G_m$ *in* $\mathbf{S}^k$.

$$\frac{{}^{1}_{3}a \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d \qquad {}^{2}_{2}b \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}{{}^{2}_{1}(a \overrightarrow{\times} b) \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}\ (L_{\overrightarrow{\times}})$$

$$\frac{{}^{2}_{1}(a \overrightarrow{\times} b) \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d \qquad {}^{3}_{1}c \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}\ (L_{\overrightarrow{\times}})$$

$$\frac{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{2}_{1}a, {}^{1}_{1}(a \overrightarrow{\times} d)}\ (R^{2}_{\overrightarrow{\times}})$$

$$\frac{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{2}_{1}a, {}^{1}_{1}(a \overrightarrow{\times} d)}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{1}_{1}(a \overrightarrow{\times} d), {}^{1}_{1}(a \overrightarrow{\times} d)}\ (R^{1}_{\overrightarrow{\times}})$$

$$\frac{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{1}_{1}(a \overrightarrow{\times} d), {}^{1}_{1}(a \overrightarrow{\times} d)}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{1}_{1}(a \overrightarrow{\times} d)}\ (R_{C})$$

$$\frac{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{1}_{1}(a \overrightarrow{\times} d)}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c), {}^{1}_{1}(\neg(a \overrightarrow{\times} d)) \Rightarrow}\ (L_{\neg})$$

$$\frac{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c), {}^{1}_{1}(\neg(a \overrightarrow{\times} d)) \Rightarrow}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c), {}^{1}_{1}(((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)) \Rightarrow}\ (L_{\wedge})$$

$$\frac{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c), {}^{1}_{1}(((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)) \Rightarrow}{{}^{1}_{1}(((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)), {}^{1}_{1}(((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)) \Rightarrow}\ (L_{\wedge})$$

$$\frac{{}^{1}_{1}(((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)), {}^{1}_{1}(((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)) \Rightarrow}{{}^{1}_{1}(((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)) \Rightarrow}\ (L_{C})$$

Fig. 3: A −2-proof in **GS**.

Corollary 12 *Let $k \in \mathbb{Z}^-$. Then there is a k-proof of ${}^{1}_{1}F \Rightarrow$ in* **GS** *iff there is a proof of ${}^{1}_{1}F \Rightarrow$ in* $\mathbf{S}^k$ *iff the degree of F in a preferred model is at most $-k$.*

My strategy in Example 2, is not only a −2-strategy but also a minmax-strategy for *Me*. This implies that *I* cannot do better than −2, i.e. the value of the game is −2. How does this translate into the proof-theoretic interpretation of Example 3? There, the minmax-strategy takes the form of invertibility of rule applications: rule applications S'/S and $(S_1, S_2)/S$ are called *invertible* iff $\deg^{\mathcal{G}}(S') = \deg^{\mathcal{G}}(S)$ and $\min\{\deg^{\mathcal{G}}(S_1), \deg^{\mathcal{G}}(S_2)\} = \deg^{\mathcal{G}}(S)$. In Example 3 only invertible rule applications are used.

In Table 2 we give a calculus $\mathbf{GS}^*$ which is equivalent to **GS** but has only invertible rules, i.e. all rule applications are invertible. The contraction rules are admissible in this system. The motivation behind this calculus is the same as in *My* maxmin-strategy: in every I-state, *I* first duplicate and then exhaustively take all the available options. Every proof produced in this system corresponds to an optimal strategy and has, therefore, an optimal degree. The below results follow directly from the invertibility of the rules:

Table 2: The proof system $\mathbf{GS}^*$ for GCL with invertible rules.

Initial Sequents

$\Gamma \Rightarrow \Delta$, where Γ and Δ consist of labeled variables

Propositional rules

$$\frac{\Gamma, {}^{k}_{l}F \Rightarrow \Delta \qquad \Gamma, {}^{k}_{l}G \Rightarrow \Delta}{\Gamma, {}^{k}_{l}(F \vee G) \Rightarrow \Delta}\ (L_{\vee}) \qquad\qquad \frac{\Gamma \Rightarrow {}^{k}_{l}F, {}^{k}_{l}G, \Delta}{\Gamma \Rightarrow {}^{k}_{l}(F \vee G), \Delta}\ (R_{\vee})$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{}^{1}_{3}a \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d \qquad {}^{2}_{2}b \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}{{}^{2}_{1}(a \overrightarrow{\times} b) \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}\,(L_{\overrightarrow{\times}}) \qquad {}^{3}_{1}c \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{2}_{1}a, {}^{1}_{2}d}\,(L_{\overrightarrow{\times}})}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \Rightarrow {}^{1}_{1}(a \overrightarrow{\times} d)}\,(R_{\overrightarrow{\times}})}{{}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c), {}^{1}_{1}\neg(a \overrightarrow{\times} d) \Rightarrow}\,(L_{\neg})}{{}^{1}_{1}(((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)) \Rightarrow}\,(L_{\wedge})
$$

