Bisimulations between Verbrugge models and Veltman models

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Abstract. Veltman semantics is the basic Kripke-like semantics for interpretability logic. Verbrugge semantics is a generalization of Veltman semantics. An appropriate notion of bisimulation between a Verbrugge model and a Veltman model is developed in this paper. We show that a given Verbrugge model can be transformed into a bisimilar Veltman model.

Keywords: Modal logic · Interpretability logic · Bisimulations.

1 Introduction

Interpretability logic is an extension of provability logic, which formalizes the notion of relative interpretability between arithmetical first-order theories. Intuitively, we say that such a theory T interprets another theory T' if there is a translation from the language of T' to the language of T such that all translations of axioms of T' are provable in T. Intepretability logic is a modal logic which, together with usual unary modality $\Box$, whose intended interpretation in this context is provability, has another modality $\rhd$, which is binary. Formulas of the form $A \rhd B$ are intended to mean that some base theory T extended with the formula A interprets the theory obtained by extending the same base theory T with the formula B. In this paper we will only deal with modal semantics of interpretability logic in general, so we omit overviewing axiomatic systems of interpretability logic (cf. e.g. [9] for this, and also for more details on arithmetical aspects).

The basic semantics of interpretability logic is defined on Veltman models, Kripke-like structures built over standard Kripke models of provability logic, which means that the accessibility relation is transitive and converse well founded, by adding a family of relations S_w between worlds R-accessible from w, for each world w in the model, satisfying certain properties, e.g. reflexivity and transitivity (a precise definition is given in the next section). Verbrugge semantics ([8], cf. also [1]) is a generalization in which relations S_w are no longer between worlds, but between worlds and sets of worlds. This semantics proved to be useful in showing some independence results which could not be proved using Veltman semantics ([11]), in proving some completeness results in cases of incompleteness w.r.t. Veltman semantics ([5], [2]), and it also enabled using filtration technique, which could not be used on Veltman models, in order to prove finite model property and consequently some decidability results ([6], [4]).

This paper addresses the following question: for a given Verbrugge model, can we obtain a Veltman model which would be closely related with the initial Verbrugge model, preferably by some appropriately defined notion of bisimulation, or at least by modal equivalence. This question is natural, since Veltman models are still appealing due to their relative simplicity, so we would like to keep them as the basic semantics and it would be nice to have a bridge by which we could possibly transfer some results from Verbrugge semantics back to Veltman semantics, or better understand why some of them cannot be transferred.

It is not surprising that this question was being addressed from the very beginning of work on Verbrugge semantics: already in [8] (cf. also [7]) a transformation of a given Verbrugge model to a modally equivalent Veltman model was provided, but using different notion of Verbrugge model than the one used in the present paper (different notions of Verbrugge models come from various possible ways to define so-called quasi-transitivity, a property of Verbrugge models which corresponds to transitivity of relations S_w in Veltman models). Similar attempt in [10] resulted in Veltman model bisimilar to a given Verbrugge model in a certain sense, but under additional conditions of image-finiteness and inverse image-finiteness. Also, this was an indirect result: bisimilarity was observed between the same kind of structures, after a transformation. In the present paper we will work with a directly defined notion of bisimulation between different kinds of structures, namely between a Verbrugge model and a Veltman model, and we will be able to avoid additional constraints on these structures.

In Section 2 we recall basic definitions and we define the notion of bisimulation between a Verbrugge model and a Veltman model. In Section 3 we provide arguments in favour of thus defined notion, e.g. we prove an analogue of Hennessy-Milner theorem. In Section 4 we obtain a Veltman model bisimilar to a given Verbrugge model. In Section 5 we conclude with some remarks on future work.

2 Bisimulation between Verbrugge and Veltman model

The alphabet of interpretability logic consists of countably many propositional variables and symbols $\bot$, $\to$ and $\rhd$. Formulas are given by

$$\varphi ::= p \,|\, \bot \,|\, \varphi_1 \to \varphi_2 \,|\, \varphi_1 \rhd \varphi_2,$$

where p ranges over the set of propositional variables. We use usual abbreviations $\top := \neg\bot$, $\neg\varphi := \varphi \to \bot$, $\varphi_1 \vee \varphi_2 := \neg\varphi_1 \to \varphi_2$, $\varphi_1 \wedge \varphi_2 := \neg(\neg\varphi_1 \vee \neg\varphi_2)$, $\varphi_1 \leftrightarrow \varphi_2 := (\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)$, $\Box\varphi := \neg\varphi \rhd \bot$, $\Diamond\varphi := \neg\Box\neg\varphi$.

A *Veltman model* is a tuple $(W, R, \{S_w : w \in W\}, \Vdash)$ such that:

- $W \neq \emptyset$ is a set called the *domain*, whose elements are called *worlds*
- $R \subseteq W \times W$ is a transitive and converse well-founded relation called the *accessibility relation*

- $S_w \subseteq R[w] \times R[w]$, where $R[w] = \{u \in W : wRu\}$, is a reflexive and transitive relation such that $wRuRv$ always implies uS_wv, for each $w \in W$
- $\Vdash$ is a relation between worlds and formulas such that for all $w \in W$ we have $w \nVdash \bot$, $w \Vdash \varphi_1 \to \varphi_2$ if and only if $w \nVdash \varphi_1$ or $w \Vdash \varphi_2$, and $w \Vdash \varphi_1 \rhd \varphi_2$ if and only of for all $u \in W$ such that wRu and $u \Vdash \varphi_1$ there is $v \in W$ such that uS_wv and $v \Vdash \varphi_2$ [1]

A *Verbrugge model* is a tuple $(W, R, \{S_w : w \in W\}, \Vdash)$, where W and R are as in the definition of Veltman models, while $S_w \subseteq R[w] \times (\mathcal{P}(R[w]) \setminus \{\emptyset\})$ such that:

- if wRu, then $uS_w\{u\}$ (*quasi-reflexivity*)
- if uS_wV and vS_wZ_v for all $v \in V$, then $uS_w \bigcup_{v \in V} Z_v$ (*quasi-transitivity*)
- if $wRuRv$, then $uS_w\{v\}$
- if uS_wV and $V \subseteq Z \subseteq R[w]$, then uS_wZ (*monotonicity*),

while $\Vdash$ is defined similarly as in the definition of Veltman model, except the following: $w \Vdash \varphi_1 \rhd \varphi_2$ if and only if for all u such that wRu and $u \Vdash \varphi_1$ there is V such that uS_wV and for all $v \in V$ we have $v \Vdash \varphi_2$ (we write $V \Vdash \varphi_2$).

