Relevant Reasoning and Implicit Beliefs

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Abstract. Combining relevant and classical modal logic is an approach to overcoming the logical omniscience problem and related issues that goes back at least to Levesque's well known work in the 1980s. The present authors have recently introduced a variant of Levesque's framework where explicit beliefs concerning conditional propositions can be formalized. However, our framework did not offer a formalization of implicit belief in addition to explicit belief. In this paper we provide such a formalization. Our main technical result is a modular completeness theorem.

Keywords. Epistemic logic, explicit belief, implicit belief, knowledge representation, modal logic, relevant logic.

1 Introduction

Formal models of epistemic notions such as belief are often based on some form of modal logic and possible-worlds semantics [4]. In this approach, beliefs of an agent are modelled by a set of accessible possible worlds, and they are expressed by means of a modal operator quantifying over possible worlds: a proposition is believed if it is true in every accessible possible world. This endows the model with many closure principles allowing to make predictions about an agent's beliefs given information about their prior beliefs. For instance, if a conjunction is believed, then so are both conjuncts since every possible world satisfying the conjunction satisfies both conjuncts as well. However, such predictions are often inaccurate when it comes to real-life agents. Such agents frequently fail to realize consequence relations occurring between pieces of information (e.g. if they do not have sufficient resources at their disposal, such as time and memory), or they prioritize relevance over consequence (when the consequences at hand are not relevant to the prior beliefs or the context in general).

The possible-worlds model provides a good rendering of what has to be true given what is believed by the agent, or what is *implicitly believed*, but it fails to model what is actively held to be true by the agent, or what is *explicitly believed*. Many adjustments of the model exist that address the issue. Hector Levesque [9] famously provided a model of explicit belief based on the logic of First Degree Entailment, FDE, the implication-free fragment of Anderson and Belnap's relevant logic of entailment E [1]. In Levesque's model, explicit beliefs are modelled by a set of situations; unlike possible worlds, situations may be incomplete or inconsistent [2], and so information supported by a situation is not closed under consequences valid in classical propositional logic. This means that, in Levesque's model, explicit belief is not closed under classical consequence, but it is closed under consequence valid in FDE. While closure under FDE is a source of some criticism [3], it makes Levesque's framework a simple model of agents who prioritize relevance over consequence[3]. An important aspect of Levesque's model is that it combines an account of explicit belief with an account of implicit belief: a proposition is believed implicitly if it holds in every possible world satisfying the agent's explicit beliefs.

Levesque's model has been extended to allow for nesting of epistemic operators [8], which makes it possible to articulate various assumptions about the interplay of explicit and implicit belief. However, the model fails to provide a satisfactory account of explicit belief concerning *conditional propositions*. This is related to the absence of a sensible conditional connective in FDE. In a recent paper [13], we offered an extension of Levesque's model using fully-fledged relevant logic instead of the implication-free fragment. However, while our framework represented explicit beliefs (truth in all accessible situations), it did not account for implicit belief (truth in all accessible worlds). In this paper we extend the framework of [13] with an account of implicit belief. Our main technical result is a modular completeness theorem applying to a range of relevant epistemic logics with implicit and explicit belief operators.

The rest of the paper is structured as follows. In Section 2 we introduce the semantic framework for relevant epistemic logic of explicit and implicit belief, and in Section 3 we provide sound and complete axiomatisations for several logics based on the semantic framework. In the concluding Section 4 we summarise the paper and point to interesting further lines of research.

2 Relevant epistemic logic with classical worlds

In this section we introduce our semantic framework, based on so-called W-models introduced in [13]. These models combine the standard semantics for relevant modal logic based on *situations* [6] with a representation of *classical possible worlds*. The point of this combination is to represent agents as reasoning according to relevant logic while being situated in classical possible worlds. In our framework, a possible world is a special kind of situation where relevant negation and implication turn out to behave like their Boolean counterparts. We define validity as satisfaction in all possible worlds, and so logics based on our framework extend classical propositional logic CPC.

When it comes to modelling explicit belief in this framework, it is crucial that any situation (not only possible worlds) can be accessible from possible worlds.

[3] Such agents can be seen as reasoning according to Harman's *clutter avoidance principle* [7] in that they do not clutter their minds with trivial but unrelated consequences of the given information.

Consequently, explicit beliefs as modelled by a relevant epistemic logic $\mathsf{C.L}$ are closed under the underlying relevant logic L:

$$\frac{\vdash_{\mathsf{L}} \varphi_1 \wedge \cdots \wedge \varphi_n \to \psi}{\vdash_{\mathsf{C.L}} \Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n \to \Box\psi}$$

Hence, relevant epistemic logics $\mathsf{C.L}$ model agents reasoning according to a relevant logic L while being situated in classical possible worlds. In this paper we add to the framework of [13] a representation of implicit belief using an additional epistemic accessibility relation on situations to obtain relevant epistemic logics $\mathsf{CI.L}$. Our semantics for implicit belief is set up with an eye to two crucial principles concerning the properties of implicit belief, namely, that implicit belief extends explicit belief and that it is closed under classical consequence:

$$\vdash_{\mathsf{CI.L}} \Box\varphi \to \Box_I\varphi \qquad \frac{\vdash_{\mathsf{CPC}} \varphi_1 \wedge \cdots \wedge \varphi_n \to \psi}{\vdash_{\mathsf{CI.L}} \Box_I\varphi_1 \wedge \cdots \wedge \Box_I\varphi_n \to \Box_I\psi}$$

Consequently, implicit belief is the classical closure of explicit belief (see Proposition 2):

$$\frac{\vdash_{\mathsf{CPC}} \varphi_1 \wedge \cdots \wedge \varphi_n \to \psi}{\vdash_{\mathsf{CI.L}} \Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n \to \Box_I\psi}$$

Definition 1 (Language). *Let $\mathcal{L}$ be generated from a countable set of atomic propositions At via the following grammar:*

$$\varphi \in \mathcal{L} \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Box_I\varphi \mid \Box_L\varphi$$

where $p \in At$. We abbreviate $(\varphi \to \psi) \wedge (\psi \to \varphi)$ as $\varphi \leftrightarrow \psi$. Moreover, $\forall, \exists, \Rightarrow, \Leftrightarrow$ will denote, respectively, universal quantification, existential quantification, implication and equivalence in the meta-language.

