Decidability of modal logics of non-k-colorable graphs

Ilya Shapirovsky[1] [0000−0001−7434−5894]

New Mexico State University, USA
ilshapir@nmsu.edu

Abstract. We consider the bimodal language, where the first modality is interpreted by a binary relation in the standard way, and the second is interpreted by the relation of inequality. It follows from Hughes (1990), that in this language, non-k-colorability of a graph is expressible for every finite k. We show that modal logics of classes of non-k-colorable graphs (directed or non-directed), and some of their extensions, are decidable.

Keywords: chromatic number · modal logic · difference modality · decidability · finite model property · filtration

1 Introduction

It is known that a non-k-colorability of a graph can be expressed by propositional modal formulas [Hug90]. In [GHV04], such formulas were used to construct a canonical logic which cannot be determined by a first-order definable class of relational structures; this gave a solution of a long-standing problem by Fine [Fin75].

In this paper, we are interested in decidability of modal logics given by axioms of non-k-colorability, and some of their extensions. We consider the bimodal language, where the first modality is interpreted by a binary relation in the standard way, and the second (difference modality) is interpreted by the relation of inequality.

The paper has the following structure. Section 2 provides preliminary syntactic and semantic facts. In Section 3, the finite model property and decidability are shown for logics of non-k-colorable graphs. In Section 4, these results are obtained for the connected non-directed case. Further results on the finite model property of logics of non-k-colorable graphs are obtained in Section 5. A discussion is given in Section 6.

2 Preliminaries

We assume that the reader is familiar with basic notions in modal logic (see, e.g., [CZ97,BdRV01] for the references). Below we briefly remind some of them.

Modal syntax and relational semantics. The set of *n-modal formulas* is built from a countable set of *variables* $\mathrm{PV} = \{p_0, p_1, \ldots\}$ using Boolean connectives $\bot, \to$ and unary connectives $\Diamond_i$, $i < n$ (*modalities*). Other logical connectives are defined as abbreviations in the standard way, in particular $\Box_i\varphi$ denotes $\neg\Diamond_i\neg\varphi$.

An *n-frame* is a structure $F = (X, (R_i)_{i<n})$, where X is a non-empty set and $R_i \subseteq X \times X$ for $i < n$. A *valuation in a frame* F is a map $\mathrm{PV} \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the set of all subsets of X. A *(Kripke) model on* F is a pair (F, θ), where θ is a valuation. The *truth* of formulas in models is defined in the usual way:

- $M, x \models p_i$ iff $x \in \theta(p_i)$;
- $M, x \not\models \bot$;
- $M, x \models \varphi \to \psi$ iff $M, x \models \varphi$ implies $M, x \models \psi$;
- $M, x \models \Diamond_i\varphi$ iff there exists y such that xR_iy and $M, y \models \varphi$.

A formula φ is *true in a model* M, in symbols $M \vDash \varphi$, if $M, x \vDash \varphi$ for all x in M. A formula φ is *valid in a frame* F, in symbols $F \vDash \varphi$, if φ is true in every model on F. For a class $\mathcal{C}$ of structures (frames or models) and a set of formulas Φ, we write $\mathcal{C} \vDash \Phi$, if $S \vDash \varphi$ for all $S \in \mathcal{C}$ and $\varphi \in \Phi$.

For the standard notions of *generated* and *point-generated subframe* and *submodel*, and *p-morphism*, we refer the reader to [CZ97, Section 3.3] or [BdRV01, Sections 2.1 and 3.3].

Modal logics. A (*propositional normal n-modal*) *logic* is a set L of n-modal formulas that contains all classical tautologies, the axioms $\neg\Diamond_i\bot$ and $\Diamond_i(p_0 \vee p_1) \to \Diamond_i p_0 \vee \Diamond_i p_1$ for each $i < n$, and is closed under the rules of modus ponens, substitution and *monotonicity*; the latter means that for each $i < n$, $\varphi \to \psi \in L$ implies $\Diamond_i\varphi \to \Diamond_i\psi \in L$.[1] We write $L \vdash \varphi$ for $\varphi \in L$. For a set Φ of n-modal formulas, $L + \Phi$ is the smallest normal logic containing $L \cup \Phi$. For a formula φ, $L + \varphi$ abbreviates $L + \{\varphi\}$. K denotes the smallest unimodal logic.