Fig. 4: A proof in $\mathbf{GS}^*$.

$$
\dfrac{\Gamma, {}^{k}_{l}F, {}^{k}_{l}G, \Rightarrow \Delta}{\Gamma, {}^{k}_{l}(F \wedge G) \Rightarrow \Delta}\,(L_{\wedge}) \qquad \dfrac{\Gamma \Rightarrow {}^{k}_{l}F, \Delta \qquad \Gamma \Rightarrow {}^{k}_{l}G, \Delta}{\Gamma \Rightarrow {}^{k}_{l}(F \wedge G), \Delta}\,(R_{\wedge})
$$

$$
\dfrac{\Gamma \Rightarrow {}^{k}_{l}F, \Delta}{\Gamma, {}^{k}_{l}\neg F \Rightarrow \Delta}\,(L_{\neg}) \qquad \dfrac{\Gamma, {}^{k}_{l}F \Rightarrow \Delta}{\Gamma \Rightarrow {}^{k}_{l}\neg F, \Delta}\,(R_{\neg})
$$

Choice rules

$$
\dfrac{\Gamma, {}^{\quad k}_{l+\mathrm{opt}(G)}F \Rightarrow \Delta \qquad \Gamma, {}^{k+\mathrm{opt}(F)}_{l}G \Rightarrow \Delta}{\Gamma, {}^{k}_{l}(F \overrightarrow{\times} G) \Rightarrow \Delta}\,(L_{\overrightarrow{\times}})
$$

$$
\dfrac{\Gamma \Rightarrow {}^{k+\mathrm{opt}(G)}_{l}F, {}^{\quad k}_{l+\mathrm{opt}(F)}G, \Delta}{\Gamma \Rightarrow {}^{k}_{l}(F \overrightarrow{\times} G), \Delta}\,(R_{\overrightarrow{\times}})
$$

Proposition 13 *Every* $\mathbf{GS}^*$*-proof of a sequent* S *has degree* $\deg^{\mathcal{G}}(S)$.

Corollary 14 *Let* $k = \deg^{\mathcal{G}}({}^{1}_{1}F \Rightarrow) \in \mathbb{Z}^-$. *Then the degree of* F *in a preferred model is equal to* $-k$. *Furthermore, a preferred model of* F *can be extracted from every* $\mathbf{GS}^*$*-proof of* ${}^{1}_{1}F \Rightarrow$.

Example 4. Figure 4 shows a $\mathbf{GS}^*$-proof of ${}^{1}_{1}((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d) \Rightarrow$. The proof is essentially a compact representation of the proof in Figure 3, and has therefore degree -2. We conclude that in a preferred model, $((a \overrightarrow{\times} b) \overrightarrow{\times} c) \wedge \neg(a \overrightarrow{\times} d)$ has degree 2. Furthermore, we can extract the preferred model $\{b\}$ from the position where the $\deg^{\mathcal{G}}$-function is minimal on the initial sequents, as computed in Example 3.

Note that the following degree-version of cut does not hold. The existence of k-strategies for $D \bigvee \mathbf{P} : F$ and $D \bigvee \mathbf{O} : F$ does not imply that a k-strategy for D exists. For example, note that the values of $\mathbf{O} : \top \bigvee \mathbf{O} : \bot \overrightarrow{\times} \top$ and $\mathbf{O}$:

$\top \bigvee \mathbf{P} : \bot \vec{\times} \top$ are -2 and 2, respectively. But the value of the "conclusion" of the cut, $\mathbf{O} : \top$, has value -1.

What is more, there is no function computing the value of the conclusion of cut from the values of the premises. To see this, note that the values of $\mathbf{O} : \bot \vec{\times} \top \bigvee \mathbf{O} : \bot \vec{\times} \top$ and $\mathbf{O} : \bot \vec{\times} \top \bigvee \mathbf{P} : \bot \vec{\times} \top$ are -2 and 2 respectively, as in the above example. However, in contrast to the above example, the conclusion of this cut, $\mathbf{O} : \bot \vec{\times} \top$, has value -2.