When we need to emphasize that $w \Vdash \varphi$ is observed in the context of a structure $\mathfrak{M}$, we will write $\mathfrak{M}, w \Vdash \varphi$.

As aforementioned, there are other variants of Verbrugge models in the literature, which differ from the above one only in the definition of quasi-transitivity and in some cases in omitting monotonicity. In this paper we work only with the above definition, since it is predominant in the literature (cf. a recent overview [1], which includes a discussion on other possibilities).

Bisimulation is the basic equivalence between modal models. It has three defining conditions: atomic equivalence between related worlds (at), the condition describing how the first model is simulated in the second one (forth), and the condition describing how the second model is simulated in the first one (back). When we work with the same kind of structures, (forth) and (back) are mutually symmetric. But now we will define the notion of bisimulation between different kinds of structures, which will therefore lack this symmetry. In fact, the direction from Verbrugge model to Veltman model (forth) will be much more complex than the opposite one.

Definition 1. *Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be a Verbrugge model and let $\mathfrak{M}' = (W', R', \{S'_{w'} : w' \in W'\}, \Vdash)$ be a Veltman model. A* **bisimulation** *between $\mathfrak{M}$ and $\mathfrak{M}'$ is any non-empty relation $Z \subseteq W \times W'$ such that:*

(at) *$\mathfrak{M}, w \Vdash p$ if and only if $\mathfrak{M}', w' \Vdash p$ for all $w \in W, w' \in W'$ such that wZw', for each propositional variable p*

(forth) *if wZw' and wRu, then there exists a non-empty $U' \subseteq W'$ such that $w'R'u'$ and uZu' for all $u' \in U'$ and for any $F : U' \to W'$ such that $u'S'_{w'}F(u')$ for all $u' \in U'$, there is V such that uS_wV and for all $v \in V$ there is $u' \in U'$ such that $vZF(u')$*

[1] Equivalently, we can define a Veltman model to be $(W, R, \{S_w : w \in W\}, V)$, where V maps each propositional variable to a subset of W, and then define satisfaction relation $\Vdash$ recursively, but this is non-essential and just a matter of style.

(back) *if wZw' and $w'R'u'$, then there exists $u \in W$ such that wRu, uZu' and for each $V \subseteq W$ such that uS_wV there are $v \in V$ and $v' \in W'$ such that $u'S'_{w'}v'$ and vZv'.*[2]

Consider an example of thus defined bisimulation.

Example 1. Consider a Verbrugge model $\mathfrak{M}$ such that:

- $W = \{0, 1, 2, 3\}$, $R = \{(0, 1), (0, 2), (0, 3)\}$, $1S_0\{2, 3\}$
- $1 \Vdash p$, $2 \Vdash q$, $3 \Vdash r$

Now, consider a Veltman model $\mathfrak{M}'$ as follows:

- $W' = \{0', 1', 1'', 2', 3'\}$, $R' = \{(0', 1'), (0', 1''), (0', 2'), (0', 3')\}$, $1'S'_{0'}2'$, $1''S'_{0'}3'$
- $1' \Vdash p$, $1'' \Vdash p$, $2' \Vdash q$, $3' \Vdash r$

Note that we omitted some pairs in S_0 and $S'_{0'}$, namely those enforced by (quasi)-reflexivity and monotonicity.

It is easy to verify that $Z = \{(0, 0'), (1, 1'), (1, 1''), (2, 2'), (3, 3')\}$ is a bisimulation.

The following proposition shows that the necessary requirement on any notion of bisimulation is satisfied: bisimilar worlds are modally equivalent.

Proposition 1. *Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be a Verbrugge model, $\mathfrak{M}' = (W', R', \{S'_{w'} : w' \in W'\}, \Vdash)$ a Veltman model and $Z \subseteq W \times W'$ a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. Then for all w, w' such that wZw' we have that w and w' are modally equivalent, i.e. $\mathfrak{M}, w \Vdash \varphi$ if and only if $\mathfrak{M}', w' \Vdash \varphi$, for each formula φ.*

Proof. The claim is proved by induction on the complexity of a formula. We only present the inductive step in the case of a formula of the form $\varphi_1 \rhd \varphi_2$.

Assume $\mathfrak{M}, w \Vdash \varphi_1 \rhd \varphi_2$ and wZw'. We need to prove $\mathfrak{M}', w' \Vdash \varphi_1 \rhd \varphi_2$. Let $u' \in W'$ such that $w'R'u'$ and $u' \Vdash \varphi_1$. Then (back) implies there is u such that wRu and uZu'. By the induction hypothesis $u \Vdash \varphi_1$. Since $w \Vdash \varphi_1 \rhd \varphi_2$, there is V such that uS_wV and $V \Vdash \varphi_2$. But (back) also implies that for any

[2] As pointed out by a reviewer, one could alternatively generalize these conditions to develop an analogous notion of bisimulation between Verbrugge models, and then establish a connection between a Verbrugge and a Veltman model by composing a bisimulation between Verbrugge models and a simple transformation from a Veltman to a Verbrugge model described at the beginning of Section 4. The present approach has an advantage that the already complex (forth) condition has an additional quantifier in case of two Verbrugge models, and (back) condition would be symmetric to (forth) when observed between two Verbrugge models, while in the present paper it is much simpler. Nevertheless, an analogous notion of bisimulation between Verbrugge models is of independent interest and is thoroughly studied in a near future paper [3].