An *L-frame* is a frame where L is valid.

For a class $\mathcal{C}$ of n-frames, the set of n-modal formulas φ such that $\mathcal{C} \vDash \varphi$ is called the *logic of* $\mathcal{C}$ and is denoted by $\mathrm{Log}\,\mathcal{C}$. It is straightforward that $\mathrm{Log}\,\mathcal{C}$ is a normal logic. Such logics are called *Kripke complete*. A logic has the *finite model property* (fmp), if it is the logic of a class of finite frames (by the cardinality of a frame or model we mean the cardinality of its domain). We say that L has the *exponential fmp*, if for every formula $\varphi \notin L$, φ is falsified in an L-frame of cardinality $\leq 2^{\ell(\varphi)}$, where $\ell(\varphi)$ is the number of subformulas of φ.

The *canonical model* $M_L = (X_L, (R_{i,L})_{i<n}, \theta_L)$ *of* L is built from maximal L-consistent sets X_L of n-modal formulas; the canonical relations and the valuation are defined in the standard way. Namely, for $\Gamma, \Delta \in X_L$, put $(\Gamma, \Delta) \in R_{i,L}$, if $\{\Diamond_i\varphi \mid \varphi \in \Delta\} \subseteq \Gamma$, and set $\theta_L(p) = \{\Gamma \in X_L \mid p \in \Gamma\}$ for $p \in \mathrm{PV}$. The following fact is well known, see e.g., [BdRV01, Chapter 4.2].

[1] For this version of the definition of normal modal logic, see, e.g., [BdRV01, Remark 4.7].

Proposition 1. *[Canonical model theorem]* $L \vdash \varphi$ *iff* $M_L \vDash \varphi$.

L is *canonical*, if L is valid in its *canonical frame* $F_L = (X_L, (R_{i,L})_{i<n})$. A formula φ is *canonical*, if $F_L \vDash \varphi$ whenever $\varphi \in L$.

Proposition 2. *Let* L *be a canonical* n*-modal logic. Then for any* n*-modal logic* $L' \supseteq L$, *we have* $F_{L'} \vDash L$.

This fact is well known and follows from a simple observation that $F_{L'}$ is a generated subframe of F_L.

Logics with the difference modality. It is known that adding the difference modality allows to increase the expressive power of propositional modal language (see, e.g., [dR92], [GG93] in the relational context, or [KS14] for topological semantics).

Is this paper we will consider bimodal ($n = 2$) and unimodal ($n = 1$) languages. We write $\Diamond$ for $\Diamond_0$, and $\langle \neq \rangle$ for $\Diamond_1$; likewise for boxes. We also use abbreviations $\exists \varphi$ for $\langle \neq \rangle \varphi \vee \varphi$ and $\forall \varphi$ for $[\neq]\varphi \wedge \varphi$.

For a unimodal frame $F = (X, R)$, let $F_{\neq}$ be the bimodal frame $(X, R, \neq_X)$, where $\neq_X$ is the inequality relation on X, i.e., the set of pairs $(x, y) \in X \times X$ such that $x \neq y$. For a class $\mathcal{F}$ of frames, put $\mathcal{F}_{\neq} = \{F_{\neq} \mid F \in \mathcal{F}\}$

For a unimodal logic L, let $L_{\neq}$ be the smallest bimodal logic that contains L and the following formulas:

$$p \to [\neq]\langle \neq \rangle p, \quad \langle \neq \rangle \langle \neq \rangle p \to \exists p, \quad \Diamond p \to \exists p. \tag{1}$$

Recall that the validity of $p \to [\neq]\langle \neq \rangle p$ in a frame (X, R, D) expresses that D is symmetric, the formula $\langle \neq \rangle \langle \neq \rangle p \to \exists p$ means that the relation $D \cup Id_X$ is transitive (Id_X denotes the diagonal relation on X), and the formula $\Diamond p \to \exists p$ expresses that $R \subseteq D \cup Id_X$; see, e.g., [dR92] for details.

