Aleatoric Propositions: Reasoning about Coins

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Abstract. Aleatoric propositions are a generalisation of Boolean propositions, that are intrinsically probabilistic, or determined by the toss of a (biased) coin. Rather than let propositions take a true/false valuation, we assume they act as a biased coin, that will sometimes land heads (*true*), and sometimes land tails (*false*) . Complex propositions then correspond to a conditional series of tosses of these coins. We extend the syntax and semantics for Aleatoric Logic to include a novel fixed-point operator that is able to represent a weak form of iteration. We examine the expressivity of the of the language, showing a correspondence to the rational functions over $(0, 1)$

Keywords: Probabilistic Reasoning, Expressivity, Correspondence Theory

1 Introduction

Logic is the study of truth and deduction. In both philosophical and mathematical contexts, a logic represents a reasoning process, where true statements are composed via some rules to infer new truths. However, the commitment to the study of *true* statements sets a very high bar for reasoning. In everyday live we are beset with uncertainty and absolute truth can rarely be assumed. Nonetheless, reason persists and we are able to act rationally and perform deductions within the bounds of our uncertainty. Furthermore, we are able to do this without ever quantifying our uncertainty: some facts are simply recognised on being contingent on things outside of our experience. As we accrue experience, our confidence in our judgements increases as does our trust in our reasoning, even though the standard of absolute truth is never attained. This is the experiential logic described by Hume [7].

As automated reasoning and artificial intelligence become more capable, there is a need for a foundation for reasoning and logic that is tolerant of the uncertainty that we find in every day life. This paper presents such a formalism: an analogue of propositional logic where the true false statements are replaced by independent probabilistic events: what may thought of as tosses of a biased coin. A coin toss is intrinsically uncertain, so while the proposition has a correspondence to the coin, we suppose that an agent only has access to this proposition via the sampling of tosses of the coin. Therefore, aleatoric propositions are defined over a series of events, where atomic events correspond to the toss of some biased coin. The complex expressions therefore describe coin-flipping protocols where the next coin chosen to flip is contingent on the outcome of the prior coin flips. Particularly, we introduce a fixed-point operator that is able to represent looping coin flipping protocols.

1.1 Related work

There has been a considerable number of works that have considered probabilistic semantics. We will briefly cover some here and attempt to categorise them in relation to this work.

Early work includes Kolmogorov's [9] axiomatization, Ramsey's [13] and de Finetti's [1] characterisation of subjective probability, These works provide a foundation for what constitutes a probability distribution, and Hailperin [4] gives a good overview of early work..

There have been a number of works applying probabilistic elements to automated reasoning and deduction, including probabilistic description logic [11], reasoning about uncertainty [5], verification of randomised programs [10]. These approaches include a modality for the probability of some event occurring. As the probabilities are explicit in the syntax, these approaches reason about the probabilities of events, in the sense that while a pair of dice landing as two ones (snake eyes) is an uncertain statement, the statement "snake eyes has probability $\frac{1}{36}$" is a Boolean (true/false) statement. Gardenfors [3] has approached the topic of probability logics in a similar way, axiomatising a logic of relative probability, which contains statements such as "A is at least as probable as B".

We are interested in reasoning probabilistically, without necessarily quantifying probabilities. In this vein also are the Fuzzy approaches to probabilistic reasoning [15]. Fuzzy logic is typically applied to describe the concept of vagueness, where the semantics allow for the increasing or decreasing of plausibility, without necessarily committing to the absolute certainty of a proposition. When the *product semantics* are used (so the plausibility of two propositions taken together are multiplied) there is a natural correspondence with independent random events like the flipping of coins. While this has many similarities to our approach, it does not have the fixed-point operator capable of expressing events of unbounded magnitude.

This work is an extension of the propositional *aleatoric calculus* presented in [2] which also considered the modal extension of the logic (not considered here) but also did not include the fixed point operator discussed here.

1.2 Overview

The following section will present the syntax and semantics for aleatoric propositions, and provide some discussion of the novel operators. The next section will consider the expressivity of the language, and give some illustrative examples, such as the expression of fractional probabilities, and conditional reasoning. The main result of the paper is to show that the aleatoric propositions coincide with the set of all rational functions that map products of the open interval $(0, 1)$

to itself to $(0, 1)$. This result is based on work by Mossel [12]. Finally we will discuss some remaining open problems, and briefly discuss how we may extend the system to talk about dependent events.

2 Syntax and Semantics

Here we present a minimal syntax and semantics for *Aleatoric Propositions*, extending the aleatoric calculus presented in [2]. Aleatoric propositions are a generalisation of Boolean propositions, and are defined over a set of atomic propositions $\mathfrak{P}$. To avoid confusion, we will not refer to propositions being *true* or *false*, but rather consider them as descriptions of sets of events (e.g. a coin landing head side up) that occurs with some probability. For this reason we use symbols $\bullet$ (*heads*) and $\circ$ (*tails*) for atoms corresponding to probability 1 (always heads) and probability 0 (always tails) respectively.

A complex proposition describes events comprised of sub-events. For example, given propositions A and B, we may consider a proposition which corresponds to events where the A-coin lands heads, and then the B-coin lands heads, *and* then the A-coin (tossed for a second time) lands heads once more.

As we consider a proposition describing the occurrence of an event, we can also consider a proposition describing an event *failing* to occur. Note, in this context, we consider the failure of an event meaning the event is explicitly tested, and that test fails. So given an event "the penny lands heads", the negation would be "the penny is flipped and and does not land heads", but it would not be "the penny is not flipped" or "the quarter is flipped and it lands heads".

The propositions can also describe conditional events: we could consider a proposition describing an event where the A-coin has the same result two times in a row, so *if* the A-coin lands heads, then it is tossed again, and lands heads again, but *if* it landed tails the first time, it is tossed again and lands tails the second time.

Finally, we also consider iterative events, where propositions are repeatedly sampled until come condition is met. A famous example is the scheme devised by von Neumann [14] to simulate a fair coin (with bias precisely $1/2$) using any coin with probability in $(0, 1)$. Here the coin is flipped twice: if we see a head followed by a tail, we report a (synthetic) head; if we see a tail followed by a head we report a (synthetic) tail; and otherwise, if we see two heads or two tails, we repeat the process. Since each coin flip is independently sampled with the same probability regardless of the bias the likelihood of the synthetic head is the same as the likelihood of the synthetic tail,

These complex propositions are built using a ternary *if-then-else* operator, a negation operator, and a *fixed-point* operator.