The modal operators $\Box$ and $\Box_I$ have a clear epistemic interpretation, formalising explicit and implicit belief, respectively. On the other hand, $\Box_L$ has a technical role in our framework, namely, internalising in the object language validity in relevant logic. The role of $\Box_L$ becomes clear after the semantics is set up.[4]

Definition 2 (L-model). *Let a* L-model *be the tuple $(S, L, \leq, {}^*, R, Q, Q_I, Q_L, V)$ such that $(S, \leq)$ is a poset; * is an anti-monotonic function on S with respect to $\leq$; R is a ternary relation on S which is downward (upward) monotone in its first and second (third) argument; Q, Q_L are binary relations on S which are downward (upward) monotone in their first (second) argument; Q_I is a binary relation on S which is downward monotone in its first argument; and $L, V(p)$ are upward-closed subsets of S, for all $p \in At$. Moreover,*

$$\forall s \exists x (x \in L \ \& \ Rxss) \tag{1}$$

$$s \in L \ \& \ Rstu \Rightarrow t \leq u \tag{2}$$

[4] $\Box_L$ can be seen as a sort of provability operator; see Lemma 7.

The definition of L-models is virtually the standard definition of models for relevant modal logic (see e.g. [6]). L-models consist of a partially ordered set of situations, or information states, ordered by the amount of information they contain (support), and each component of the L-model satisfies the usual monotonicity condition with respect to $\leq$. We say that $\leq$ models an information order on situations in that $s \leq t$ means that t contains (supports) at least as much information as s. The unary operation * is the "Routley star", mapping each state s to its maximally compatible state, i.e. the state which is maximal with respect to the information order $\leq$ among those states that do not support the negation of any formula supported by s. R is the usual ternary relation interpreting $\to$, where $Rstu$ means that the result of combining the information contained in s with that contained in t contains at least as much information as that contained in u. L is the designated set of *logical situations*, containing situations carrying logical information, with Conditions (1-2) enforcing, as usual, the semantic deduction theorem with respect to relevant implication $\to$. As in Levesque's semantics, explicit beliefs are modelled by a set of situations. In particular, Q is the epistemic accessibility relation associated with explicit belief, associating with each state s the *epistemic state* $Q(s)$ of the (contextually fixed) agent according to the (information contained in) situation s. More specifically, $Q(s)$ consists of the situations that contain the information that is explicitly believed by the agent according to the information in s. In comparison to standard epistemic relevant models, L-models feature two further accessibility relations, Q_L and Q_I, associated with $\Box_L$ and $\Box_I$, respectively.

Definition 3 (W-model). *Let a* W-model *be the following tuple.*

$$\mathfrak{M} = (S, W, L, 0, 1, \leq, {}^*, R, Q, Q_I, Q_L, V)$$

- $(S, L, \leq, {}^*, R, Q, Q_I, Q_L, V)$ *is a L-model;*
- $W \subseteq S$ *such that for all* $w \in W$ *and* $s, t \in S$

$$w^* = w \tag{3}$$
$$Rwww \tag{4}$$
$$Rwst \Rightarrow s = 0 \text{ or } w \leq t \tag{5}$$
$$Rwst \Rightarrow t = 1 \text{ or } s \leq w \tag{6}$$
$$Q_I ws \Rightarrow s \in W \tag{7}$$
$$Q_I(w) \subseteq Q(w) \tag{8}$$
$$Q_L(W) = L \tag{9}$$

- $0 \notin V(p)$, $1 \in V(p)$ *for all* $p \in At$*, such that for all* $s, t \in S$, $Q_{(LI)} \in \{Q, Q_L, Q_I\}$

$$0 \leq s \leq 1 \tag{10}$$
$$1^* = 0 \;\&\; 0^* = 1 \tag{11}$$
$$Q_{(LI)}00 \tag{12}$$

$$Q_{(LI)}1s \Rightarrow s = 1 \tag{13}$$

$$R010 \tag{14}$$

$$R1st \Rightarrow (s = 0 \text{ or } t = 1) \tag{15}$$

where for any relation A, $A(x) = \{y \mid Axy\}$ and $A(B) = \{y \mid \exists x \in B(Axy)\}$.

As already mentioned, possible worlds are seen as a special kind of situations; see Conditions (3-8). Conditions (3-6) enforce classical behaviour of negated and implicative formulas when evaluated at possible worlds, as clarified by Lemma 3. Conditions (7–8) concerning Q_I yield the intended interpretation of $Q_I(w)$, which represents implicit beliefs of the agent in possible world w. In particular, by Condition (7), $Q_I(w)$ contains only possible worlds and so implicit beliefs are closed under classical consequence, while by Condition (8) the "implicit" epistemic state of the agent at w is a subset of the "explicit" epistemic state, and so every explicit belief is an implicit belief. Note also that, contrary to Q and Q_L, we do not assume that Q_I is upward monotone in its second argument[5]. Finally, Condition (9) plays a fundamental role in connecting the classical and the relevant layers of our semantics. Stipulating that the set of logical states L is exactly the set of Q_L-accessible states from W yields a modified version of the semantic deduction theorem, as clarified by Lemma 4 (item 1).

The last component of W-models are the bounds $0, 1$, which represent the empty situation and the full situation, respectively (the terminology is clarified by Lemma 2). The bounds were used in [14] to provide a general frame semantics for relevant modal logic. In our setting the bounds play a technical role that will be clarified in the completeness proof; see also their discussion in [13].

Definition 4 (Satisfaction). *Let the* satisfaction *relation in a W-model $\mathfrak{M}$ (notation $\models$) be a binary relation between states of $\mathfrak{M}$ and formulas of $\mathfrak{L}$ defined recursively (on $\mathfrak{L}$) as follows.*

$$\begin{array}{lcl}
\mathfrak{M}, s \models p & \iff & s \in V(p) \\
\mathfrak{M}, s \models \neg\varphi & \iff & \mathfrak{M}, s^* \not\models \varphi \\
\mathfrak{M}, s \models \varphi \wedge \psi & \iff & \mathfrak{M}, s \models \varphi \;\; \& \;\; \mathfrak{M}, s \models \psi \\
\mathfrak{M}, s \models \varphi \vee \psi & \iff & \mathfrak{M}, s \models \varphi \textit{ or } \mathfrak{M}, s \models \psi \\
\mathfrak{M}, s \models \varphi \rightarrow \psi & \iff & Rstu,\, \mathfrak{M}, t \models \varphi \Rightarrow \mathfrak{M}, u \models \psi \\
\mathfrak{M}, s \models \Box\varphi & \iff & Qst \Rightarrow \mathfrak{M}, t \models \varphi \\
\mathfrak{M}, s \models \Box_I\varphi & \iff & Q_I st \Rightarrow \mathfrak{M}, t \models \varphi \\
\mathfrak{M}, s \models \Box_L\varphi & \iff & Q_L st \Rightarrow \mathfrak{M}, t \models \varphi
\end{array}$$

Let the proposition expressed by a formula φ in a W-model $\mathfrak{M}$ be $[\![\varphi]\!]^{\mathfrak{M}} = \{s \mid \mathfrak{M}, s \models \varphi\}$. Let a formula φ be valid *in a W-model $\mathfrak{M}$, written $\mathfrak{M} \models \varphi$, iff for all $w \in W$ we have that $\mathfrak{M}, w \models \varphi$. Let a formula φ be* entailed *by a set of formulas*

[5] This condition has to do with the canonical model construction (see Section 3), since in the canonical model Q_I^c will not be upward monotone.