In particular, it follows that we have the following characterization of bimodal point-generated frames that validate $\mathrm{K}_{\neq}$:

Proposition 3. $F = (X, R, D)$ *is a point-generated* $\mathrm{K}_{\neq}$*-frame iff* $\neq_X \subseteq D$.

The formulas (1) are Sahlqvist formulas, and hence are canonical (see, e.g., [CZ97, Theorem 10.30]). In particular, it follows that $\mathrm{K}_{\neq}$ is Kripke complete. It is well-known that this logic has the finite model property: for every non-theorem φ of $\mathrm{K}_{\neq}$, consider a submodel M of the canonical model of $\mathrm{K}_{\neq}$ generated by a point x where φ is refuted, and take a filtration of M.

Proposition 4 ([dR92]). $\mathrm{K}_{\neq}$ *is the logic of the class of all (finite) frames of the form* $(X, R, \neq_X)$.

This proposition follows from Proposition 3 and the following standard move that "repairs" D-reflexive points. For a point-generated $\mathrm{K}_{\neq}$-frame $F = (X, R, D)$, let $F^{(\neq)}$ be the frame $(Y, S, \neq_Y)$, where

$$\begin{aligned} Y &= \{(x, 0) : x \in X\} \cup \{(x, 1) : x \in X \,\&\, xDx\}, \\ (x, i)S(y, j) &\text{ iff } xRy. \end{aligned}$$

Let $f : X \to Y$ be the map defined by $f(x, i) = x$. Readily, f is a p-morphism from $F^{(\neq)}$ onto F. Now Proposition 4 follows from the p-morphism lemma (see, e.g., [BdRV01, Theorem 3.14(i)]).

The frame $F^{(\neq)}$ will be used later; we will call it the *repairing of* F.

3 Logics of non-k-colorable graphs

By a *graph* we mean a unimodal frame (X, R) in which R is symmetric. A *directed graph* is a unimodal frame. As usual, a *partition* $\mathcal{A}$ of a set X is a family of non-empty pairwise disjoint sets such that $X = \bigcup \mathcal{A}$.

Definition 1. Let X be a set, $R \subseteq X \times X$. A partition $\mathcal{A}$ of X is *proper*, if $\forall A \in \mathcal{A}\, \forall x \in A\, \forall y \in A\, \neg xRy$. Let

$$C(X, R) = \{|\mathcal{A}| : \mathcal{A} \text{ is a finite proper partition of } X\}.$$

Let $\chi(X, R)$ be the least k in $C(X, R)$, if $C(X, R) \neq \varnothing$, and ∞ otherwise.

In the case when R is symmetric, $\chi(X, R)$ is called the *chromatic number of the graph* (X, R).

Put

$$\chi_k^{>} = \forall \bigvee_{i<k} (p_i \wedge \bigwedge_{i \neq j < k} \neg p_j) \to \exists \bigvee_{i<k} (p_i \wedge \Diamond p_i).$$

Proposition 5 ([Hug90,GHV04]). *Let $F = (X, R, D)$ be a point-generated* $\mathrm{K}_{\neq}$*-frame. Then $\chi(X, R) > k$ iff $F \vDash \chi_k^{>}$.*

Remark 1. Formulas considered in [Hug90,GHV04] are formally different.

Proof. The premise of $\chi_k^{>}$ says that non-empty values of p_i's form a partition of X, the conclusion says that this partition is not proper. □

In particular, it follows that for every graph G,

$$\text{the chromatic number of } G > k \text{ iff } G_{\neq} \vDash \chi_k^{>}.$$

To show that logics of non-k-colorable graphs have the finite model property, we will use filtrations.