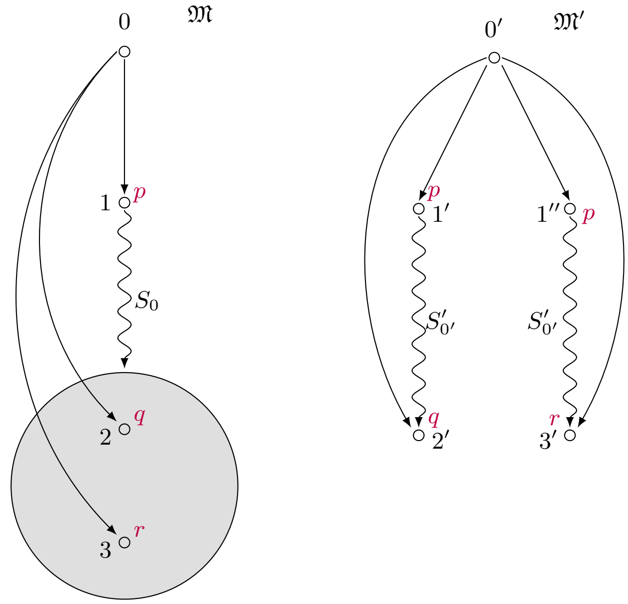

Fig. 1. Illustration of Example 1

S_w-successor of u, thus also for V, there are $v \in V$ and $v' \in W'$ such that vZv' and $u'S_{w'}v'$. Again by the induction hypothesis $v' \Vdash \varphi_2$, as desired.

Conversely, assume $\mathfrak{M}', w' \Vdash \varphi_1 \rhd \varphi_2$ and wZw' and prove $\mathfrak{M}, w \Vdash \varphi_1 \rhd \varphi_2$. Let $u \in W$ such that wRu and $u \Vdash \varphi_1$. Then by (forth) there is $U' \neq \emptyset$ such that $w'R'u'$ and uZu', and thus by the induction hypothesis $u' \Vdash \varphi_1$, for all $u' \in U'$, such that for any choice of one $S'_{w'}$-successor for each world in U', there is V such that uS_wV and each world in V is bisimilar to some of those $S'_{w'}$-successors. Now for all $u' \in U'$, since $w' \Vdash \varphi_1 \rhd \varphi_2$ and $u' \Vdash \varphi_1$, there is v' such that $u'S_{w'}v'$ and $v' \Vdash \varphi_2$. For such a world v', put $F(u') = v'$. Since the above holds for any choice F of one $S'_{w'}$-successor for each $u' \in U'$, it holds in particular for the choice F. Thus there is V such that uS_wV and each $v \in V$ is bisimilar to some $F(u')$, so by the induction hypothesis $v \Vdash \varphi_2$ for all $v \in V$, as desired. ∎

Example 2. Since Z defined in Example 1 is a bisimulation, the previous proposition implies that 0 and $0'$ are modally equivalent (as are all pairs in Z).

3 Hennessy-Milner theorem

As aforementioned, the first requirement of any notion of bisimulation is that it implies modal equivalence. That requirement shows that the definition is not too weak, i.e. structural relation between two models is strong enough to ensure modal formulas cannot distinguish them. But on the other hand, some proposed relation between models can be too strong. For example, isomorphism of course implies modal equivalence, but it is obviously unnecessarily strong, i.e. much weaker structural relations can imply modal equivalence. In other words, one would like to have a converse of the previous proposition, to see that the defined notion of bisimulation is just enough strong. Unfortunately, it is well known that the direct converse never holds (a counterexample can easily be constructed from some of the known counterexamples for basic modal logic). But, there are some approximations, notably Hennessy-Milner-like theorems, which say that the converse holds in case of image-finite models. If Hennessy-Milner analogue holds for some proposed notion, then it is a good sign that the notion is just about as strong as bisimulation should be.

Theorem 1. *Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be a Verbrugge model and let $\mathfrak{M}' = (W', R', \{S'_{w'} : w' \in W'\}, \Vdash)$ be a Veltman model such that relations R and R' are image-finite, i.e. for all $w \in W, w' \in W'$ we have that $R[w]$ and $R'[w']$ are finite.*

Then any $w \in W$ and $w' \in W'$ are modally equivalent if and only if there is a bisimulation Z between $\mathfrak{M}$ and $\mathfrak{M}'$ such that wZw'.

Proof. Let $Z \subseteq W \times W'$ be the modal equivalence between worlds of $\mathfrak{M}$ and $\mathfrak{M}'$, i.e. wZw' if and only if w and w' satisfy exactly the same formulas. We will prove that Z is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. Together with the previous proposition, this clearly implies the claim.

Obviously (at) holds. Assume (back) does not hold, i.e. there are w, w', u' such that wZw' and $w'R'u'$ and for all u such that wRu and uZu' there is V such that uS_wV and for all $v \in V$ and v' such that $u'S_{w'}v'$ we have that v and v' are not modally equivalent.

For any x such that wRx which is not modally equivalent to u' there is a formula φ_x such that $u' \Vdash \varphi_x$ and $x \nVdash \varphi_x$. Since there are only finitely many such worlds x, there is a finite conjunction φ of one such formula for each x, so $u' \Vdash \varphi$. Observe now that for any u such that wRu we have uZu' if and only if $u \Vdash \varphi$.