Γ in a W-model $\mathfrak{M}$, written $\Gamma \models_{\mathfrak{M}} \varphi$ iff for all $s \in S$, $\mathfrak{M}, s \models \varphi$ if $\mathfrak{M}, s \models \psi$ for all $\psi \in \Gamma$. Let a formula φ be classically entailed *by a set of formulas Γ in a W-model $\mathfrak{M}$, written $\Gamma \models^{c}_{\mathfrak{M}} \varphi$ iff for all $w \in W$, $\mathfrak{M}, w \models \varphi$ if $\mathfrak{M}, w \models \psi$ for all $\psi \in \Gamma$.*

The intended properties of the semantics are highlighted in the following series of lemmas. We omit reference to $\mathfrak{M}$ whenever it is clear from the context.

Lemma 1 (Heredity). *For every W-model $\mathfrak{M}$, $s, t \in S$ and $\varphi \in \mathfrak{L}$: $s \in [\![\varphi]\!]^{\mathfrak{M}}$ & $s \leq t \Rightarrow t \in [\![\varphi]\!]^{\mathfrak{M}}$.*

Proof. By induction on the structure of φ. The base case holds by the fact that $V(p)$ is upward monotone. The cases involving $\wedge, \vee$ are trivial, while the cases involving $\neg, \rightarrow, \Box, \Box_I, \Box_L$ hold thanks to monotonicity properties of the corresponding accessibility relations (i.e., $^*, R, Q, Q_I, Q_L$, respectively). □

Lemma 2 (Full empty). *For every W-model $\mathfrak{M}$ and $\varphi \in \mathfrak{L}$: $\mathfrak{M}, 1 \models \varphi$ and $\mathfrak{M}, 0 \not\models \varphi$.*

Proof. The proof is by induction on the structure of φ, as given in [13]. The new case of $\varphi = \Box_I \psi$ is established as follows. Assuming $Q_I 1 s$, we have by (13) that $s = 1$, hence by induction hypothesis (IH) $s \models \psi$, by which we conclude that $1 \models \Box_I \psi$. Moreover, by (12) $Q_I 00$, hence there is s, namely 0, such that $Q_I 0 s$ and (by IH) $s \not\models \psi$, by which we conclude that $0 \not\models \Box_I \psi$. □

Lemma 3 (Worlds extensionality). *For every W-model $\mathfrak{M}$, $w \in W$ and $\varphi, \psi \in \mathfrak{L}$:*

$$\mathfrak{M}, w \models \neg\varphi \iff \mathfrak{M}, w \not\models \varphi$$
$$\mathfrak{M}, w \models \varphi \rightarrow \psi \iff \mathfrak{M}, w \not\models \varphi \text{ or } \mathfrak{M}, w \models \psi$$

Proof. The first claim follows from (3). The second claim follows by (4) in one direction, while the other is established by case distinction, assuming $Rwst$ and $s \models \varphi$. If $w \not\models \varphi$, by (6) either $t = 1$ (by which we conclude by Lemma 2 that $t \models \psi$), or $s \leq w$, (by which we conclude by Lemma 1 that $w \models \varphi$, which is a contradiction). If $w \models \psi$, by (5) $w \leq t$, hence by Lemma 1 we conclude that $t \models \psi$. □

Lemma 4 ((Classical) entailment). *For every W-model $\mathfrak{M}$ and $\varphi, \psi \in \mathfrak{L}$:*

1. *$\varphi \models_{\mathfrak{M}} \psi \Leftrightarrow \mathfrak{M} \models \Box_L(\varphi \rightarrow \psi)$;*
2. *$\varphi \models^{c}_{\mathfrak{M}} \psi \Leftrightarrow \mathfrak{M} \models \varphi \rightarrow \psi$.*

Proof. The first item follows from (1-2, 9) and Lemma 1, while the second from Lemma 3.

Distinguishing between explicit and implicit beliefs has interesting applications to the problem of logical omniscience. W-models help to identify the origin of logical omniscience and circumvent the problem to some extent. To recall, the logical omniscience problem for an epistemic logic extending classical propositional logic lies in the fact that whenever a set of formulas Γ classically entails φ and the agent believes each formula in Γ, then the agent automatically believes φ [5].

In the spirit of Levesque's [9], omniscience is avoided since it is possible that φ classically entails ψ in a model $\mathfrak{M}$ without $[\![\varphi]\!]^{\mathfrak{M}} \subseteq [\![\psi]\!]^{\mathfrak{M}}$. Crucially, Q is allowed to "reach out" to non-worldly situations from possible worlds, thus providing counterexamples to classically valid entailments. On the other hand, the situation with implicit belief is different: since Q_I connects possible worlds only with possible worlds by Condition (7), it cannot reach counterexamples to classically valid entailments. Thus, logical omniscience is restored, as clarified by Proposition 1. We stress that this is a welcome result, since implicit belief captures the (classical) consequences of explicit belief, i.e. what an ideal, unbounded agent would explicitly believe; see Proposition 2 at the end of the section.

Proposition 1 (Logical omniscience). *For all $\Gamma, \{\varphi\} \subseteq \mathfrak{L}$ and all W-models $\mathfrak{M}$:*

1. $\Gamma \models^c_{\mathfrak{M}} \varphi \not\Rightarrow \Box\Gamma \models^c_{\mathfrak{M}} \Box\varphi$;
2. $\Gamma \models^c_{\mathfrak{M}} \varphi \Rightarrow \Box_I\Gamma \models^c_{\mathfrak{M}} \Box_I\varphi$.

where $\Box_{(I)}\Gamma = \{\Box_{(I)}\psi \mid \psi \in \Gamma\}$ *for* $\Box_{(I)} \in \{\Box, \Box_I\}$.