For a model $M = (W, (R_i)_{i<n}, \theta)$ and a set of n-modal formulas Γ, put

$$x \sim_\Gamma y \text{ iff } \forall \psi \in \Gamma\ (M, x \models \psi \text{ iff } M, y \models \psi).$$

For a formula φ, let $\mathrm{Sub}\,\varphi$ be the set of all subformulas of φ. A set Γ of formulas is Sub-*closed*, if $\mathrm{Sub}\,\varphi \subseteq \Gamma$ whenever $\varphi \in \Gamma$.

Definition 2. Let Γ be a Sub-closed set of formulas. A Γ*-filtration* of a model $M = (X, (R_i)_{i<n}, \theta)$ is a model $\widehat{M} = (\widehat{X}, (\widehat{R}_i)_{i<n}, \widehat{\theta})$ such that

1. $\widehat{X} = X/{\sim}$ for some equivalence relation $\sim$ such that ${\sim} \subseteq {\sim_\Gamma}$;

2. $\widehat{M}, [x] \models p$ iff $M, x \models p$ for all $p \in \Gamma$. Here $[x]$ is the $\sim$-class of x.
3. For all $i < n$, we have $(R_i)_\sim \subseteq \widehat{R}_i \subseteq (R_i)^\Gamma_\sim$, where

$$[x]\,(R_i)_\sim\,[y] \text{ iff } \exists x' \sim x\ \exists y' \sim y\ (x'\,R_i\,y'),$$
$$[x]\,(R_i)^\Gamma_\sim\,[y] \text{ iff } \forall \psi\ (\Diamond_i \psi \in \Gamma\ \&\ M, y \models \psi \Rightarrow M, x \models \Diamond_i \psi).$$

The relations $(R_i)_\sim$ are called the *minimal filtered relations*.

If $\sim = \sim_\Psi$ for some finite set of formulas $\Psi \supseteq \Gamma$, then $\widehat{M}$ is called a *definable Γ-filtration* of the model M.

The following fact is well known, see, e.g., [CZ97]:

Proposition 6 (Filtration lemma). *Suppose that Γ is a finite* Sub-*closed set of formulas and $\widehat{M}$ is a Γ-filtration of a model M. Then, for all points x in M and all formulas $\varphi \in \Gamma$, we have:*

$$M, x \models \varphi \textit{ iff } \widehat{M}, [x] \models \varphi.$$

For a bimodal formula φ, let $[\varphi]$ be the set of bimodal formulas that are substitution instances of φ (the axiom scheme).

Lemma 1. *Let $M = (X, R, D, \theta)$ be a bimodal model, $k < \omega$, $M \vDash [\chi_k^>]$, and let Γ be a finite* Sub-*closed set of bimodal formulas. Then for every finite $\Psi \supseteq \Gamma$, for every Γ-filtration $\widehat{M} = (X/{\sim_\Psi}, \widehat{R}, \widehat{D}, \widehat{\theta})$ of M, we have $\chi(X/{\sim_\Psi}, \widehat{R}) > k$.*

Remark 2. We do not make the assumption that (X, R, D) is a $\mathrm{K}_{\neq}$-frame or even that $M \vDash \mathrm{K}_{\neq}$. We also do not assume that $\chi(X, R) > k$: in general, $M \vDash [\chi_k^>]$ is a weaker condition.

Proof. Let $\widehat{X} = X/{\sim_\Psi}$. Since Ψ is finite, for every $A \in \widehat{X}$ there is a modal formula ψ_A such that

$$M, x \vDash \psi_A \text{ iff } x \in A. \tag{2}$$

Hence, for every $B \subseteq \widehat{X}$, for the formula $\varphi_B = \bigvee_{A \in B} \psi_A$ we have:

$$M, x \vDash \varphi_B \text{ iff } x \in \bigcup B. \tag{3}$$

We say that φ_B *defines* B.