Definition 1. *The* syntax of aleatoric propositions *is given by:*

$$\alpha ::= \bullet \mid A \mid \neg\alpha \mid (\alpha ? \alpha : \alpha) \mid \mathbb{F}X\alpha$$

where $A, X \in \mathfrak{P}$, *and* X *is* linear in α. *The set of aleatoric propositions is denoted as* $\mathcal{L}$.

We let $\mathrm{var}(\alpha)$ be the set of atomic propositions appearing in α, and say $X \in \mathrm{var}(\alpha)$ is *free in* α ($X \in \mathrm{free}(\alpha)$) if X does not appear in the scope of a $\mathbb{F}X$ operator. If $X \in \mathrm{var}(\alpha)$ does appear in the scope of a $\mathbb{F}X$ operator, we say X is *bound* in α ($X \in \mathrm{bnd}(\alpha)$).

The fixed point operator required that X *is linear in* α, and this is to ensure that the fixed point is unique and non-ambiguous.

Definition 2. *An atomic proposition* $X \in \mathfrak{P}$ is linear in α *if and only if for every subformula* $(\beta?\gamma_1:\gamma_2)$ *of* α*, there is no occurrence of* X *appearing in* β*.*

The meaning and significance of linearity will be discussed once the semantics have been presented.

A brief description of these operators is as follows:

- $\bullet$ (*heads*) describes an event that invariably occurs (i.e. a coin that always lands heads, or a double headed coin).
- A is an *atom* that describes an event that occurs with some probability $p \in (0,1)$.
- $\neg\alpha$, (*not* α) describes the *failure* of an event to occur. That is, the event is explicitly tested for, and that test fails.
- $(\alpha?\beta:\gamma)$ (*if* α *then* β *else* γ) describes a *conditional* event where the event described by α is tested (or sampled) and if it occurs, then an event described by β occurs, but if the alpha event does not occur, then an event described by γ occurs.
- $\mathbb{F}X\alpha$ (X *where* $X = \alpha$) is the *fixed point* proposition, and it describes an event with probability x such that if the event corresponding to the atom X had likelihood x, so would α. An alternative way to consider this operation is a description of a recursive event, so that whenever the proposition, X, is to be tested instead a test of the proposition $\mathbb{F}X\alpha$ is (recursively) substituted. If this process continues forever, (e.g., in the evaluation of $\mathbb{F}XX$, its value is deemed to be $\frac{1}{2}$. See the proof of Lemma 1 for a discussion of this.

As an example of this syntax, we can represent the scheme of von Neumann, mentioned above, as:

$$\textit{fair-coin} = \mathbb{F}X(A?(A?X:\bullet):(A?\circ:X)).$$

We note that the notion of uncertainty is intrinsic in these operators. That an atomic proposition A "happened" does not mean that A is *true* in the common sense: we could repeat the process and the subsequent event satisfies $\neg A$. The propositions, themselves are mercurial and transient, so it does not make sense to say a proposition *is* true. Rather, we are interested in the *probability* that an event described by the proposition *will occur*, and this is what is presented in the following semantics.

The semantics of these operators is given over an *interpretation*, $\mathcal{I}$, which assigns a probability between 0 and 1 to every atomic proposition.

Definition 3. *An* interpretation *for propositional aleatoric logic is a function* $\mathcal{I} : \mathfrak{P} \longrightarrow (0,1)$*. Given an interpretation,* $\mathcal{I}$*, an atomic proposition* $X \in \mathfrak{P}$

and some $p \in (0,1)$*, we let the interpretation* $\mathcal{I}[X : p]$ *be such that for all* $Y \in \mathfrak{P}\backslash\{X\}$*,* $\mathcal{I}[X:p](Y) = \mathcal{I}(Y)$ *and* $\mathcal{I}[X:p] = p$*.*

We will use the notation $\alpha[X\backslash\beta]$ to represent the proposition α with all free occurrences of X in α replaced by β, and we say β *is free of* X *in* α if for every free variable Y in β, X is not in the scope of an operator $\mathbb{F}Y$ in α. We can now define the semantics as below[1]:

Definition 4. *Given an interpretation,* $\mathcal{I}$*, and some aleatoric proposition* α*, the interpretation assigns the probability* $\mathcal{I}(\alpha)$ *inductively as follows:*

$$\begin{aligned}
\bullet^{\mathcal{I}} &= 1\\
A^{\mathcal{I}} &= \mathcal{I}(A)\\
(\neg\alpha)^{\mathcal{I}} &= 1 - \alpha^{\mathcal{I}}\\
(\alpha?\beta:\gamma)^{\mathcal{I}} &= \alpha^{\mathcal{I}}\cdot\beta^{\mathcal{I}} + (1-\alpha^{\mathcal{I}})\cdot\gamma^{\mathcal{I}}\\
(\mathbb{F}X\alpha)^{\mathcal{I}} &= \begin{cases} 1 \textit{ if } \alpha^{\mathcal{I}} = 1\\ 0 \textit{ if } \alpha^{\mathcal{I}} = 0\\ x \textit{ if } x \textit{ is the unique value such that } \alpha^{\mathcal{I}[X:x]} = x\\ 1/2 \textit{ if } \forall x \in (0,1),\ \alpha^{\mathcal{I}[X:x]} = x \end{cases}
\end{aligned}$$

We must show that the semantic interpretation of the fixed point operator is well defined; that is, the fixed point always exists and has uniquely defined value..

Lemma 1. *The semantic interpretation of the fixed point operator is well defined. Given any* α *where* X *is linear in* α*, given any interpretation* $\mathcal{I}$*, either* $\alpha^{\mathcal{I}} \in \{0,1\}$*, or there is a unique* $x \in (0,1)$ *such that* $\alpha^{\mathcal{I}[X:x]} = x$*, or for every* $x \in (0,1)$*,* $\alpha^{\mathcal{I}[X:x]} = x$*.*

Proof. This proof will be given by induction over the complexity of formulas, and will also provide an alternative semantic definition for the fixed point operator.

The induction hypothesis is, for every aleatoric proposition $\alpha \in \mathcal{L}$, for each atomic proposition X where X is linear in α, given any interpretation $\mathcal{I}$, there are unique values $h_\alpha,\ i_\alpha^X \in [0,1]$ such that $\alpha^{\mathcal{I}[X:x]} = h_\alpha + i_\alpha^X \cdot x$. For simplicity we will assume that free(α) and bnd(α) are disjoint sets. The induction is given over the complexity of formulas as follows:

- for $\psi = \bullet$, $h_\psi = 1$ and $i_\psi^X = 0$.
- for $\psi = A \in$ var, where $A \notin$ bnd(α), let $h_\psi = \mathcal{I}(A)$ and $i_\psi^X = 0$. Since $A^{\mathcal{I}[X;x]} = \mathcal{I}(A)$, it is clear that $h_\psi = \mathcal{I}(A)$ and $i_\psi^X = 0$ are the only values that satisfy the induction hypothesis.
- for $\psi = X$, where $X \in$ bnd(α), let $h_\psi = 0$ and $i_\psi^X = 1$. Since $X^{\mathcal{I}[X:x]} = x$, it is clear that $h_\psi = 0$ and $i_\psi^X = 1$ are the only values that satisfy the induction hypothesis.