Proof. Item (1) follows from the fact that, for $\Gamma = \{\psi_i \mid i \in K\}$, $\bigcap_{i\in K}([\![\psi_i]\!] \cap W) \subseteq [\![\varphi]\!] \cap W$ does not in general imply $\bigcap_{i\in K}([\![\psi_i]\!] \cap Q(W)) \subseteq [\![\varphi]\!] \cap Q(W)$. For example, consider the formulas $\neg p \vee q$ and $p \to q$, which are true in the same possible worlds for all W-model $\mathfrak{M}$, hence $\neg p \vee q \models^c_{\mathfrak{M}} p \to q$ but the two formulas may not be true in the same situations. In particular, take the W-model $\mathfrak{M}$ with $S = \{s, t\}$ such that $s^* = t$, $t \notin V(p), s \in V(p), s \notin V(q), Qss$ and $Rsss$ (the remaining components can be specified so that $\mathfrak{M}$ is indeed a W-model). In $\mathfrak{M}$, we have $s \models \Box(\neg p \vee q)$ but $s \not\models \Box(p \to q)$. Item (2) follows from the fact that, thanks to (7) we have that $Q_I(W) \subseteq W$, by which we conclude that $\bigcap_{i\in K}([\![\psi_i]\!] \cap W) \subseteq [\![\varphi]\!] \cap W$ does imply $\bigcap_{i\in K}([\![\psi_i]\!] \cap Q_I(W)) \subseteq [\![\varphi]\!] \cap Q_I(W)$. $\Box$

We note that, thanks to the above proposition, the logic of W-models is hyperintensional, in that agents can distinguish between logically equivalent propositions. The fact that in W-models agents are not logically omniscient with respect to explicit belief has other interesting consequences. Most notably, agents' belief bases are not *cluttered* by irrelevant information. That is, explicit belief is not closed under some implications valid in classical logic where the consequent introduces information that is unrelated to the information expressed in the antecedent. In our framework, "irrelevant" is seen simply as "not following by relevant logic". For instance, the following clutter principles fail for explicit belief, but they do hold for implicit belief:

$$\Box\varphi \to \Box(\psi \to \varphi) \tag{16}$$

$$\Box\varphi \to \Box(\psi \vee \neg\psi) \tag{17}$$

$$\Box(\varphi \wedge \neg\varphi) \to \Box\psi \tag{18}$$

Avoidance of epistemic clutter in our framework is mediated by the fact that relevant logics satisfy the *variable sharing principle*: an implication $\varphi \to \psi$ is provable only if φ and ψ share at least one propositional variable. This means that cases where φ and ψ are "totally unrelated" have counterexamples which can then be exploited in our framework to give counterexamples to $\Box\varphi \to \Box\psi$. However, we note that some aspects of epistemic clutter, as one may understand the notion, are preserved in our framework as, for instance, $\Box\varphi \to \Box(\varphi \vee \psi)$ is valid, for all φ and ψ (even if ψ is "totally unrelated" to φ).

We conclude this section by commenting on the relation of explicit and implicit belief in our framework. Proposition 2 says that, in a specific sense, implicit beliefs of an agent are the *classical closure* of the agent's explicit beliefs.

Lemma 5 (Implicit-explicit). *For every W-model* $\mathfrak{M}$ *and* $\varphi \in \mathfrak{L}$*:* $\Box\varphi \models^c_{\mathfrak{M}} \Box_I\varphi$.

Proof. This follows from Condition (8).

Lemma 6 (Classical-implicit). *For every W-model* $\mathfrak{M}$ *and* $\varphi_1, \dots, \varphi_n, \psi \in \mathfrak{L}$*:* $\varphi_1, \dots, \varphi_n \models^c_{\mathfrak{M}} \psi \Rightarrow \Box_I\varphi_1, \dots, \Box_I\varphi_n \models^c_{\mathfrak{M}} \Box_I\psi$.

Proof. This follows from Condition (7).

Proposition 2 (Classical closure). *For all* $\varphi_1, \dots, \varphi_n, \psi \in \mathfrak{L}$ *without occurrences of modal operators, the following are equivalent:*

1. $\varphi_1 \wedge \ldots \wedge \varphi_n \to \psi$ *is a classical tautology;*
2. $\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box_I\psi$ *is valid in all W-models.*

Proof. 1 implies 2: If $\bigwedge_{i\leq n} \varphi_i \to \psi$ is a classical tautology, then $\bigwedge_{i\leq n} \varphi_i \models^c_{\mathfrak{M}} \psi$ for all W-models $\mathfrak{M}$ by Lemma 3. Then, $\bigwedge_{i\leq n} \Box_I\varphi_i \models^c_{\mathfrak{M}} \Box_I\psi$ by Lemma 6, which entails $\bigwedge_{i\leq n} \Box\varphi_i \models^c_{\mathfrak{M}} \Box_I\psi$ by Lemma 5. Consequently, $\bigwedge_{i\leq n} \Box\varphi_i \to \Box_I\psi$ is valid in all W-models $\mathfrak{M}$ by Lemma 4 (item 2).

2 implies 1: If $\bigwedge_{i\leq n} \varphi_i \to \psi$ is a propositional formula that is not a classical tautology, then there is a classical valuation v such that $v(\varphi_i) = 1$ for all φ_i and $v(\psi) = 0$. We may turn this valuation into a W-model $\mathfrak{M}$ with the set of states $S = \{0, v, 1\}$ and V such that $v \in V(p)$ iff $v(p) = 1$ for all $p \in At$. Moreover, we assume that $W = \{v\}$, $Q_{(I)}vv$, and the rest is added so that this structure is indeed a W-model[6]. It is obvious that $\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box_I\psi$ is not valid in $\mathfrak{M}$.

From a semantic point of view, implicit belief is stronger than the classical closure of explicit belief, as Conditions (7-8) ensure only that $Q_I(w) \subseteq Q(w) \cap W$ for all $w \in W$ and not the stronger condition $Q_I(w) = Q(w) \cap W$. However, the above proposition tells us that this does not matter in general. The present weaker semantics is more amenable to the canonical model technique.

[6] We can define $\mathfrak{M}$ similarly as in the +-construction used in the proof of Proposition 7, with the proviso that we do not add a new possible world w since v itself is seen as the only possible world in the model.