Let $\mathcal{B}$ be a partition of $\widehat{X}$ and $|\mathcal{B}| = n \leq k$. Then $\{\bigcup B : B \in \mathcal{B}\}$ is a partition of X. Let $\varphi_0, \dots \varphi_{n-1}$ be formulas that define elements of $\mathcal{B}$. For $n - 1 < i < k$, let $\varphi_i = \bot$. By (3), we have

$$M \vDash \forall \bigvee_{i<k} (\varphi_i \wedge \bigwedge_{i \neq j < k} \neg \varphi_j).$$

The result of substitution of φ_i's for p_i's in $\chi_k^>$ is true in M, so

$$M \vDash \exists \bigvee_{i<k} (\varphi_i \wedge \Diamond \varphi_i).$$

It follows from (3) that for some i, for some $x, y \in \bigcup B_i$ we have xRy. Let $[x]_\Psi$ denote the $\sim_\Psi$-class of x. We have $[x]_\Psi, [y]_\Psi \in B_i$. Since $\widehat{R}$ contains the minimal filtered relation, $[x]_\Psi \widehat{R} [y]_\Psi$. So $\mathcal{B}$ is not a proper partition of $(\widehat{X}, \widehat{R})$. □

Recall that the modal formula $p \to \Box\Diamond p$ expresses the symmetry of a binary relation. Let KB be the smallest unimodal logic containing this formula. It is well known that this logic is canonical.

Theorem 1. *For each $k < \omega$, the logics $\mathrm{K}_{\neq} + \chi_k^{>}$ and $\mathrm{KB}_{\neq} + \chi_k^{>}$ have the exponential finite model property and are decidable.*

Proof. Let $M_1 = (X_1, R_1, D_1, \theta_1)$ and $M_2 = (X_2, R_2, D_2, \theta_2)$ be the canonical models of the logics $\mathrm{K}_{\neq} + \chi_k^{>}$ and $\mathrm{KB}_{\neq} + \chi_k^{>}$, respectively. By Proposition 2, the canonical frames (X_1, R_1, D_1) and (X_2, R_2, D_2) validate the logic $\mathrm{K}_{\neq}$, and also R_2 is symmetric.

Let L be one of these logics, $\varphi \notin L$. Then φ is false at a point x in the canonical model of L. Let $M = (Y, R, D, \theta)$ be its submodel generated by x. By Proposition 3, for all $y, z \in Y$ we have:

$$\text{if } y \neq z, \text{ then } yDz. \tag{4}$$

Let $\Gamma = \mathrm{Sub}\,\varphi$, $\sim\, = \,\sim_\Gamma$. Put $\widehat{Y} = Y/{\sim}$, and consider the filtration $\widehat{M} = (\widehat{Y}, R_\sim, D_\sim, \widehat{\theta})$. Clearly, the size of $\widehat{Y}$ is bounded by $2^{\ell(\varphi)}$

By Filtration lemma (Proposition 6), φ is falsified in $\widehat{M}$. Let us show that the frame $(\widehat{Y}, R_\sim, D_\sim)$ validates L.

From (4), it follows that $(\widehat{Y}, R_\sim, D_\sim)$ validates the logic $K_{\neq}$. In the case of symmetric R, the minimal filtered relation $R_\sim$ is also symmetric. Finally, by Lemma 1, $\chi(\widehat{Y}, R_\sim) > k$. By Proposition 5, $(\widehat{Y}, R_\sim, D_\sim)$ validates L.

Hence L is complete with respect to its finite frames. ◻

Theorem 2. *Let $\mathcal{G}^{>k}$ be the class of graphs G such that $\chi(G) > k$, and let $\mathcal{D}^{>k}$ be the class of directed graphs G such that $\chi(G) > k$. Then $\mathrm{Log}\,\mathcal{G}^{>k}_{\neq} = \mathrm{KB}_{\neq} + \chi_k^{>}$, and $\mathrm{Log}\,\mathcal{D}^{>k}_{\neq} = \mathrm{K}_{\neq} + \chi_k^{>}$.*

Proof. By Theorem 1, the logics $\mathrm{K}_{\neq} + \chi_k^{>}$ and $\mathrm{KB}_{\neq} + \chi_k^{>}$ are complete with respect to their finite point-generated frames.