[1] We use the notation where given the probabilities x and y, $x \cdot y$ is interpreted as the *product* of x and y.

- for $\psi = (\beta ? \gamma_1 : \gamma_2)$, $h_\psi = h_\beta \cdot h_{\gamma_1} + (1 - h_\beta) \cdot h_{\gamma_2}$, and $i^X_\psi = h_\beta \cdot i^X_{\gamma_1} + (1 - h_\beta) \cdot i^X_{\gamma_2}$. Since $\mathcal{I}(\psi) = \mathcal{I}(\beta) \cdot \mathcal{I}(\gamma_1) + (1 - \mathcal{I}(\beta)) \cdot \mathcal{I}(\gamma_2)$, it follows that $h_\psi = h_\beta \cdot h_{\gamma_1} + (1 - h_\beta) \cdot h_{\gamma_2}$ and $i^X_\psi = h_\beta \cdot i^X_{\gamma_1} + (1 - h_\beta) \cdot i^X_{\gamma_2}$, noting that as X is linear in α, by the induction hypothesis$i^X_\beta = 0$. The values for h_ψ and i^X_ψ are unique since these calculations are deterministic.
- for $\psi = \neg\beta$, let $h_\psi = 1 - h_\beta$, $i^X_\psi = -i^X_\beta$. This derivation follows from the induction hypothesis; as $\beta^{\mathcal{I}[X:x]}$ is described with respect to x by the function $\beta^{\mathcal{I}[X:x]} = h_\beta + i^X_\beta \cdot x$, it follows that

$$1 - \beta^{\mathcal{I}[X:x]} = 1 - (h_\beta + i^X_\beta \cdot x) = (1 - h_\beta) + (-i^X_\beta) \cdot x)$$

  as required.
- for $\psi = \mathbb{F}Y\beta$, $h_\psi = \frac{h_\beta}{1 - i^Y_\beta}$ or $h_\psi = 1/2$ if $i^Y_\beta = 1$, and $i^X_\psi = \frac{i^X_\beta}{1 - i^Y_\beta}$ or 0 if $i^Y_\beta = 1$. For any y, we have $\beta^{\mathcal{I}[X:x,Y:y]} = h^X_\beta + i^X_\beta \cdot x + i^Y_\beta \cdot y$, noting the linearity of both X and Y in β. As $(\mathbb{F}Y\beta)^{\mathcal{I}[X:x]} = y$ where $y = h^X_\beta + i^X_\beta \cdot x + i^Y_\beta \cdot y$, solving for y, provided $i^Y_\beta \neq 1$, the unique solution $(\mathbb{F}Y\beta)^{\mathcal{I}[X:x]} = \frac{h^X_\beta}{1 - i^Y_\beta} + \frac{i^X_\beta \cdot x}{1 - i^Y_\beta}$ gives the definition of h^X_ψ and i^X_ψ. If $i^Y_\beta = 1$, the it must be the case that $h_\beta = 0$ and $\beta^{\mathcal{I}[Y:y]} = y$. Therefore, for every $y \in (0, 1)$ we have $\beta^{\mathcal{I}[Y:y]} = y$ so the fixed point semantics gives $(\mathbb{F}Y\beta)^{\mathcal{I}[X:x]} = 1/2$. It follows that $h_\psi = 1/2$ and $i^X_\psi = 0$.

These definitions are complete and deterministic, and from the induction hypothesis, it follows $(\mathbb{F}X\alpha)^{\mathcal{I}} = \frac{h_\alpha}{1 - i^X_\alpha}$ or $\frac{1}{2}$ if $i^X_\alpha = 1$. In either case, the semantic interpretation of the fixed point operator is well defined.

The fixed point is a genuine fixed point and the fact that in the formula $\mathbb{F}X\alpha$, X is always linear in α means that the fixed point is always unique (see Figure 1, which also demonstrates $\neg\mathbb{F}X\alpha(X) = \mathbb{F}X\neg\alpha(\neg X)$). This gives an alternative semantic formulation of the fixed point operator, and a convenient way to visualise the fixed point as the intersection point of a line with intercept h_α and gradient i^X_α with the line with intercept 0 and gradient 1. This also motivates, Definition 2, where X is linear in α if α is a linear function of the interpretation of X, when all other arguments of α are fixed.

Definition 5. *Given an interpretation $\mathcal{I}$, and some $\alpha \in \mathcal{L}$ the* functional semantics *for propositional aleatoric logic assigns a value $h_\alpha \in [0, 1]$ and a value*

i^X_α *for each* $X \in \text{bnd}(\alpha)$ *as follows:*

$\psi = \bullet :$	$h_\psi = 1$	$i^X_\psi = 0$
$\psi = A \in \text{free}(alpha) :$	$h_\psi = \mathcal{I}(A)$	$i^X_\psi = 0$
$\psi = X \in \text{bnd}(\alpha) :$	$h_\psi = 0$	$i^X_\psi = 1$
$\psi = Y \in \text{bnd}(\alpha) :$	$h_\psi = 0$	$i^X_\psi = 0$
$\psi = (\alpha ? \beta : \gamma) :$	$h_\psi = h_\alpha \cdot h_\beta + (1 - h_\alpha) \cdot h_\gamma$	$i^X_\psi = h_\alpha \cdot i^X_\beta + (1 - h_\alpha) \cdot i^X_\gamma$
$\psi = \neg\alpha :$	$h_\psi = 1 - h_\alpha$	$i^X_\psi = -i^X_\alpha$
$\psi = \mathbb{F}X\alpha,\ i^X_\alpha \neq 1 :$	$h_\psi = \frac{h_\alpha}{1 - i^X_\alpha}$	$i^X_\psi = 0$
$\psi = \mathbb{F}X\alpha,\ i^X_\alpha = 1 :$	$h_\psi = 1/2$	$i^X_\psi = 0$
$\psi = \mathbb{F}Y\alpha,\ i^Y_\alpha \neq 1 :$	$h_\psi = \frac{h_\alpha}{1 - i^Y_\alpha}$	$i^X_\psi = \frac{i^X_\alpha}{1 - i^Y_\alpha}$
$\psi = \mathbb{F}Y\alpha,\ i^Y_\alpha = 1 :$	$h_\psi = 1/2$	$i^X_\psi = 0$