3 Axiomatization

In this section we introduce a Hilbert-style axiomatisation for our logic of explicit and implicit belief and prove that it is sound and complete with respect to the class of W-models. In fact, we provide a modular soundness and completeness result for a family of several logics $\mathsf{CI.L}$, where L ranges over a number of relevant logics, extending our basic system at the propositional and modal level. The methods employed here are the same as the ones used in [13]. In particular, we use a Henkin-style canonical model construction (see Definition 8) which combines the usual strategies for completeness in classical propositional logic (defining worlds as maximally consistent $\mathsf{CI.L}$-theories) and relevant modal logics (defining information states as prime L-theories). We note that a crucial step in the proof is a model construction allowing to transform every L-model into a suitable W-model, so that $\vdash_{\mathsf{L}} \varphi \Rightarrow \vdash_{\mathsf{CI.L}} \Box_L \varphi$ is an admissible meta-rule (see Lemma 7 Item (1)). A similar result for the framework without implicit belief was proven in [13].

We begin by recalling Fuhrmann's axiomatization of the basic conjunctively regular relevant modal logic $\mathsf{BM.C}$ [6]. In our formulation, the logic contains three modal operators, not one.

Definition 5 (Axiom system $\mathsf{BM.C}$). *Let $\mathsf{BM.C}$ be a conjunctively regular multi-modal axiom system comprising the following axioms and rules:*

- *The following axioms and rules of the propositional relevant logic BM [12]:*

$$\begin{array}{llll}
(\mathsf{BM1}) & \varphi \to \varphi & (\mathsf{BM8}) & \neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi) \\
(\mathsf{BM2}) & (\varphi \wedge \psi) \to \varphi & (\mathsf{BM9}) & (\neg\varphi \wedge \neg\psi) \to \neg(\varphi \vee \psi) \\
(\mathsf{BM3}) & (\varphi \wedge \psi) \to \psi & (\mathsf{BM10}) & ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi)) \\
(\mathsf{BM4}) & \varphi \to (\varphi \vee \psi) & (\mathsf{BM11}) & ((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi) \\
(\mathsf{BM5}) & \psi \to (\varphi \vee \psi) & (\mathsf{BM12}) & (\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi)) \\
(\mathsf{BM6}) & \dfrac{\varphi \qquad \varphi \to \psi}{\psi} & (\mathsf{BM13}) & \dfrac{\varphi \to \chi \qquad \psi \to \xi}{(\varphi \to \psi) \to (\chi \to \xi)} \\
(\mathsf{BM7}) & \dfrac{\varphi \qquad \psi}{\varphi \wedge \psi} & (\mathsf{BM14}) & \dfrac{\varphi \to \psi}{\neg\psi \to \neg\varphi}
\end{array}$$

- *The following axioms and rules, for $\Box_{(IC)} \in \{\Box, \Box_I, \Box_L\}$:*

$$\begin{array}{ll}
(\Box_{(IC)}.\mathsf{C}) & \Box_{(IC)}\varphi \wedge \Box_{(IC)}\psi \to \Box_{(IC)}(\varphi \wedge \psi) \\
(\Box_{(IC)}.\mathsf{M}) & \dfrac{\varphi \to \psi}{\Box_{(IC)}\varphi \to \Box_{(IC)}\psi}
\end{array}$$

Figure 1 lists further axioms and rules one may add to $\mathsf{BM.C}$ in order to obtain well-known relevant axiom systems (see [6] for a taxonomy).

Our goal is to set up a general framework that allows the user to use the relevant logic that is most suitable given their intuitions or the situation at hand. Obvious candidates include modal extensions of the strong relevant logics E or R, which received detailed discussion and motivation in [1] and [10], respectively. In particular, the logic E.C (the conjunctively regular modal extension of E) is obtained by adding (L1–L8) and (L11) to BM.C, and R.C results from E.C by adding (L9).

In what follows, we use the variable L for an axiom system extending BM.C with axioms and rules of Figure 1, and we stipulate that $Rstuv := \exists x(Rstx \ \& \ Rsuv)$, $Rs(tu)v := \exists x(Rsxv \ \& \ Rtux)$, $RQ_{(I)}stu := \exists x(Rstx \ \& \ Q_{(I)}xu)$ and $Q_{(I)}Rstu := \exists x(Q_{(I)}sx \ \& \ Rxtu)$. Moreover, we assume that in (L12)-(L17) the frame condition with the suitable accessibility relation $Q_{(I)} \in \{Q, Q_I\}$ corresponds to the $\Box_{(I)}$-variant of each. Let a L-*model* be a W-model satisfying the frame conditions corresponding to the axioms and rules of L from Figure 1. Finally, let the $\Box_L$-*version* of a L-axiom (L-rule) be obtained by prefixing $\Box_L$ to the axiom (to each of the premises and the conclusion of the rule).

Axiom/rule		**Frame condition**
(L1)	$\varphi \leftrightarrow \neg\neg\varphi$	$s^{**} = s$
(L2)	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	$Rstu \Rightarrow Rsu^*t^*$
(L3)	$((\varphi \to \psi) \wedge (\psi \to \chi)) \to (\varphi \to \chi)$	$Rstu \Rightarrow Rs(st)u$
(L4)	$\varphi \vee \neg\varphi$	$s \in L \Rightarrow s^* \leq s$
(L5)	$(\varphi \to \neg\varphi) \to \neg\varphi$	Rss^*s
(L6)	$(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$	$Rstuv \Rightarrow Rs(tu)v$
(L7)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	$Rstuv \Rightarrow Rt(su)v$
(L8)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	$Rstu \Rightarrow Rsttu$
(L9)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	$Rstuv \Rightarrow Rsutv$
(L10)	$\varphi \to (\varphi \to \varphi)$	$Rstu \Rightarrow (s \leq u \vee t \leq u)$
(L11)	$\dfrac{\varphi}{(\varphi \to \psi) \to \psi}$	$\exists x(x \in L \ \& \ Rsxs)$
(L12)	$\dfrac{\varphi}{\Box\varphi}$	$(x \in L \ \& \ Qxs) \Rightarrow s \in L$
(L13)	$\Box_{(I)}(\varphi \to \psi) \to (\Box_{(I)}\varphi \to \Box_{(I)}\psi)$	$RQ_{(I)}stu \Rightarrow \exists x(Q_{(I)}tx \ \& \ Q_{(I)}Rsxu)$
(L14)	$\Box_{(I)}\varphi \to \varphi$	$Q_{(I)}ss$
(L15)	$\Box_{(I)}\neg\varphi \to \neg\Box_{(I)}\varphi$	$\exists x(Q_{(I)}sx^* \ \& \ Q_{(I)}s^*x)$
(L16)	$\Box_{(I)}\varphi \to \Box_{(I)}\Box_{(I)}\varphi$	$(Q_{(I)}st \ \& \ Q_{(I)}tu) \Rightarrow Q_{(I)}su$
(L17)	$\neg\Box_{(I)}\varphi \to \Box_{(I)}\neg\Box_{(I)}\varphi$	$(Q_{(I)}s^*u \ \& \ Q_{(I)}st) \Rightarrow Q_{(I)}t^*u$

Fig. 1. Frame conditions with the corresponding axioms and rules for L.