Now, let $u \in W$ such that wRu and uZu'. By the assumption, there is $V_u \subseteq R[w]$ such that uS_wV_u and no $v \in V_u$ is modally equivalent to any v' such that $u'S_{w'}v'$. For each $y \in R[w]$ which is not modally equivalent to any v' such that $u'S_{w'}v'$, and for each such v', there is a formula $\psi_{y,v'}$ which is satisfied at y, but not at v'. Since S_w-successors of u' are R-successors of w, there are only finitely many of them, so there is a finite conjunction ψ_y of one such formula for each v', and we have $y \Vdash \psi_y$ and $v' \nVdash \psi_y$. But now we clearly have $V_u \Vdash \psi$, where ψ is the disjunction of all ψ_y, where $y \in R[w]$ is not modally equivalent to u'. Hence, $w \Vdash \varphi \rhd \psi$. Since wZw', we have $w' \Vdash \varphi \rhd \psi$. Since $u' \Vdash \varphi$, there is v' such that $u'S_{w'}v'$ and $v' \Vdash \psi$, so $v' \Vdash \psi_y$ for some y and thus $v' \Vdash \psi_{y,v'}$, which is a contradiction.

It remains to prove (forth). Assume it does not hold, i.e. there are w, w', u such that wZw', wRu and for any $U' \neq \emptyset$ such that $w'R'u'$ and uZu' for all $u' \in U'$, there is a choice $F : U' \to W'$ of one $S'_{w'}$-successor for each $u' \in U'$

such that for all V such that uS_wV there is $v \in V$ not equivalent to $F(u')$ for any $u' \in U'$.

In particular, this holds if we take U' to be the set of all $u' \in R'[w']$ such that uZu'. Further in this proof U' will denote that set.

Similarly as in the proof of (back), we can show that there is a formula φ such that $u \Vdash \varphi$ and for all $u' \in R'[w']$ we have uZu' if and only if $u' \Vdash \varphi$. Furthermore, for any $u' \in U'$ and any V such that uS_wV, there is a formula $\psi_{u',V}$ which is satisfied at $F(u')$, but not at some $v \in V$. For each $u' \in U'$, let $\psi_{u'}$ be the conjunction of all $\psi_{u',V}$, ranging over all V such that uS_wV. Again, this is clearly a finite conjunction, and we have $F(u') \Vdash \psi_{u'}$.

Let ψ be the disjunction of all $\psi_{u'}$, where $u' \in U'$. Clearly $w' \Vdash \varphi \rhd \psi$. Now wZw' implies $w \Vdash \varphi \rhd \psi$. Since $u \Vdash \varphi$, there is V such that uS_wV and $V \Vdash \psi$. Hence, $V \Vdash \psi_{u'}$ for some $u' \in U'$. But then $V \Vdash \psi_{u',V}$, which is a contradiction, since there is $v \in V$ not satisfying $\psi_{u',V}$. ■

The reader may be surprised to see that, in the definition of bisimulations between Verbrugge models and Veltman models, the condition (forth) demands the existence of a set of worlds U' instead of just the existence of at least one world as usual. But without this, the notion of bisimulation would not be useful. To see this, note that a seemingly more natural (forth) would demand: if wZw' and wRu, then there is u' such that $w'R'u'$, uZu' and for all v' such that $u'S'_{w'}v'$ there is V such that uS_wV and vZv' for all $v \in V$. It is easily checked that this does imply modal equivalence, but nevertheless it is too restrictive, since it has a consequence that all worlds in V are mutually modally equivalent, which practically collapses Verbrugge semantics to Veltman semantics.

The following example illustrates why we need (forth) to be as complex as it is, and also provides an idea how to proceed with the main goal of the paper: find a bisimilar Veltman model for a given Verbrugge model.

Example 3. To illustrate usefulness of seemingly too complicated (forth), consider again the bisimulation Z defined in Example 1. Let us consider just one part of the verification that Z is a bisimulation, namely (forth) for $0R1$ and $0Z0'$. Then the good choice for U' is $\{1', 1''\}$. Then e.g. for F defined by $F(1') = 2'$, $F(1'') = 3'$, we have $1S_0\{2, 3\}$, $2Z2'$, $3Z3'$.

With the aforementioned more restrictive definition of bisimulation, we would not have a bisimulation in this example, thus we can use it as a counterexample for Hennessy-Milner analogue in that case. Namely, in the situation illustrated above, for $0R1$ and $0Z0'$ the restrictive (forth) would force us to choose just one R'-successor of $0'$ bisimilar to 1. Then for both possible choices we would not be able to satisfy the remaining requirement of (forth), e.g. for $1'$ and its $S'_{0'}$-successor $2'$, there is no V such that $1S_0V$ and all elements of V are bisimilar to $2'$.

4 Obtaining a Veltman model bisimilar to a given Verbrugge model

It is straightforward to obtain a Verbrugge model from a given Veltman model $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$: we use the same W and R, and define $uS'_w V$ if and only if $uS_w v$ for some $v \in V$. It is very easy to see that $Z = \{(w, w) : w \in W\}$ is a bisimulation between thus obtained Verbrugge model and $\mathfrak{M}$.

Although it is very simple, our running example already illustrates that the opposite direction is much more involved. The basic idea is that each world from a given Verbrugge model will have multiple copies in the associated Veltman model, to make it possible for S_w-connections with sets of worlds to be simulated by connections with worlds which are representatives of these sets.

So, let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be a Verbrugge model. We will define a Veltman model associated with $\mathfrak{M}$, which we will denote by $Vel(\mathfrak{M}) = (W', R', \{S'_{w'} : w' \in W'\}, \Vdash)$.

First we introduce some notation and terminology.

4.1 $\overline{S_w}$-paths

When we consider subsets of W, it will often be essential that they are represented as certain unions. We will keep track of such information in the following way: instead of some $X \subseteq W$, we will consider a family $\overline{X} = \{X_i : i \in I\}$ such that $X = \bigcup_{i \in I} X_i$. We will use the notation $\overline{X}$ for the sake of simplicity, although, of course, $\overline{X}$ is not uniquely determined by X. It will, however, always be clear form the context what we mean by $\overline{X}$.