Corollary 1. *For any proposition* $\alpha \in \mathcal{L}$*, and any interpretation* $\mathcal{I}$*, we have* $\alpha^\mathcal{I} = h_\alpha$*, where* h_α *is given in Definition 5.*

This corollary follows directly from the proof of Lemma 1

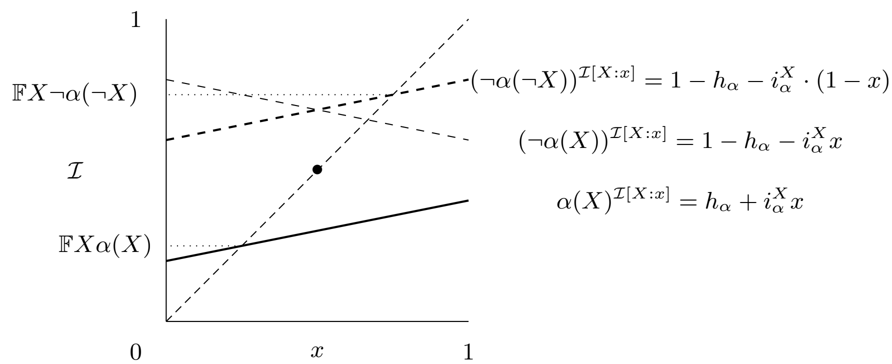

Fig. 1. The semantic interpretation of $\mathbb{F}X\alpha(X)$, showing how the value of $\mathbb{F}X\alpha$ corresponds to $\alpha^{\mathcal{I}[X:x]}$ with respect to x. The lower thick line is the function $\alpha(X)^{\mathcal{I}[X:x]}$, with intercept h_α and gradient i^X_α. The thick dashed line corresponds to the function $\neg\alpha(\neg X)^{\mathcal{I}[X:x]}$. The fixed point of each function is the point where the line crosses the diagonal.

2.1 Abbreviations

Within these semantics we may define conventional logic operators where the semantics loosely align with fuzzy logic using the product t-norm [15].

Table 2.1 contains some useful abbreviations. Note the notion of frequency in this set of abbreviations. The formula $\alpha^{\frac{n}{m}}$ or α at least n out of m times refers to

Abbreviation	Expression	Description
$\bigcirc$	$\neg\bullet$	*tails*
$\LEFTcircle$	$\mathbb{F}XX$	*fair coin flip*
$\alpha \wedge \beta$	$(\alpha?\beta:\bigcirc)$	α *and* β
$\alpha \vee \beta$	$(\alpha?\bullet:\beta)$	α *or* β
$\alpha \to \beta$	$(\alpha?\beta:\bullet)$	α *implies* β
$\alpha \leftrightarrow \beta$	$(\alpha?\beta:\neg\beta)$	α *if and only if* β
$\alpha^{\frac{0}{m}}$	$\bullet$	α 0 *out of* m *times.*
$\alpha^{\frac{n}{0}}$	$\bigcirc$	α n *out of* 0 *times* ($n > 0$).
$\alpha^{\frac{n}{m}}$	$(\alpha?\alpha^{\frac{n-1}{m-1}}:\alpha^{\frac{n}{m-1}})$	α *at least* n *out of* m *times.*

Table 1. Some useful abbreviations for aleatoric propositions.

the event that α when sampled m times, α occurred at least n times. This does not suggest that probability of α is at least $\frac{n}{m}$. It simply describes an event: if $\Pr(\alpha) = 0.1$ then $\alpha^{\frac{10}{10}}$ is simply a very unlikely event. In experiential logic, this gives a proxy for truth: $\alpha^{\frac{100}{100}}$ is the case only when we are very confident in α.

2.2 Motivation and discussion

With the semantics established and shown to be well-founded it is worth taking some time to motivate the semantics choices made. We will consider the following motivating example:

Example 1. Suppose that Venus and Serena are playing a game of tennis, and are involved in a tie break. The tie break works by Venus serving first, then Serena serving twice, and then Venus serving twice, and so on, until one of them is two points ahead of the other. In tennis, it is often supposed the server has the advantage, so we let V be the probability Venus wins on her serve, and S be the probability that Serena wins on her serve. Then Venus winning the tie break can be represented as the following aleatoric proposition:

$$\mathbb{F}X(V?(S?(S?(V?X:\bigcirc):(V?\bullet:X)):\bullet):(S?\bigcirc:(S?(V?X:\bigcirc):(V?\bullet:X))))$$

Applying Corollary 1 (and several algebraic reductions) we can describe the probability of Venus winning the tie-break as a function of V and S:

$$\mathit{VenusWins}(V,S) = \frac{V - S\cdot V}{S + V - 2\cdot S\cdot V} \tag{1}$$

with the contour plot given in Figure 2.

This first example is effectively a representation of a Markov decision process. Where the probabilities correspond to discrete events. However, aleatoric propositions can also be used to represent situations with unknown variables, such as "Venus has an injury", which are either true or false, but unknown to the reasoner. The following example demonstrates how aleatoric propositions could be applied to these epistemic variables

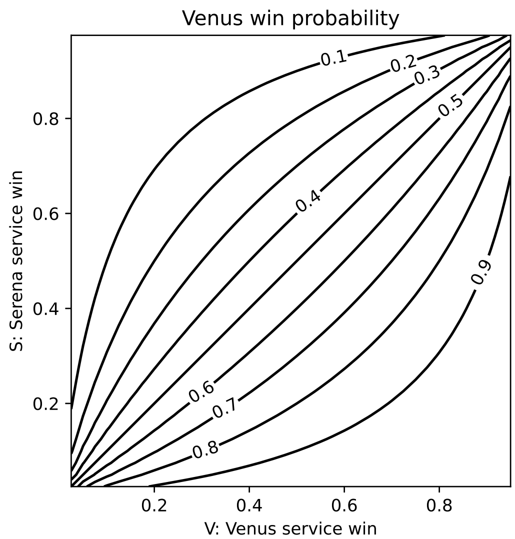

Fig. 2. A contour diagram of the probability of Venus winning a tie break (1, given Venus has probability V of winning on her own serve, and a probability of $(1-S)$ of winning on Serena's serve.

Example 2. It is Alex's turn to get dinner and Blake is speculating what Alex may do. If Alex is not too tired, a home cooked meal is likely, but if Alex can afford it, Alex may (50%) order dinner from a food delivering service.