Definition 6 (Axiom system CI.L). *Let the logic* CI.L *consist of the following:*

- *an axiomatisation of classical propositional logic (*CPC*);*
- *the* $\Box_L$*-versions of axioms and rules of* L*;*
- *the following axioms and rules:*

$(\Box\Box_I)$ $\quad \Box\varphi \to \Box_I\varphi$

$(\Box_I.\mathsf{K})$ $\quad \Box_I(\varphi \to \psi) \to (\Box_I\varphi \to \Box_I\psi)$

$(\Box_I.\mathsf{N})$ $\quad \dfrac{\varphi}{\Box_I\varphi}$

$$\text{(BR)} \quad \frac{\Box_L(\varphi \to \psi)}{\varphi \to \psi}$$

Let provability of a formula φ in CI.L*, written $\vdash_{\mathsf{CI.L}} \varphi$, be defined as usual.*

Theorem 1 (CI.L soundness). *For every* L*-model $\mathfrak{M}$: $\vdash_{\mathsf{CI.L}} \varphi \Rightarrow \mathfrak{M} \models \varphi$.*

Proof. By induction on the length of CI.L-proofs. The axioms of CPC are valid thanks to Lemma 3. The fact that, for each L, all L-axioms are satisfied in all states $s \in L$ in all L-models is established as usual in relevant modal logic; see [13] for details. The cases corresponding to the $\Box_I$-variants of (L13) – (L17) are established similarly as their $\Box$-variant. Then, we infer that $\Box_L\varphi$ is valid in each L-model for each L-axiom φ using (9). The fact that the $\Box_L$-versions of L-rules preserve validity is established similarly. The cases corresponding to the remaining axioms and rules is established as follows. For ($\Box_I$.K), assume $w \models \Box_I(\varphi \to \psi)$ and $w \models \Box_I\varphi$ for all $w \in W$. Hence, for all s such that Q_Iws, $s \models \varphi \to \psi$ and $s \models \varphi$. By (7) we have $s \in W$, hence $s \models \psi$, by which we conclude that $w \models \Box_I\psi$. For ($\Box\Box_I$), assume $w \models \Box\varphi$ and Q_Iws for some $w \in W$ and $s \in S$. By (8) we have Qws, hence by $w \models \Box\varphi$ we have $s \models \varphi$. Hence, we conclude that $w \models \Box_I\varphi$. For ($\Box_I$.N), assume $w \models \varphi$ for all $w \in W$ and Q_Iws for some arbitrary $s \in S$. By (7) we have $s \in W$, hence $s \models \varphi$, by which we conclude that $\models \Box_I\varphi$. For (BR), assume $w \models \Box_L(\varphi \to \psi)$ for all $w \in W$. By Lemma 4 we have $[\![\varphi]\!] \subseteq [\![\psi]\!]$, hence by $W \subseteq S$ and Lemma 3 we conclude that $w \models \varphi \to \psi$. □

The following lemma clarifies the relationship between the logics L and CI.L. In particular, by item (1), the modal operator $\Box_L$ expresses L-provability within CI.L, and items (2) and (3) state some interesting consquences of item (1). We note that item (1) will be crucial in establishing that Condition (9) holds in the canonical model.

Lemma 7 (L-CI.L). *For every $\varphi \in \mathfrak{L}$:*

1. $\vdash_{\mathsf{L}} \varphi \Leftrightarrow \vdash_{\mathsf{CI.L}} \Box_L\varphi$;
2. $\vdash_{\mathsf{L}} \varphi \Rightarrow \vdash_{\mathsf{CI.L}} \varphi$*, for* L *not containing* (L12);
3. $\vdash_{\mathsf{CI.L}} \Box_L\varphi \Rightarrow \vdash_{\mathsf{CI.L}} \varphi$*, for* L *not containing* (L12).

Proof. For item (1), one direction is established by induction on the length of L-proofs. If φ is an L-axiom, by definition of CI.L $\Box_L\varphi$ is a CI.L-axiom. If φ is obtained by a L-inference rule with premises $\varphi_1, \ldots, \varphi_n$, by IH $\vdash_{\mathsf{CI.L}} \Box_L\varphi_1, \ldots, \vdash_{\mathsf{CI.L}} \Box_L\varphi_n$, hence by application of the $\Box_L$-version of the rule we conclude $\vdash_{\mathsf{CI.L}} \Box_L\varphi$. For the other direction we construct a W-model $\mathfrak{M}^+$ from a L-model $\mathfrak{M} = (S, \leq, L, R, *, Q, Q_I, V)$ such that if $\mathfrak{M} \not\models \varphi$, then $\mathfrak{M}^+ \not\models \Box_L\varphi$ (the result then follows by 1). Let $\mathfrak{M}^+ = (S^+, \leq^+, W^+, R^+, *^+, Q^+, Q_L^+, Q_I^+, V^+)$ be defined as follows:

$$S^+ = S \cup \{w, 0, 1\}$$

$$\leq^+ = \leq \cup \{(w,w)\} \cup \{(s,1) \mid s \in S^+\} \cup \{(0,s) \mid s \in S^+\}$$
$$W = \{w\}$$
$$L^+ = L \cup \{w, 1\}$$
$$R^+ = R \cup \{(w,w,w)\} \cup \{(0,s,t),(s,0,t),(s,t,1) \mid s,t \in S^+\}$$
$$*^+ = * \cup \{(w,w)\} \cup \{(0,1),(1,0)\}$$
$$Q^+ = Q \cup \{(w,w)\} \cup \{(s,1) \mid s \in S^+\} \cup \{(0,s) \mid s \in S^+\}$$
$$Q_L^+ = Q_L \cup \{(w,w)\} \cup \{(w,s) \mid s \in L\} \cup \{(s,1) \mid s \in S^+\} \cup \{(0,s) \mid s \in S^+\}$$
$$Q_I^+ = Q_I \cup \{(w,w)\} \cup \{(0,s) \mid s \in S^+\}$$
$$V^+(p) = V(p) \cup \{1\} \text{ for all } p$$