Lastly, we demonstrate that $\mathbf{GS}$ and $\mathbf{S}^k$ are useful systems, i.e. that computing the degree of initial sequents is easier than the degree of general sequents.

Proposition 15 *Deciding whether* $\deg^{\mathcal{G}}(\Gamma \Rightarrow \Delta) \trianglerighteq k$ *is* coNP*-hard in general. If* $\Gamma \Rightarrow \Delta$ *is initial, then* $\deg^{\mathcal{G}}(\Gamma \Rightarrow \Delta)$ *can be computed in polynomial time.*

Proof. coNP-hardness of deciding $\deg^{\mathcal{G}}(\Gamma \Rightarrow \Delta) \trianglerighteq k$ follows by coNP-hardness of the validity problem in classical logic: if F is a classical formula, then it holds that $\deg^{\mathcal{G}}(\Rightarrow {}_1^1F) \in \mathbb{Z}^+$ if and only if F is valid (true under all interpretations).

We now show that $\deg^{\mathcal{G}}(\Gamma \Rightarrow \Delta)$ can be computed in polynomial time if $\Gamma \Rightarrow \Delta$ is initial. We start with the empty interpretation $\mathcal{I} = \emptyset$. Now, go through every variable x occurring in $\Gamma \Rightarrow \Delta$. Consider $\Gamma_x \Rightarrow \Delta_x$ where ${}_k^l x \in \Gamma_x$ iff ${}_k^l x \in \Gamma$ and ${}_k^l x \in \Delta_x$ iff ${}_k^l x \in \Delta$. If we have $\deg^{\mathcal{G}}_{\{x\}}(\Gamma_x \Rightarrow \Delta_x) \triangleleft \deg^{\mathcal{G}}_{\emptyset}(\Gamma_x \Rightarrow \Delta_x)$ then let $\mathcal{I} = \mathcal{I} \cup \{x\}$, otherwise leave $\mathcal{I}$ unchanged. In other words, since $\Gamma \Rightarrow \Delta$ is initial, we can simply choose the 'better' option for any given variable x without side effects. Thus, this procedure gives us the minimal $\mathcal{I}$ for $\Gamma \Rightarrow \Delta$. $\square$

5 Conclusion and Future Work

In this paper, we investigate the notion of validity in choice logics. Specifically, we lift a previously established [10] semantic game $\mathbf{NG}$ for the language of QCL to a provability game $\mathbf{DG}$. This allows us to examine formulas with respect to *all* interpretations. Similar to truth, validity in choice logic comes in degrees. We show that the value of $\mathbf{DG}$ adequately models these validity degrees. Strategies for *Me* in $\mathbf{DG}$ correspond to proofs in an analytic labeled sequent calculi $\mathbf{GS}$. The unique feature of this system is that its proofs have degrees that represent the degree of validity. We give two variants of $\mathbf{GS}$ – $\mathbf{GS}^*$ with invertible rules corresponding to *My* optimal strategy, and the more orthodox system $\mathbf{S}^k$ where proofs do not have degrees, but a "degree-profile" is guessed similar to [3].

For future work, it will be interesting to adapt $\mathbf{NG}$ to capture related logics such as Conjunctive Choice Logic [5] or Lexicographic Choice Logic [4], both of which introduce another choice connective in place of ordered disjunction. Using the methods established in this paper, provability games for these semantic games can then be derived. Indeed, our systems are quite modular in this sense, since most aspects of our provability game and our calculi require no adaptation if ordered disjunction were to be exchanged with another choice connective.

Acknowledgments

We thank all our anonymous reviewers for their valuable feedback. This work was funded by the Austrian Science Fund (FWF) under grants P32830 and P32684, the Vienna Science and Technology Fund (WWTF) under grant ICT19-065, and partially funded by the EU (Marie Skłodowska-Curie RISE) project MOSAIC, grant 101007624.