For non-empty $U, V \subseteq W$ such that $U = \bigcup_{i \in I} U_i$ and $V = \bigcup_{u \in U} V_u$, we write $\overline{U}\,\overline{S_w}\,\overline{V}$ if $uS_w V_u$ for all $u \in U$. Observe that the quasi-transitivity can now be expressed in the following way: if $uS_w V$ and $\overline{V}\,\overline{S_w}\,\overline{Z}$, then $uS_w Z$.

Observe also that $uS_w V$ is equivalent to $\overline{\{u\}}\,\overline{S_w}\,\overline{V}$, where $\overline{\{u\}} = \{\{u\}\}$ and $\overline{V} = \{V\}$ are singleton families, which is of course more complicated notation, but useful for considering some sequences as $\overline{S_w}$-paths, as follows.

Definition 2. *Consider a finite path of the form* $\overline{\{u\}}\,\overline{S_w}\,\overline{V}_1\,\overline{S_w}\,\overline{V_2 S_w} \ldots \overline{S_w}\,\overline{V_k}$. *We call the sequence* $\overline{\{u\}}, \overline{V}_1, \overline{V_2}, \ldots, \overline{V_k}$ *an* $\overline{S_w}$**-path starting with** u*, or simply an* $\overline{S_w}$**-path** *if it is clear from the context what it starts with.*

In what follows, to avoid repeating all properties each time we mention such paths, and since we will not consider any other kind of paths, when we shortly say that something is an $\overline{S_w}$- path, we will always mean that it is a finite path starting from a singleton family which has a singleton set as its only element. If wRu, we will consider just $\overline{\{u\}}$ to be an $\overline{S_w}$-path (of length zero).

If $\overline{\{u\}}, \overline{V_1}, \overline{V_2}, \ldots, \overline{V_k}, \overline{V_{k+1}}, \ldots, \overline{V_{k+l}}$ is an $\overline{S_w}$-path, then for a given $v \in V_k$ we denote by $\overline{\{v\}}, \overline{V'_1}, \overline{V'_2}, \ldots, \overline{V'_l}$ the $\overline{S_w}$-path such that $\overline{V'_i} \subseteq \overline{V_{k+i}}$, $i = 1, 2, \ldots, l$, which is uniquely determined in the way that V'_1 is the element of the family $\overline{V_{k+1}}$ which is determined by v, i.e. obtained as an S_w-successor of v, and $\overline{V'_{i+1}}$ is the subfamily of $\overline{V_{k+i+1}}$ consisting exactly of elements determined by elements of V'_i, for $i = 1, 2, \ldots, l-1$. We say that thus obtained $\overline{S_w}$-path is *induced* by v.

4.2 Well defined choice of representatives

For an $\overline{S_w}$-path $\{u\}, \overline{V_1}, \overline{V_2}, \ldots, \overline{V_k}$, we say that $(u, v_1, v_2, \ldots, v_k)$ is a *well defined sequence of representatives* if $v_i \in V_i$ for all $i = 1, \ldots, k$, and v_{i+1} is a world from the element of the family $\overline{V_{i+1}}$ which is determined by v_i, i.e. obtained as an S_w-successor of v_i, for $i = 1, \ldots, k-1$.

Definition 3. *Let $w \in W$ and $x \in W$ such that xRw, and let f_x be a function which maps each $\overline{S_x}$-path starting with w to a well defined sequence of representatives (when needed, if $f_x(\overline{\{w\}}, \overline{V_1}, \ldots, \overline{V_n}) = (w, v_1, \ldots, v_n)$, we write $f_x(\overline{\{w\}}, \overline{V_1}, \ldots, \overline{V_n})_i = v_i$, $i = 1, \ldots, n$, and $f_x(\overline{\{w\}}, \overline{V_1}, \ldots, \overline{V_k})_0 = w$). Then we say that f_x is a* **well defined choice of representatives of $\overline{S_x}$-paths starting with** w*, if the following conditions hold:*

- *$f_x(\overline{\{w\}}, \overline{V_1}) = (w, v_1)$, where $v_1 \in V_1$ is arbitrarily chosen, for each $\overline{S_x}$-path of length 1*
- *if $f_x(\overline{\{w\}}, \overline{V_1}, \ldots, \overline{V_n})$ is defined for each $\overline{S_x}$-path of length n, then for each $\overline{S_x}$-put of length $n+1$ we have the following cases:*
  - *$f_x(\overline{\{w\}}, \overline{V_1}, \overline{V_2}, \ldots, \overline{V_{n+1}})_{i+1} = f_x(\overline{\{w\}}, \overline{V_2}, \ldots, \overline{V_{n+1}})_i$ for $i = 1, \ldots, n$, if $V_1 = \{u\}$ is singleton and wRu*
  - *$f_x(\overline{\{w\}}, \overline{V_1}, \ldots, \overline{V_{n+1}}) = (w, v_1, \ldots, v_{n+1})$, otherwise, where we have that $f_x(\overline{\{w\}}, \overline{V_1}, \ldots, \overline{V_n}) = (w, v_1, \ldots, v_n)$, and v_{n+1} is arbitrarily chosen so that $(w, v_1, \ldots, v_{n+1})$ is a well defined sequence of representatives and $f_x(\overline{\{w\}}, \overline{V_1}, \ldots, \overline{V_n}, \overline{V_{n+1}})_{n+1} = f_x(\overline{\{w\}}, \overline{V_1}, \ldots, \overline{V_n}, \overline{V_{n+1}}')_{n+1}$ whenever the element of the family $\overline{V_{n+1}}$ determined by v_n equals the element of the family $\overline{V_{n+1}}'$ determined by v_n*

4.3 A Veltman model associated with a given Verbrugge model

Now we are ready to define $Vel(\mathfrak{M})$ and to prove that it is indeed a Veltman model.