It suffices to prove (i) that $\mathfrak{M}^+$ is a W-model; (ii) that for all $s \in S$, $\mathfrak{M}, s \models \varphi \Leftrightarrow \mathfrak{M}^+, s \models \varphi$; and (iii) that each frame condition of Figure 1 holds in $\mathfrak{M}^+$ whenever it holds in $\mathfrak{M}$. Putting (i)-(iii) together, we conclude that if there is $l \in L$ such that $\mathfrak{M}, l \not\models \varphi$, then $(\mathfrak{M}^+, w) \not\models \Box_L \varphi$. (i) is established as in [13], with the new cases involving (9) and the monotonicity property of Q_I^+ holding by inspection of the definition of $\mathfrak{M}^+$. (ii) is established by induction on the structure of φ, as in [13], with the new case $\varphi = \Box_I \psi$ established as follows. If $\mathfrak{M}, s \not\models \Box_I \psi$, then $\mathfrak{M}^+, s \not\models \Box_I \psi$ by $Q_I \subseteq Q_I^+$ and IH. Conversely, if $\mathfrak{M}^+, s \not\models \Box_I \psi$, then there is t such that $Q_I^+ st$ and $\mathfrak{M}^+, t \not\models \psi$. By inspection of the definition of Q_I^+, $t \in S$ and so $Q_I st$, which implies using IH that $\mathfrak{M}, s \not\models \Box_I \psi$. (iii) is established virtually as in [13] thanks to the observation that $s \in L^+$ iff $Q_L^+ ws$ (the case corresponding to the $\Box_I$-variants of (L12) – (L17) is almost identical as their $\Box$-variants).

Item (2) is established by induction on the length of L-proofs. All implicational axioms and rules of L are provable (preserve provability) in CI.L by item (1) and (BR), and (Adj) preserves provability thanks to $\vdash_{\mathsf{CPC}} \varphi \to (\psi \to (\varphi \land \psi))$ for all φ, ψ[7]. Item (3) follows from items (1) and (2). □

Definition 7 (Theories, Pairs). *A* L-theory *is a set of formulas T closed under provable implications and under conjunction, i.e. for all $\varphi, \psi \in \mathfrak{L}$ (i) $\varphi \in T$ and $\vdash_{\mathsf{L}} \varphi \to \psi$ implies $\psi \in T$ and (ii) $\varphi, \psi \in T$ implies $\varphi \land \psi \in T$. A* L*-theory is* regular *if it contains all theorems of* L*;* prime *if for all $\varphi, \psi \in \mathfrak{L}$ $\varphi \lor \psi \in T$ implies $\varphi \in T$ or $\psi \in T$;* proper *if it does not contain all formulas of $\mathcal{L}$.*

A pair of sets of formulas (Γ, Δ) is (C).L-independent *(for (C.)L $\in \{$CI.L, L$\}$) iff there are no finite non-empty sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash_{\mathsf{(C).L}} \bigwedge \Gamma' \to \bigvee \Delta'$.*

Lemma 8 (Extension Lemma). *If (Γ, Δ) is* L*-independent (*CI.L*-independent and both Γ and Δ are non-empty), then there is a prime* L*-theory (non-empty proper prime* CI.L*-theory) Σ such that $\Gamma \subseteq \Sigma$ and $\Delta \cap \Sigma = \emptyset$.*

Proof. [11]. □

[7] Note that (L12) is problematic since φ in general is not an implication, so we cannot use item (1) and (BR).

Definition 8 (Canonical model). *Let the* canonical $\mathsf{CI.L}$-model *be the following tuple:*

$$\mathfrak{M}^c = (S^c, W^c, L^c, 0^c, 1^c \leq^c, R^c, *^c, Q^c, Q^c_L, V^c)$$

- S^c *is the set of prime* L*-theories;*
- W^c *is the set of non-empty proper prime* $\mathsf{CI.L}$*-theories;*
- L^c *is the set of regular prime* L*-theories;*
- $0^c = \emptyset$ *and* $1^c = \mathfrak{L}$*;*
- $\leq^c = \subseteq$*;*
- $\varphi \in s^{*^c}$ *iff* $\neg\varphi \notin s$*;*
- $R^c stu$ *iff* $\varphi \to \psi \in s$ & $\varphi \in t \Rightarrow \psi \in u$*;*
- $Q^c st$ *iff* $\Box\varphi \in s \Rightarrow \varphi \in t$*;*
- $Q^c_L st$ *iff* $\Box_L\varphi \in s \Rightarrow \varphi \in t$*;*
- $Q^c_I st$ *iff* $\begin{cases} \Box_I\varphi \in s \Rightarrow \varphi \in t & \text{if } s \notin W^c \\ (\Box_I\varphi \in s \Rightarrow \varphi \in t) \ \& \ t \in W^c & \text{if } s \in W^c \end{cases}$
- $s \in V^c(p)$ *iff* $p \in s$*.*

In what follows, we omit the superscript from the canonical $\mathsf{CI.L}$-model $\mathfrak{M}^c$ whenever the context allows this.

Lemma 9 (Canonical model). $\mathfrak{M}^c$ *is a* L*-model.*

Proof. First, the canonical model is well-defined since $W \subseteq S$ ($\vdash_{\mathsf{L}} \varphi \to \psi$ implies $\vdash_{\mathsf{CI.L}} \Box_L(\varphi \to \psi)$ by the first item of Lemma 7, and $\vdash_{\mathsf{CI.L}} \Box_L(\varphi \to \psi)$ implies $\vdash_{\mathsf{CI.L}} \varphi \to \psi$ using (BR)). The monotonicity properties of $*, R, Q, Q_L$ hold by inspection of the definition of $\mathfrak{M}$. To show that Q_I is downward monotone in its first argument, assume $Q_I st$, $u \leq s$ and $\Box_I\varphi \in u$. If $s \in W$, then by $Q_I st$ we have $t \in W$ and by $u \leq s$ we have $\Box_I\varphi \in s$, by which we conclude that $\varphi \in t$. If $s \notin W$, then by $u \leq s$ we have $\Box_I\varphi \in s$, by which we conclude by $Q_I st$ that $\varphi \in t$. The proof for 1-6 and 10-15 is as in [13][8]. The remaining conditions are established as follows. (7) holds since, assuming $Q_I st$ and $s \in W$, by definition of Q_I we have that $t \in W$. (8) holds since, assuming $Q_I st$, $\Box\varphi \in s$ and $s \in W$, by ($\Box\Box_I$) we have $\Box_I\varphi \in s$, hence by definition of Q_I we conclude that $\varphi \in t$. (9) holds by the following argument. By contraposition, assume $s \notin L$. i.e. $\varphi \notin s$ for some φ such that $\vdash_{\mathsf{L}} \varphi$. By Lemma 7 $\vdash_{\mathsf{CI.L}} \Box_L\varphi$, hence $\Box_L\varphi \in w$ for all $w \in W$, which together with $\varphi \notin s$ implies that not $Q_L ws$ for all $w \in W$. Hence, $s \notin Q_L(W)$. Conversely, assume $s \in L$. We have to prove that there is $w \in W$ such that $Q_L ws$. If $s = 1$, then it is sufficient to show that W is non-empty. This