References

1. Benferhat, S., Sedki, K.: Alert correlation based on a logical handling of administrator preferences and knowledge. In: Fernández-Medina, E., Malek, M., Hernando, J. (eds.) SECRYPT 2008, Porto, Portugal, July 26-29, 2008, SECRYPT is part of ICETE. pp. 50–56. INSTICC Press (2008)
2. Benferhat, S., Sedki, K.: Two alternatives for handling preferences in qualitative choice logic. Fuzzy Sets Syst. **159**(15), 1889–1912 (2008). https://doi.org/10.1016/j.fss.2008.02.014
3. Bernreiter, M., Lolic, A., Maly, J., Woltran, S.: Sequent calculi for choice logics. In: Blanchette, J., Kovács, L., Pattinson, D. (eds.) Automated Reasoning - 11th International Joint Conference, IJCAR 2022, Haifa, Israel, August 8-10, 2022, Proceedings. Lecture Notes in Computer Science, vol. 13385, pp. 331–349. Springer (2022). https://doi.org/10.1007/978-3-031-10769-6_20, https://doi.org/10.1007/978-3-031-10769-6_20
4. Bernreiter, M., Maly, J., Woltran, S.: Choice logics and their computational properties. Artif. Intell. **311**, 103755 (2022). https://doi.org/10.1016/j.artint.2022.103755, https://doi.org/10.1016/j.artint.2022.103755
5. Boudjelida, A., Benferhat, S.: Conjunctive choice logic. In: ISAIM 2016, Fort Lauderdale, Florida, USA, January 4-6, 2016 (2016)
6. Brewka, G., Benferhat, S., Berre, D.L.: Qualitative choice logic. Artif. Intell. **157**(1-2), 203–237 (2004). https://doi.org/10.1016/j.artint.2004.04.006
7. Brewka, G., Niemelä, I., Syrjänen, T.: Logic programs with ordered disjunction. Comput. Intell. **20**(2), 335–357 (2004). https://doi.org/10.1111/j.0824-7935.2004.00241.x
8. Fermüller, C.G., Metcalfe, G.: Giles's game and the proof theory of Łukasiewicz logic. Studia Logica **92**(1), 27–61 (2009). https://doi.org/10.1007/s11225-009-9185-2
9. Freiman, R.: Games for hybrid logic. In: Silva, A., Wassermann, R., de Queiroz, R. (eds.) Wollic 2021, Virtual Event, October 5-8, 2021, Proceedings, pp. 133–149. Springer Verlag (2021)
10. Freiman, R., Bernreiter, M.: Truth and preferences - A game approach for qualitative choice logic. In: M-PREF@IJCAI (2022), available at https://doi.org/10.48550/arXiv.2209.12777
11. Hintikka, J.: Logic, Language-Games and Information: Kantian Themes in the Philosophy of Logic. Oxford, Clarendon Press (1973)
12. Liétard, L., Hadjali, A., Rocacher, D.: Towards a gradual QCL model for database querying. In: Laurent, A., Strauss, O., Bouchon-Meunier, B., Yager, R.R. (eds.) IPMU 2014, Montpellier, France, July 15-19, 2014, Proceedings, Part III. Communications in Computer and Information Science, vol. 444, pp. 130–139. Springer (2014). https://doi.org/10.1007/978-3-319-08852-5_14

13. Pavlova, A., Freiman, R., Lang, T.: From semantic games to provability: The case of Gödel logic. Studia Logica (10 2021). https://doi.org/10.1007/s11225-021-09966-x
14. Pigozzi, G., Tsoukiàs, A., Viappiani, P.: Preferences in artificial intelligence. Ann. Math. Artif. Intell. **77**(3-4), 361–401 (2016). https://doi.org/10.1007/s10472-015-9475-5
15. Sedki, K., Lamy, J., Tsopra, R.: Learning preferences in prioritized qualitative choice logic. In: 32nd IEEE International Conference on Tools with Artificial Intelligence, ICTAI 2020, Baltimore, MD, USA, November 9-11, 2020. pp. 368–375. IEEE (2020). https://doi.org/10.1109/ICTAI50040.2020.00065, https://doi.org/10.1109/ICTAI50040.2020.00065
16. Sedki, K., Lamy, J., Tsopra, R.: Qualitative choice logic for modeling experts recommendations of antibiotics. In: Barták, R., Keshtkar, F., Franklin, M. (eds.) Proceedings of the Thirty-Fifth International Florida Artificial Intelligence Research Society Conference, FLAIRS 2022, Hutchinson Island, Jensen Beach, Florida, USA, May 15-18, 2022 (2022). https://doi.org/10.32473/flairs.v35i.130677, https://doi.org/10.32473/flairs.v35i.130677