Consider a point-generated $\mathrm{K}_{\neq}$-frame $F = (X, R, D)$ and its repairing $F^{(\neq)} = (Y, S, \neq_Y)$. Recall that F is a p-morphic image of $F^{(\neq)}$. Let $\mathcal{A}$ be a partition of Y, $|\mathcal{A}| \leq k$. Consider the following partition $\mathcal{B}$ of X: $B \in \mathcal{B}$ iff there is $A \in \mathcal{A}$ such that $B = \{x : (x, 0) \in A\}$ and $B \neq \varnothing$.

Assume that $\chi(X, R) > k$. It follows that for some $B \in \mathcal{B}$ and some $x, y \in B$ we have xRy. Then for some $A \in \mathcal{A}$ we have $(x, 0), (y, 0) \in A$ and $(x, 0)S(y, 0)$. Thus, $\mathcal{A}$ is not a proper partition of (Y, S). Hence, $\chi(Y, S) > k$. This completes the proof in the directed case: $\mathrm{Log}\,\mathcal{D}^{>k}_{\neq} = \mathrm{K}_{\neq} + \chi_k^{>}$.

Clearly, if R is symmetric, then S is symmetric is well. This observation completes the proof in the non-directed case. ◻

Remark 3. These theorems can be extended for the case of graphs where relation is irreflexive, if instead of the formula $\Diamond p \to \exists p$ in the definition of $L_{\neq}$ we use the formula $\Diamond p \to \langle\neq\rangle p$. Then in any frame (X, R, D) validating this version of $L_{\neq}$, the second relation contains R, and so if a point is R-reflexive, it is also

D-reflexive. In this case, the repairing $F^{(\neq)}$ should be modified in the following way:

$$\begin{aligned} Y &= \{(x,0) : x \in X\} \cup \{(x,i) : x \in X \,\&\, xDx \,\&\, 0 < i \leq k\}, \\ (x,i)S(y,j) &\text{ iff } xRy \,\&\, ((x,i) \neq (y,j)). \end{aligned}$$

Then S is irreflexive, the map $(x,i) \mapsto x$ remains a p-morphism, and R-reflexive points in F become cliques of size $> k$. Also, it follows that $\chi(Y,S) > k$ whenever $\chi(X,R) > k$.

4 Logics of connected graphs

A frame $F = (X,R)$ is *connected*, if for any points x,y in X, there are points $x_0 = x, x_1, \ldots, x_n = y$ such that for each $i < n$, $x_i R x_{i+1}$ or $x_{i+1} R x_i$.

Let Con be the following formula:

$$\exists p \wedge \exists \neg p \to \exists (p \wedge \Diamond \neg p). \tag{5}$$

Proposition 7. *Let $F = (X,R,D)$ be a point-generated* $\mathrm{KB}_{\neq}$*-frame. Then (X,R) is connected iff $F \vDash$* Con.

Proof. Assume that (X,R) is connected and M is a model on F such that $\exists p \wedge \exists \neg p$ is true (at some point) in M. Hence there are points x,y in M such that $M,x \vDash p$ and $M,y \vDash \neg p$. Then there are $x_0 = x, x_1, \ldots, x_n = y$ such that $x_i R x_{i+1}$ for each $i < n$. Let $k = \max\{i : M, x_i \vDash p\}$. Then $M, x_k \vDash p \wedge \Diamond \neg p$. Hence Con is valid in F.