[8] The presence of the bounds $0, 1$ is necessary for the following reason. The bound-free versions of Conditions (5-6) are sufficient for Lemma 3, but these simpler versions do not hold in the canonical model. For instance, $\emptyset$ is a perfectly legitimate prime L-theory, and $Rw\emptyset t$ obviously holds for all $w \in W^c$ and $t \in S^c$. Hence, $Rwst \Rightarrow w \subseteq t$ fails. (The argument that $Rwst \Rightarrow s \subseteq w$ fails is similar, exploiting the possibility that $t = \mathfrak{L}$.) In this situation we can either add extensional truth constants to the language, and so rule out $\emptyset$ and $\mathfrak{L}$ as legitimate L-theories, or work with $\emptyset$ and $\mathfrak{L}$ as special kinds of states in the model while modifying the frame conditions (5-6) so that they refer to these special states. We chose the second option.

follows from the fact that each $\mathsf{CI.L}$ considered here is consistent by Theorem 1. If $s \neq 1$, then we reason as follows. Consider the pair $(\{\psi \mid \vdash_{\mathsf{CI.L}} \psi\}, \{\Box_L \varphi \mid \varphi \notin s\})$ and note that both sets in the pair are non-empty. The pair is $\mathsf{CI.L}$-independent, since otherwise

- $\vdash_{\mathsf{CI.L}} \bigvee_{i<n} \Box_L \varphi_i$ for some $n > 0$ only if (by $\vdash_{\mathsf{CI.L}} \Box_L \varphi \vee \Box_L \psi \to \Box_L(\varphi \vee \psi)$)
- $\vdash_{\mathsf{CI.L}} \Box_L \bigvee_{i<n} \varphi_i$ only if (by Lemma 7)
- $\vdash_{\mathsf{L}} \bigvee_{i<n} \varphi_i$ only if (by $s \in L$)
- $\bigvee_{i<n} \varphi_i \in s$ only if (since s is prime)
- $\varphi_i \in s$ for some $i < n$

which contradicts $\varphi_i \notin s$. It follows using Lemma 8 that there is a non-empty proper prime $\mathsf{CI.L}$-theory w such that $Q_L ws$. Finally, the fact that the frame conditions corresponding to ($\mathsf{L1}$)-($\mathsf{L17}$) are canonical is established as in [13], where the new cases of the conditions corresponding to the $\Box_I$-variants of ($\mathsf{L12}$)-($\mathsf{L17}$) are virtually identical to their $\Box$-variants. □

Lemma 10 (Truth). *For every $\varphi \in \mathfrak{L}$: $\varphi \in s \Leftrightarrow \mathfrak{M}^c, s \models \varphi$.*

Proof. By induction on the structure of φ. The proof employs the standard arguments of relevant modal logic (using the fact that $\Box, \Box_L$ are conjunctively regular modalities, see e.g. [6]), except for the case $\varphi := \Box_I \psi$, which we show as an illustration. For one direction, assume $\Box_I \varphi \in s$ and $Q^c st$. Hence, $\varphi \in t$, by which we conclude that $t \models \varphi$ by IH. For the other direction, assume $\Box_I \psi \notin s$ and consider the pair $t_0 = (\{\chi \mid \Box_I \chi \in s\}, \{\psi\})$. In case $s \in W^c$, we have to show that t_0 is $\mathsf{CI.L}$-independent. This holds, since otherwise

- $\vdash_{\mathsf{CI.L}} \chi_1 \wedge \cdots \wedge \chi_n \to \psi$ only if (by ($\Box_I$.C) and ($\Box_I$.M), which are derivable using ($\Box_I$.K) and ($\Box_I$.N) in the usual way)
- $\vdash_{\mathsf{CI.L}} \Box_I \chi_1 \wedge \cdots \wedge \Box_I \chi_n \to \Box_I \psi$ only if (by construction of t_0)
- $\Box_I \psi \in s$

contradicting $\Box_I \psi \notin s$. Hence, by Lemma 8 there is, $t \in W^c$ such that $Q_I st$ and $\psi \notin t$. If $s \notin W^c$, then the argument is similar – we just need to show that t_0 is L-independent. In both cases, $s \not\models \Box_I \psi$, using the induction hypothesis. □

Theorem 2 (Completeness). *For all $\varphi \in \mathfrak{L}$: If $\mathfrak{M} \models \varphi$ for every* L*-model $\mathfrak{M}$, then $\vdash_{\mathsf{CI.L}} \varphi$.*

Proof. The theorem follows from Lemmas 9 and 10. □

4 Conclusion

This paper extends our framework from [13] with a formalization of implicit belief. In the spirit of Levesque [9], we model explicit beliefs of an agent by a set of accessible situations that may contain counterexamples to classically valid entailments, and we model implicit beliefs by a subset of accessible situations that behave like classical possible worlds. In our setting, explicit belief is closed under the underlying relevant logic, while implicit belief is closed under classical logic and corresponds to the classical closure of explicit belief. The framework is best seen as formalizing agents that reason using a relevant logic because they prioritize relevance over classical consequence with the goal of not cluttering their belief bases by irrelevant consequences of their information. Our main technical result is a modular completeness theorem for a family of relevant epistemic logics based on the framework, extending the completeness result of [13].

We note that undecidability of some relevant logics L (such as E and R, for instance [15]) implies undecidability of $\mathsf{CI.L}$ in view of Lemma 7. We conjecture that, conversely, if L is decidable, then so is $\mathsf{CI.L}$.

Natural topics for future research include a study of extensions of the present framework with a formalization of group-epistemic notions (common and distributed belief) and with a formalization of epistemic dynamics (public announcement, or action models in general). Another topic is a deeper investigation of possible applications in knowledge representation.

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