Definition 4. *Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be a Verbrugge model. By $Vel(\mathfrak{M}) = (W', R', \{S'_{w'} : w' \in W'\}, \Vdash)$ we denote a structure* **associated with** $\mathfrak{M}$*, defined as follows:*

- *W' consists of all ordered pairs (w, f), where $w \in W$, and f is a function which maps each $x \in W$ such that xRw to a function f_x which is a well defined choice of representatives of $\overline{S_x}$-paths starting with w*
- *$(w, f)R'(u, g)$ if and only if wRu and for all x such that xRw, for each $\overline{S_x}$-path $\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_k}$ (observe that then $\overline{\{w\}}, \overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_k}$ is an $\overline{S_x}$-path starting with w) we have*

$$f_x(\overline{\{w\}}, \overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_k})_{i+1} = g_x(\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_k})_i, \quad i = 1, \ldots, k$$

- *$(u, g)S'_{(w,f)}(v, h)$ if and only if $(w, f)R'(u, g)$, $(w, f)R'(v, h)$ and there is an $\overline{S_w}$-path $\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_k}$ such that $v = g_w(\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_k})_k$, for some $k \geqslant 0$,* *and for any continuation, i.e. for any $\overline{S_w}$-path $\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_k}, \overline{V_{k+1}}, \ldots, \overline{V_{k+l}}$ we have*

$$g_w(\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_{k+l}})_{k+i} = h_w(\overline{\{v\}}, \overline{V_1'}, \ldots, \overline{V_l'})_i, \quad i = 1, \ldots, l$$

where $\overline{S_w}$-path $\overline{\{v\}}, \overline{V_1'}, \ldots, \overline{V_l'}$ is induced by v

– $Vel(\mathfrak{M}), (w, f) \Vdash p$ *if and only if* $\mathfrak{M}, w \Vdash p$*, for all* $(w, f) \in W'$*, for each propositional variable* p

Proposition 2. *Let $\mathfrak{M}$ be a Verbrugge model. Then $Vel(\mathfrak{M})$ is a Veltman model.*

Proof. Obviously R' is converse well founded. To show that it is also transitive, let $(w, f)R'(u, g)R'(v, h)$. Then $wRuRv$ and therefore wRv. Furthermore, for an arbitrary x such that xRw and an arbitrary $\overline{S_x}$-path $\overline{\{v\}}, \overline{V_1}, \ldots, \overline{V_k}$ we have

$$h_x(\overline{\{v\}}, \overline{V_1}, \ldots, \overline{V_k})_i = g_x(\overline{\{u\}}, \overline{\{v\}}, \overline{V_1}, \ldots, \overline{V_k})_{i+1}$$

$$= f_x(\overline{\{w\}}, \overline{\{u\}}, \overline{\{v\}}, \overline{V_1}, \ldots, \overline{V_k})_{i+2} = f_x(\overline{\{w\}}, \overline{\{v\}}, \overline{V_1}, \ldots, \overline{V_k})_{i+1}, \quad i = 1, \ldots, k$$

(the last equality holds since f_x is a well defined choice of representatives).

Now we verify properties of the relation $S'_{(w,f)}$ for an arbitrary (w, f). The reflexivity trivially follows from the convention that $\overline{\{u\}}$ is considered to be an $\overline{S_w}$-path of length 0. To prove the transitivity, assume $(u, g)S'_{(w,f)}(v, h)S'_{(w,f)}(z, s)$. Since $(u, g)S'_{(w,f)}(v, h)$, there is an $\overline{S_w}$-path $\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_k}$ such that $v = v_k = g_w(\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_k})_k$ and other properties from the definition of the relation $S'_{(w,f)}$ hold. Also, since $(v, h)S'_{(w,f)}(z, s)$, there is an $\overline{S_w}$-path $\overline{\{v\}}, \overline{V_1'}, \ldots, \overline{V_l'}$ such that $z = z_l = h_w(\overline{\{v\}}, \overline{V_1'}, \ldots, \overline{V_l'})_l$, with other properties from the definition of $S'_{(w,f)}$.

Put $V_{k+j} = (V_k \setminus \{v\}) \cup V_j'$ and $\overline{V_{k+j}} = \{\{x\} : x \in V_k \setminus \{v\}\} \cup \overline{V_j'}$, $j = 1, \ldots, l$. Since vS_wV_1' i $xS_w\{x\}$ for all $x \in V_k \setminus \{v\}$, we have $\overline{V_k}\, \overline{S_w}\, \overline{V_{k+1}}$. Similarly, since $\overline{V_j'}\, \overline{S_w}\, \overline{V_{j+1}'}$ for $j = 1, \ldots, l$, we have that $\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_{k+l}}$ is an $\overline{S_w}$-path. Then $(u, g)S'_{(w,f)}(v, h)$ implies $g_w(\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_{k+l}})_{k+l} = h_w(\overline{\{v\}}, \overline{V_1'}, \ldots, \overline{V_l'})_l = z$.

To check the remaining condition needed to conclude $(u, g)S'_{(w,f)}(z, s)$, take any finite sequence $\overline{Z_1}, \ldots, \overline{Z_m}$ such that $\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_{k+l}}, \overline{Z_1}, \ldots, \overline{Z_m}$ is an $\overline{S_w}$-path. Consider the $\overline{S_w}$-path $\overline{\{v\}}, \overline{V_1'}, \ldots, \overline{V_l'}, \overline{Z_1'}, \ldots, \overline{Z_m'}$ induced by v and the $\overline{S_w}$-path $\overline{\{z\}}, \overline{Z_1''}, \ldots, \overline{Z_m''}$ induced by z. Then $(u, g)S'_{(w,f)}(v, h)S'_{(w,f)}(z, s)$ implies $g_w(\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_{k+l}}, \overline{Z_1}, \ldots, \overline{Z_m})_{k+l+j} = h_w(\overline{\{v\}}, \overline{V_1'}, \ldots, \overline{V_l'}, \overline{Z_1'}, \ldots, \overline{Z_m'})_{l+j} = s_w(\overline{\{z\}}, \overline{Z_1''}, \ldots, \overline{Z_m''})_j$, for all $j = 1, \ldots, m$.