Blake considers Alex's current state base on discussion they have had during the week, and imagines *alex_tired* and *alex_rich* as two coins with biases reflecting Blake's assessment of Alex's current state. Blake is then able to synthesise a new coin representing whether there will be a home cooked meal:

$$home\_cooked = (alex\_rich ? (◐ ? \neg alex\_tired : \bot) : \neg alex\_tired)$$

In this example, if Alex is rich, there is a 50% chance that they will get a food delivery service. Otherwise, if Alex is not too tired Alex will cook a home cooked meal. Note that it is possible to similarly devise a coin for whether they will order dinner from a service, but the outcomes of the coins will not necessarily be mutually exclusive, nor necessarily sum to 1 (if Alex is poor and tired breakfast cereal may be an option).

The use of aleatoric propositions can be thought of as mental simulations. Given an agent's experiences, they may imagine how the world might be: this is a test. Blake can ponder "Is Alex rich", and he may imagine it to be so or not. These mental simulations reflect Blake's belief. The structure of an aleatoric proposition describes this mental simulation process: what beliefs are considered and in what order.

3 A correspondence for aleatoric propositions

In this section we will investigate the expressivity of aleatoric propositions, and give a correspondence result. The correspondence is based on earlier work by Keane and O'Brien [8] and Elchanan Mossel and Yuval Peres [12]. Keane and O'Brien originally showed that "Bernoulli factories", which are essentially coin flipping protocols that transform one probability to another[2], can simulate any continuous function over $(0,1)$, and Mossel and Peres showed that with some restrictions on the coin flipping protocols, the resulting set of functions correspond to the set of rational functions over $(0,1)$. We will follow Mossel's and Peres's presentation here.

To be precise, we can consider a proposition $\alpha \in \mathcal{L}$ to be a function that given an interpretation, $\mathcal{I}$, returns a probability $\alpha^{\mathcal{I}}$. In turn, the interpretation, $\mathcal{I}$, with respect to α is simply an assignment from $\mathrm{var}(\alpha)$ to $(0,1)$, so any aleatoric proposition, α, may be considered as a function from $(0,1)^{var(\alpha)}$ to $(0,1)$. To characterise the expressivity of aleatoric propositions, we will first describe the set of *rational functions* from $(0,1)^{\mathrm{var}(\alpha)}$ to $(0,1)$. The next subsection will express the semantics of aleatoric propositions as functions, and establish a normal form for aleatoric propositions, showing every aleatoric proposition corresponds to a rational function. The final subsection completes the correspondence by showing that every rational function from $(0,1)^{\mathcal{X}}$ to $(0,1)$ agrees with the semantics of some aleatoric proposition defined over the atomic propositions $\mathcal{X}$.

In this section we suppose that $\alpha \in \mathcal{L}$ is an aleatoric proposition where $\mathrm{var}(\alpha) = \{X_1, ..., X_n\} = \mathcal{X}$, and $f_\alpha : (0,1)^{\mathcal{X}} \longrightarrow (0,1)$ is a function such that $f_\alpha(X_1 \mapsto X_1^{\mathcal{I}}, ..., X_n \mapsto X_n^{\mathcal{I}}) = \alpha^{\mathcal{I}}$.

In general, rational functions over $\mathcal{X}$ maybe thought of as fractions of polynomials.

Definition 6. *A* rational function of degree k *from* $(0,1)^{\mathcal{X}}$ *to* $[0,1]$ *is a function of the form:*

$$f(\overline{x}) = \frac{\sum_{\overline{a}\in\sigma^k_{\mathcal{X}})} \ell_{\overline{a}} \prod_{x\in\mathcal{X}} x^{a_x}}{\sum_{\overline{a}\in\sigma^k_{\mathcal{X}}} m_{\overline{a}} \prod_{x\in\mathcal{X}} x^{a_x}}$$

where $\sigma^k_{\mathcal{X}} = \{\overline{a} \in \{0, ..., k\}^{\mathcal{X}} \mid \sum_{x\in\mathcal{X}} a_x = k\}$ *for all* $\overline{a} \in \sigma^k_{\mathcal{X}}$, $\ell_{\overline{a}}, m_{\overline{a}} \in \mathbb{Z}$, *and for all* $\overline{x} \in (0,1)^{\mathcal{X}}$, $f(\overline{x}) \in (0,1)$.

3.1 Aleatoric functions

In this subsection we will define a special form for aleatoric propositions, and through a series of semantically invariant transformations, show that every aleatoric proposition can be represented in this form.

In this section we will suppose that we are dealing with formulas consisting of the free variables $A_1, ..., A_n$, and the fixed point variables $X_1, ..., X_m$, which are disjoint with the free variables.

The definition of *block normal form* for aleatoric propositions is as follows.

[2] For example by twice flipping a coin with bias p, we construct an event with probability p^2.

Definition 7. *A formula of aleatoric propositional logic is in* k-block normal form *if it satisfies the following syntax for* γ*:*

$$
\begin{aligned}
\alpha_1^0 &::= \CIRCLE \mid \Circle \mid X_0 \\
\alpha_i^{j+1} &::= (A_i ? \alpha_i^j : X_0) \mid (A_i ? X_0 : \alpha_i^j) \\
\alpha_{i+1}^1 &::= (A_{i+1} ? \alpha_i^k : X_0) \mid (A_{i+1} ? X_0 : \alpha_i^k) \\
\beta_0 &::= \alpha_n^k \\
\beta_{i+1} &::= (\LEFTcircle ? \beta_i : \beta_i) \\
\gamma &::= \mathbb{F} X_0 \beta_\ell
\end{aligned}
$$

A representation of a formula in block form is given in Figure 3. Here, the formula $\mathbb{F}X(\neg(A \wedge B) \rightarrow (A \wedge X))$ is converted to the 2-block normal form. As the 2-block normal form uses the conditional statements $(\alpha ? \beta : \gamma)$, where α is guaranteed to be either a propositional atom A_i or $\LEFTcircle$, we use the convention of drawing the formulas as a tree where β is on the left branch and γ is on the right branch. The cut off branches at the α_i^j levels are shorthand for X_0. Similarly, from the definition of k-block normal form the leaves are all labelled with $\CIRCLE$, $\Circle$ or X_0.

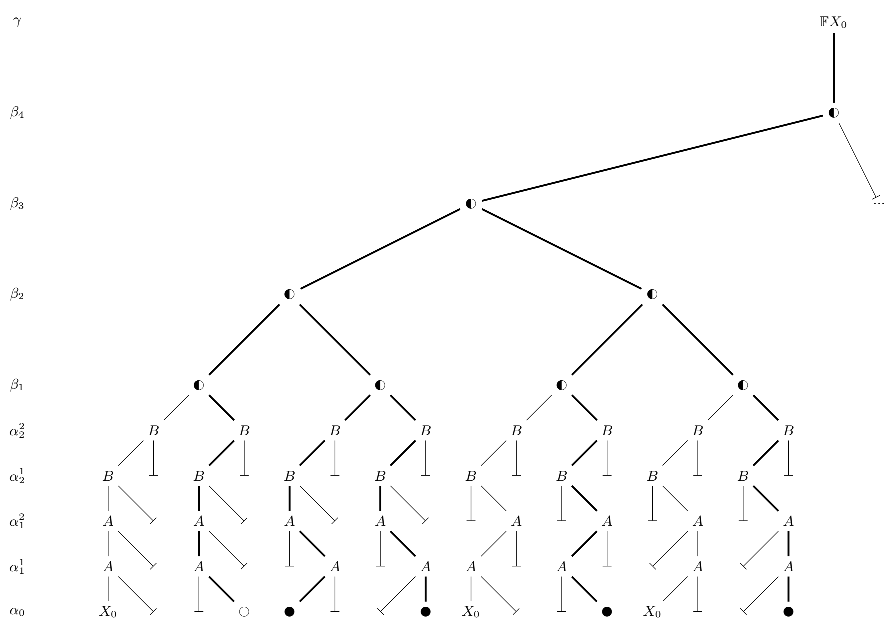

Fig. 3. A 2-block normal form representation of the formula $\mathbb{F}X(\neg(A \wedge B) \rightarrow (A \wedge X))$, Each internal node represents a proposition $(A_i ? \alpha_1 : \alpha_2)$ where α_1 is the left child, and α_2 is the right child, except for the root which is a fixed point proposition. The entire tree is repeated again as the right branch of the node at level β_3.

Lemma 2. *Every aleatoric proposition, where each distinct free variable occurs at most k times is semantically equivalent to an aleatoric proposition in k-block-form. Specifically, there is a function $\tau : \mathcal{L} \longrightarrow \mathcal{L}$, such that:*

1. *for all $\alpha \in \mathcal{L}$, $\tau(\alpha)$ is in k-block-form.*
2. *for all interpretations, $\mathcal{I}$, $\mathcal{I}(\alpha) = \mathcal{I}(\tau(\alpha))$.*

Proof. The proof is given by construction where we give a set of semantically valid transformations that: push negations down to only apply in the context of abbreviation$\bigcirc$; modify conditional statements $(\alpha ? \beta : \gamma)$ so that α is either some free variable (A_i) or $◐$; modify conditional statements $(A_i ? \beta : \gamma)$ so that either β or γ or both are X_0; order the atomic propositions, so in the subformulas $(A_i ? \beta : \gamma)$, β and γ can only contain free variables A_j where $j < i$; and combine all fixed point operators into a single fixed point operator at the highest level.

This is achieved through the following transformations, that preserve the interpretation of the formulas. We write $\alpha \Rightarrow \beta$ to indicate that $\alpha^{\mathcal{I}} = \beta^{\mathcal{I}}$, and the form of α is a defect that needs to be corrected to move into the normal form.

1. To move negations to occur only in the context $\bigcirc$, we note:
   - $\neg \mathbb{F} X \alpha(X) \Rightarrow \mathbb{F} X \neg \alpha(\neg X)$ (see fig 1);
   - $\neg(\alpha ? \beta : \gamma) \Rightarrow (\alpha ? \neg\beta : \neg\gamma)$;
   - $(\neg\alpha ? \beta : \gamma) \Rightarrow (\alpha ? \gamma : \beta)$;
   - $(\alpha ? \neg A_i : \beta) \Rightarrow (\alpha ? (A_i ? \bigcirc : \bullet) : \beta)$.

   These can all be checked with basic algebraic reductions.
2. To ensure the internal branching nodes are only free variables or instances of $◐$, we apply the following transformations:
   - $((\alpha ? \beta_1 : \beta_2) ? \gamma_1 : \gamma_2) \Rightarrow (\alpha ? (\beta_1 ? \gamma_1 : \gamma_2) : (\beta_2 ? \gamma_1 : \gamma_2))$, when β_1 and β_2 are not bound variables;
   - $((\alpha ? X : \beta) ? \gamma_1 : \gamma_2) \Rightarrow (\alpha ? X : (\beta ? \gamma_1 : \gamma_2))$;
   - $((\alpha ? \beta : X) ? \gamma_1 : \gamma_2) \Rightarrow (\alpha ? (\beta ? \gamma_1 : \gamma_2) : X)$;
   - $(\mathbb{F} X \alpha(X) ? \beta : \gamma) \Rightarrow \mathbb{F} X (\alpha ? \beta : \gamma)$, under the assumption that X does not appear free in β or γ (or is renamed to a fresh variable if it does).
3. The previous two transformations are sufficient to give a tree structure, where subformulas $(\alpha ? \beta : \gamma)$ are such that $\alpha = A_i$ or $\alpha = ◐$. The next defect to address is fixed points appearing anywhere other than the root. To address this we apply the transformations:
   - $(\alpha ? \mathbb{F} X \beta : \gamma) \Rightarrow \mathbb{F} X (\alpha ? \beta : \beta[\bullet, \bigcirc \backslash \gamma])$;
   - $(\alpha ? \beta : \mathbb{F} X \gamma) \Rightarrow \mathbb{F} X (\alpha ? \gamma[\bullet, \bigcirc \backslash \beta] : \gamma)$.

   The idea of this transformation is to move the fixed point operator to the root of the conditional statement, so that when X is encountered (i.e. α was heads, and the evaluation of β was X), the entire statement is re-evaluated from the root. This would make favour the branch containing γ, so that branch is similarly scaled by including the evaluation of β, but with every occurrence of $\bullet$ or $\bigcirc$ replaced by γ. Effectively, this ensures that both branches are equally likely to be reevaluated from the root, so no branch is advantaged. To show this is the case, we apply the function semantics given in Definition 5:

$$\begin{aligned}(\mathbb{F}X(\alpha?\beta:\beta[\bullet,\bigcirc\backslash\gamma]))^{\mathcal{I}} &= \frac{h_\alpha\cdot h_\beta+(1-h_\alpha)\cdot(1-i^X_\beta)\cdot h_\gamma}{1-(h_\alpha\cdot i^X_\beta+(1-h_\alpha)\cdot i^X_\beta)}\\ &= \frac{h_\alpha\cdot h_\beta+(1-h_\alpha)\cdot(1-i^X_\beta)\cdot h_\gamma}{1-h_\alpha\cdot i^X_\beta-i^X_\beta+h_\alpha\cdot i^X_\beta}\\ &= \frac{h_\alpha\cdot h_\beta+(1-h_\alpha)\cdot(1-i^X_\beta)\cdot h_\gamma}{1-i^X_\beta}\\ &= h_\alpha\cdot\frac{h_\beta}{1-i^X_\beta}+(1-h_\alpha)\cdot h_\gamma\\ &= (\alpha?\mathbb{F}X\beta:\gamma)^{\mathcal{I}}\end{aligned}$$

The second reduction is shown in a similar manner. Note that the first line of the derivation used the property that $(\beta[\bullet,\bigcirc\backslash\gamma])^{\mathcal{I}}=(1-i^X_\beta)\cdot h_\gamma$, which assumes that all defects have been already removed from β.