Finally, assume $(w, f)R'(u, g)R'(v, h)$ and show $(u, g)S'_{(w,f)}(v, h)$. First, we have $wRuRv$, so $uS_w\{v\}$, i.e. $\overline{\{u\}}, \overline{\{v\}}$ is an $\overline{S_w}$-path and obviously it must be $g_w(\overline{\{u\}}, \overline{\{v\}})_1 = v$. Note that for any continuation $\overline{\{u\}}, \overline{\{v\}}, \overline{V}_2, \ldots, \overline{V_{l+1}}$, the $\overline{S_w}$-path induced by v is actually $\overline{\{v\}}, \overline{V}_2, \ldots, \overline{V_{l+1}}$. Furthermore, since $(u, g)R'(v, h)$, by the definition of the relation R' applied to the path $\overline{\{v\}}, \overline{V}_2, \ldots, \overline{V_{l+1}}$, we have $g_w(\overline{\{u\}}, \overline{\{v\}}, \overline{V_2}, \ldots, \overline{V_{l+1}})_{i+1} = h_w(\overline{\{v\}}, \overline{V_2}, \ldots, \overline{V_{l+1}})_i$, $i = 1, \ldots, l$, which is exactly what we need to conclude $(u, g)S'_{(w,f)}(v, h)$. ■

4.4 The main result

Theorem 2. *Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be a Verbrugge model. Put $wZ(x, f)$ if and only if $w = x$. Then Z is a bisimulation between $\mathfrak{M}$ and $Vel(\mathfrak{M})$.*

Proof. The condition (at) holds by the definition of satisfaction in $Vel(\mathfrak{M})$.

To show (back), choose any $(w, f) \in W'$ and suppose $(w, f)R'(u, g)$. Then wRu and $uZ(u, g)$. Let $V \subseteq W$ such that uS_wV. Then for $v = g_w(\overline{\{u\}}, \overline{V})_1$ we have $v \in V$. It remains to define some h such that $(u, g)S'_{(w,f)}(v, h)$. First, to ensure $(w, f)R'(v, h)$, for all x such that xRw put $h_x(\overline{\{v\}}, \overline{V_1}, \overline{V_2}, \dots)_i = f_x(\overline{\{w\}}, \overline{\{v\}}, \overline{V_1}, \overline{V_2}, \dots)_{i+1}$, $i = 1, 2, \dots$, for each $\overline{S_w}$-path $\overline{\{v\}}, \overline{V_1}, \overline{V_2}, \dots$

Now, we define h_w as follows: for each $\overline{S_w}$-path $\overline{\{v\}}, \overline{V_1}, \overline{V_2}, \dots$, if $\overline{S_w}$-path $\overline{\{u\}}, \overline{V}, \overline{V_1}', \overline{V_2}', \dots$ is such that $\overline{\{v\}}, \overline{V_1}, \overline{V_2}, \dots$ is induced by v with respect to it, put $h_w(\overline{\{v\}}, \overline{V_1}, \overline{V_2}, \dots)_i = g_w(\overline{\{u\}}, \overline{V}, \overline{V_1}', \overline{V_2}', \dots)_{i+1}$, $i = 1, 2, \dots$, which is not ambiguous, i.e. does not depend on a choice of $\overline{S_w}$-path $\overline{\{u\}}, \overline{V}, \overline{V_1}', \overline{V_2}', \dots$, due to the definition of well defined choice of representatives.

To be more precise, to conclude that h_w is well defined, we need to show that for any $\overline{S_w}$-paths $\overline{\{u\}}, \overline{V}, \overline{V_1'}, \dots, \overline{V_k'}$ and $\overline{\{u\}}, \overline{V}, \overline{V_1''}, \dots, \overline{V_k''}$ such that the $\overline{S_w}$-path $\overline{\{v\}}, \overline{V_1}, \dots, \overline{V_k}$ is induced by v with respect to both of those paths, we have $g_w(\overline{\{u\}}, \overline{V}, \overline{V_1'}, \dots, \overline{V_k'}) = g_w(\overline{\{u\}}, \overline{V}, \overline{V_1''}, \dots, \overline{V_k''})$. We prove this by induction on k. For $k = 1$, since $\overline{\{v\}}, \overline{V_1}$ is induced by v with respect to both $\overline{\{u\}}, \overline{V}, \overline{V_1'}$ and $\overline{\{u\}}, \overline{V}, \overline{V_1''}$, the element of $\overline{V_1'}$ determined by v equals the element of $\overline{V_1''}$ determined by v.

So, by definition of well defined choice of representatives, $g_w(\overline{\{u\}}, \overline{V}, \overline{V_1'})_2 = g_w(\overline{\{u\}}, \overline{V}, \overline{V_1''})_2$. Also, of course $g_w(\overline{\{u\}}, \overline{V}, \overline{V_1'})_1 = g_w(\overline{\{u\}}, \overline{V}, \overline{V_1''})_1 = v$, so $g_w(\overline{\{u\}}, \overline{V}, \overline{V_1'}) = g_w(\overline{\{u\}}, \overline{V}, \overline{V_1''})$. Assume now that we have proved the claim for $k = n$ and let us prove it for $k = n + 1$. Let $\overline{\{v\}}, \overline{V_1}, \dots, \overline{V_{n+1}}$ be induced by v with respect both to $\overline{\{u\}}, \overline{V}, \overline{V_1'}, \dots, \overline{V_{n+1}'}$ and $\overline{\{u\}}, \overline{V}, \overline{V_1'}, \dots, \overline{V_{n+1}'}$. By induction hypothesis, we have $g_w(\overline{\{u\}}, \overline{V}, \overline{V_1'}, \dots, \overline{V_n'}) = g_w(\overline{\{u\}}, \overline{V}, \overline{V_1''}, \dots, \overline{V_n''})$. Since $\overline{\{v\}}, \overline{V_1}, \dots, \overline{V_{n+1}}$ is induced by v with respect to both paths, and since $g_w(\overline{\{u\}}, \overline{V})_1 = v$, the elements of families $\overline{V_{n+1}'}$ and $\overline{V_{n+1}''}$ determined by the world $g_w(\overline{\{u\}}, \overline{V}, \overline{V_1'}, \dots, \overline{V_n'})_{n+1}$ must be equal, because they are determined in the same way by the induced path. Hence, by the definition of well defined choice of representatives, $g_w(\overline{\{u\}}, \overline{V}, \overline{V_1'}, \dots, \overline{V_{n+1}'})_{n+2} = g_w(\overline{\{u\}}, \overline{V}, \overline{V_1''}, \dots, \overline{V_{n+1}''})_{n+2}$, as needed.