4. Having enforced a tree structure, and moved all fixed point operators to the root, the next defect to address is to ensure that, for all conditional statements, $(\alpha?\beta:\gamma)$, either $\alpha=◐$ or one of β or γ is X_0. To do this, given $\alpha=A_i$ we can apply the following transformation:

$$(A_i?\beta:\gamma)\Rightarrow\mathbb{F}X_0(◐?(A_i?\beta:X_0):(A_i?X_0:\gamma)).$$

This transformation uses a fair coin flip a fixed point operators to turn the conditional statement into a series of independent tests. Essentially, a fair coin is flipped to see whether we test the case where A_i is heads or the case where A_i is tails. In each instance we first test if the hypothesis is right (e.g. A_i is heads) and if it is not, we repeat the process. If A_i is heads, we then continue to test β, and similarly for when the fair coin lands tails, we apply a similar process to test, given that A_i is tails, γ. Using the notation of Lemma 1 we have, when $\psi=(◐?(A_i?\beta:X_0):(A_i?X_0:\gamma))$, $h_\psi=(h_\alpha\cdot h_\beta+(1-h_\alpha)\cdot h_\gamma)/2$, and $i^{X_0}_\psi=1/2$. As $h_{\mathbb{F}X\psi}=h_\psi/(1-i_\psi)$ the result follows.

5. To order the free variables we use the identities, for all interpretations, $\mathcal{I}$:

$$(\alpha?(\beta?\gamma_1:\gamma_2):(\beta?\delta_1:\delta_2))^{\mathcal{I}}=(\beta?(\alpha?\gamma_1:\delta_1):(\alpha?\gamma_2:\delta_2))^{\mathcal{I}} \quad (2)$$

$$\alpha=(\beta?\alpha:\alpha) \quad (3)$$

These identities are proven and discussed in [2]. Using these identities we can apply the following transformations:

$$(A_j?(A_k?\beta_1:\beta_2):\gamma)\Rightarrow(A_k?(A_j?\beta_1:\gamma):(A_j?\beta_2:\gamma)) \quad \text{when } k>j \quad (4)$$

$$(A_j?\gamma:(A_k?\beta_1:\beta_2))\Rightarrow(A_k?(A_j?\gamma:\beta_1):(A_j?\gamma:\beta_2)) \quad \text{when } k>j \quad (5)$$

$$(A_j?(◐?\beta_1:\beta_2):\gamma)\Rightarrow(◐?(A_j?\beta_1:\gamma):(A_j?\beta_2:\gamma)) \quad (6)$$

$$(A_j?\gamma:(◐?\beta_1:\beta_2))\Rightarrow(◐?(A_j?\gamma:\beta_1):(A_j?\gamma:\beta_2)) \quad (7)$$

$$(A_j?\beta:\gamma)\Rightarrow(A_j?(A_j?\beta:\gamma):(A_j?\beta:\gamma)) \quad (8)$$

This allows us to organise the tree representation of Figure 3 so that all paths from the leaves to the root go through exactly k instances of each free variable in order, and then the nodes labelled by ◐.

6. Finally, we combine the fixed point operators into one, noting that they have all moved to the root, and we can apply the transformation $\mathbb{F}X\mathbb{F}Y\alpha(X,Y) \Rightarrow \mathbb{F}X\alpha(X,X)$.

These transformations can be applied repeatedly until the formula is in k-block normal form. As each transformation can be shown to preserve the interpretation of the formula, this completes the proof.

From this formula we are able to define the notion of an aleatoric function:

Definition 8. *Suppose that α is a formula in k-block normal form, defined over the free atomic propositions* $\mathrm{var}(\alpha) = \mathcal{X} = \{A_1, ..., A_n\}$. *Let the functions $h_\alpha(A_1, ..., A_n)$ and $i_\alpha(A_1, ..., A_n)$ be defined as follows:*

$$
\begin{array}{rlrl}
h_{\bullet}(\mathcal{X}) &= 1 & i_{\bullet}(\mathcal{X}) &= 0\\
h_{\circ}(\mathcal{X}) &= 0 & i_{\circ}(\mathcal{X}) &= 0\\
h_{A_j}(\mathcal{X}) &= A_j & i_{A_j}(\mathcal{X}) &= 0\\
h_{X_0}(\mathcal{X}) &= 0 & i_{X_0}(\mathcal{X}) &= 1\\
h_{(A_j?\alpha\beta)}\mathcal{X} &= A_j \cdot h_\alpha + (1-A_j)\cdot h_\beta & i_{(A_j?\alpha\beta)}\mathcal{X} &= A_j \cdot i_\alpha + (1-A_j)\cdot i_\beta\\
h_{(◐?\alpha\beta)}\mathcal{X} &= (1/2)\cdot h_\alpha + (1/2)\cdot h_\beta & i_{(◐?\alpha\beta)}\mathcal{X} &= (1/2)\cdot i_\alpha + (1/2)\cdot i_\beta\\
h_{\mathbb{F}X_0\alpha}(\mathcal{X}) &= h_\alpha(\mathcal{X})/(1-i_\alpha(\mathcal{X})) & &
\end{array}
$$

Given any proposition $\alpha \in \mathcal{L}$ where k is the maximum number of times a single atomic proposition appears in α, the aleatoric function of α *is the function $f_\alpha(\mathcal{X}) = h_{\tau(\alpha)}(\mathcal{X})$, where $\tau(\alpha)$ is the k-block normal form reduction of α.*

The following corollary is just a special case of Corollary 1

Corollary 2. *Given any aleatoric proposition $\alpha \in \mathcal{L}$ and any aleatoric interpretation $\mathcal{I}$:*

$$\mathcal{I}(\alpha) = f_\alpha(1, \mathcal{I}(X_1), ..., \mathcal{I}(X_n)).$$

3.2 Positive rational functions

Here, we show that every rational function from $(0,1)^{\mathcal{X}}$ to $(0,1)$ is equivalent to some aleatoric proposition defined of the atomic propositions $\mathcal{X}$. This subsection follows the analysis of coin flipping polynomials by Mossel and Peres [12]. Particularly, the following Lemma is based on Lemma 2.7 of [12].

Lemma 3. *Let $f : (0,1)^{\mathcal{X}} \longrightarrow (0,1)$ be a rational function. Then there exists polynomials ℓ and m:*

$$
\begin{aligned}
\ell(\mathcal{X}) &= \textstyle\sum_{a\in\rho^k_{\mathcal{X}}} \ell_a \prod_{x\in\mathcal{X}} x^{a_{(x,+)}} \cdot (1-x)^{a_{(x,-)}}\\
m(\mathcal{X}) &= \textstyle\sum_{a\in\rho^k_{\mathcal{X}}} m_a \prod_{x\in\mathcal{X}} x^{a_{(x,+)}} \cdot (1-x)^{a_{(x,-)}}
\end{aligned}
$$

where $\rho^k_{\mathcal{X}} = \{\overline{a} \in \{0, ..., k\}^{\mathcal{X}\times\{+,-\}} \mid \sum_{x\in\mathcal{X}} a_{(x,+)} + a_{(x,-)} = k\}$, for all $a \in \rho^k_{\mathcal{X}}$, ℓ_a and m_a are integers such that $\ell_a < m_a$ and $f(\mathcal{X}) = \ell(\mathcal{X})/m(\mathcal{X})$.

Proof. As $f(\mathcal{X})$ is a rational function over $(0,1)$ we may assume that it may be written $L(\mathcal{X})/M(\mathcal{X})$, where L and M are relatively prime polynomials with integer coefficients. We may therefore write $L(\mathcal{X}) = \sum_{a\in A} L_a \cdot \prod_{x\in\mathcal{X}} x^{a_x}$ and $M(\mathcal{X}) = \sum_{b\in B} M_b \cdot \prod_{x\in\mathcal{X}} x^{b_x}$, where $A, B \subset \{0,...,k\}^{\mathcal{X}}$, for some k. We may define homogeneous polynomials of degree $k \cdot |\mathcal{X}|$, by defining

$$L'(\mathcal{X},\mathcal{X}') = \sum_{a\in A} L_a \prod_{x\in\mathcal{X}} x^{a_x} \cdot (x+x')^{k-a_x}$$

and

$$M'(\mathcal{X},\mathcal{X}') = \sum_{b\in B} M_b \prod_{x\in\mathcal{X}} x^{b_x} \cdot (x+x')^{k-b_x}$$

so that $L'(\mathcal{X}, 1-\mathcal{X}) = L(\mathcal{X})$ and $M'(\mathcal{X}, 1-\mathcal{X}) = M(\mathcal{X})$.

Then $L'(\mathcal{X},\mathcal{X}')$ and $M'(\mathcal{X},\mathcal{X}')$ can be written as, respectively

$$L'(\mathcal{Y}) = \sum_{c\in C} L'_c \prod_{y\in\mathcal{Y}} y^{c_y} \quad \text{and} \quad M'(\mathcal{Y}) = \sum_{c\in C} M'_c \prod_{y\in\mathcal{Y}} y^{c_y},$$

where $\mathcal{Y} = \mathcal{X} \cup \mathcal{X}$ and $C = \{c \in \{0,...,k\}^{\mathcal{Y}} \mid \sum_{y\in\mathcal{Y}} c_y = k\cdot|\mathcal{X}|\}$. We note that $L'(\mathcal{Y})$, $M'(\mathcal{Y})$ and $M'(\mathcal{Y}) - L'(\mathcal{Y})$ are all homogeneous positive polynomials.

From Pólya [6] we have the following result:

> Given $f : (0,1)^{\mathcal{Y}} \longrightarrow (0,1)$, a homogeneous and positive polynomial, for sufficiently large n, all the coefficients of $(\sum_{y\in\mathcal{Y}} y)^n \cdot f(\mathcal{Y})$ are positive.

It follows that for some n,

$$(\sum_{y\in\mathcal{Y}} y)^n \cdot L'(\mathcal{Y}),\ (\sum_{y\in\mathcal{Y}} y)^n \cdot M'(\mathcal{Y}),\ \text{and}\ (\sum_{y\in\mathcal{Y}} y)^n \cdot (M'(\mathcal{Y}) - L'(\mathcal{Y}))$$

all have positive coefficients. The result follows from the observation that

$$f(\mathcal{X}) = L'(\mathcal{X}, 1-\mathcal{X})/M'(\mathcal{X}, 1-\mathcal{X}).$$

The correspondence now follows from the functional representation of aleatoric propositions in block normal form.

Theorem 1. *1. For every aleatoric proposition $\alpha \in \mathcal{L}$ defined over the free variables in $\mathcal{X}$, $f_\alpha(\mathcal{X})$ is a rational function from $(0,1)$ to $(0,1)$.*
2. For every rational function $f(\mathcal{X})$ from $(0,1)^{\mathcal{X}}$ to $(0,1)$, there is some aleatoric proposition α such that $f_\alpha(\mathcal{X}) = f(\mathcal{X})$.

The first part is immediate from the Definition 8, and the second part follows by noting the form of the polynomials $\ell(\mathcal{X})$ and $m(\mathcal{X})$ in the proof of Lemma 3 agrees with the numerator and denominator of f_α in Definition 8 Particularly, noting the form of the tree in Figure 3, constructing a formula for a given $\ell(\mathcal{X})$ and $m(\mathcal{X})$ can be thought of as labelling the leaves of the tree so that

1. exactlyℓ_a a-paths have the leaf labelled with ● for each $a \in \rho^k_{\mathcal{X}}$,
2. exactly $m_a - \ell_a$ a-paths have the leaf labelled with ○ for each $a \in \rho^k_{\mathcal{X}}$,
3. and all other paths are labelled with X_0,

where an a-path is a branch of the tree with exactly $a_{(x,+)}$ positive instances of x, for each $x \in \mathcal{X}$.

4 Conclusion and future work

This paper has given a description of aleatoric propositions, extending the work of [2], introducing the fixed point operator $\mathbb{F}X\alpha$, and establishing a correspondence with rational functions over $(0, 1)$. While this correspondence is based on earlier work in [12], the presentation as a logical system is novel, and this provides a foundation for future work on the logical aspects of this approach.

Future work will, taking aleatoric propositions as a base, extend the formalism to first order concepts, or aleatoric predicates. In this setting, we suppose that there is a probability space of domain elements, and a set of boolean predicates given over these domain elements. An *expectation* operator allows us to express the expectation a proposition will be true when an element is drawn randomly from the domain. The analogy is an urn of marbles, where the marbles are labelled and predicates are defined over those labels.

We will also consider axiomatisations of these logics, the satisfiability problem, and combinations with modal necessity operators.

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