For all other x such that xRv we can choose h_x arbitrarily. It is easy to see that for thus defined h we have $(u, g)S'_{(w,f)}(v, h)$.

It remains to show (forth). Let $w \in W$, $(w, f) \in W'$ and u such that wRu. Let $U' = \{(x, g) \in W' : x = u \text{ and } (w, f)R'(x, g)\}$. It is easy to see that $U' \neq \emptyset$. We claim that this is a good choice of U' which shows that (forth) holds, i.e. that for any choice of one $S'_{(w,f)}$-successor for each world in U' there is V such that uS_wV and each $v \in V$ is bisimilar to some of those $S'_{(w,f)}$-successors, i.e. the first component of some of them equals v (we will shortly say that such v is *covered*). Assume the opposite, i.e. there exists a choice of one $S'_{(w,f)}$-successor for each world in U' such that for any V such that uS_wV there is $v \in V$ which is not bisimilar to any of those $S'_{(w,f)}$-successors. Let $F : U' \to W'$ be such a choice of $S'_{(w,f)}$-successors.

We will show that there exists a well defined choice of representatives of $\overline{S_w}$-paths starting with u such that each representative on any path is not covered, i.e. does not equal the first component of any $(v, h) \in F(U')$. For each path $\overline{\{u\}}, \overline{V_1}, \ldots, \overline{V_n}$, denote by V_1' the set of all uncovered elements in V_1, and by V_1'' the set of all covered elements in V_1. Then denote by V_2' the set of all uncovered worlds belonging to those elements of the union $\overline{V_2}$ which are determined by elements from V_1', i.e. they are their S_w-successors, and denote by V_2'' the set of all such worlds which are covered. Analogously define $V_3', V_3'', \ldots, V_n', V_n''$. Obviously, a desired well defined choice of representatives will exist if for any path all sets $V_1', V_2', \ldots, V_n'$ are non-empty. Now, the assumption implies $V_1' \neq \emptyset$. Assuming $V_2' = \emptyset$, by quasi-transitivity $vS_w\{v\}$ for all $v \in V_1''$ and $\overline{V_1'}\,\overline{S_w}\,\overline{V_2''}$ would imply $uS_w(V_1'' \cup V_2'')$, hence we would find an S_w-successor of u with all of its elements covered, contrary to the assumption. Similarly, for any k we can see that, if $V_k' = \emptyset$ while all before it are non-empty, we would have $uS_w(V_1'' \cup \cdots \cup V_k'')$, where $V_1'' \cup \cdots \cup V_k''$ is covered, which contradicts the assumption.

Thus we proved that there is a well defined choice of representatives g_w such that all representatives on each path are uncovered. For all x such that xRw we can choose g_x such that the condition from the definition of R' holds, and for all other $x \in W$ we can choose g_x arbitrarily. In this way we obtain g such that $(w, f)R'(u, g)$. But, by the definition of $S'_{(w,f)}$, the world $F(u, g)$ is obtained as a representative determined by g_w, which is impossible, since all representatives determined by g_w are uncovered. ∎

Corollary 1. *For any formula φ and for all $(w, f) \in W'$ we have:*
$\mathfrak{M}, w \Vdash \varphi$ *if and only if* $Vel(\mathfrak{M}), (w, f) \Vdash \varphi$.

5 Further work

By analogy to other notions of bisimulation, it is to be expected that finite approximations of bisimulation, so-called n-bisimulations, where n is a natural number, can be defined, as well as bisimulation games and n-games, with desirable properties: (n-)bisimilarity is equivalent to the existence of Defender's winning strategy in bisimulation (n-)game, and n-bisimilarity implies n-modal equivalence, i.e. the equivalence w.r.t. formulas of modal depth at most n. Furthermore, we conjecture that the converse in the case of finite alphabet would also hold. Together with Hennessy-Milner analogue we proved in Section 3, these results would round up arguments in favour of the definition of bisimulation between Verbrugge and Veltman models presented in this paper, but this exceeds limits and purpose of this paper, the main purpose being to show how we can transform a Verbrugge model to a bisimilar Veltman model.

More important further line of research, closely related to this purpose, is to explore how this transformation behaves with respect to particular classes of Verbrugge models and Veltman models, with additional constraints related to various principles of interpretability, which are used as additional axioms of many systems of interpretability logic in the literature. For example, if a given Verbrugge model $\mathfrak{M}$ is an ILM-model, i.e. belongs to the characteristic class of Verbrugge models related to so-called Montagna's principle, does $Vel(\mathfrak{M})$ belong to the corresponding characteristic class of Veltman models? An analogous question may be addressed system by system, or more generally, if possible, conditions may be provided under which such a preservation works. Or if it does not work, can we modify the construction of $Vel(\mathfrak{M})$ for a particular principle or set of principles to make it work? Certainly, there is no general positive answer, since obviously for systems complete w.r.t. Verbrugge semantics but incomplete w.r.t. Veltman semantics, a transformation from a Verbrugge model to a modally equivalent Veltman model does not exist (cf. [5] for the case of the system ILP_0).

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