Assume that (X,R) is not connected. Then there are x,y in X such that $(x,y) \notin R^*$, where R^* is the reflexive transitive closure of R. Put $\theta(p)$ $=\{z : (x,z) \in R^*\}$s. In the model $M = (F,\theta)$, we have $M \vDash \exists p \wedge \exists \neg p$. On the other hand, at every point z in M we have $M,z \vDash p \to \Box p$, so the conclusion of Con is not true in M. So Con is not valid in F. ◻

In particular, it follows that for every graph G,

$$G \text{ is connected iff } G_{\neq} \vDash \textsc{Con}.$$

Remark 4. There are different ways to express connectedness in propositional modal languages [She90]. In particular, in the directed case, the connectedness can be expressed by the following modification of (5):

$$\exists p \wedge \exists \neg p \to \exists (p \wedge \Diamond \neg p) \vee \exists (\neg p \wedge \Diamond p);$$

Following the line of [She90], one can modally express the property of a graph to have at most n connected components for each finite n.

It is known that in many cases, adding axioms of connectedness preserves the finite model property [She90,GH18]. The following lemma shows that this is the case in our setting as well.

Lemma 2. *Assume that (X,R,D) is a point-generated* $\mathrm{KB}_{\neq}$*-frame. Let* $M = (X,R,D,\theta)$ *be a model such that* $M \vDash [\textsc{Con}]$*, and let* Γ *be a finite* Sub*-closed set of bimodal formulas. Then for every finite* $\Psi \supseteq \Gamma$*, for every* Γ*-filtration* $\widehat{M} = (X/{\sim_\Psi}, \widehat{R}, \widehat{D}, \widehat{\theta})$ *of* M*,* $(X/{\sim_\Psi}, \widehat{R})$ *is connected.*

Remark 5. Similarly to Lemma 1, connectedness of (X,R) does not follow from $M \vDash [\textsc{Con}]$.

Proof. Let $\widehat{X} = X/{\sim_\Psi}$, c the number of elements in $\widehat{X}$. We recursively define c distinct elements $A_0, \ldots, A_{c-1}$ of $\widehat{X}$, and auxiliary sets $\widehat{Y}_n = \{A_0, \ldots, A_n\}$, $\widehat{R}_n = \widehat{R} \cap (\widehat{Y}_n \times \widehat{Y}_n)$ for $n < c$ such that

$$\text{the restriction } (\widehat{Y}_n, \widehat{R}_n) \text{ of } (\widehat{X}, \widehat{R}) \text{ to } \widehat{Y}_n \text{ is connected.} \tag{6}$$

Let A_0 be any element of $\widehat{X}$. The frame $(\widehat{Y}_0, \widehat{R}_0)$ is connected, since it is a singleton.

Assume $0 < n < c$ and define A_n. By the same reasoning as in Lemma 1, there is a formula φ_n such that

$$M, x \vDash \varphi_n \text{ iff } x \in A_i \text{ for some } i < n. \tag{7}$$

The formula

$$\exists \varphi_n \wedge \exists \neg \varphi_n \to \exists(\varphi_n \wedge \Diamond \neg \varphi_n). \tag{8}$$

is a substitution instance of Con, so it is true in M. Let $V = \bigcup \widehat{Y}_{n-1}$. The set $\widehat{Y}_{n-1}$ has $n < c$ elements, so there are points x, y in X such that $x \in V$, and $y \notin V$. So $M, x \vDash \varphi_n$ and $M, y \vDash \neg\varphi_n$. By Proposition 3, the premise of (8) is true in M. Hence we have $M, z \vDash \varphi_n \wedge \Diamond\neg\varphi_n$ for some z in M. Then $z \in V$ and there exists u in $X \setminus V$ with zRu. Since $\widehat{R}$ contains the minimal filtered relation, $[z]_\Psi \widehat{R} [u]_\Psi$. We put $A_n = [u]_\Psi$. By the hypothesis (6), $(\widehat{Y}_{n-1}, \widehat{R}_{n-1})$ is connected, and so $(\widehat{Y}_n, \widehat{R}_n)$ is connected as well.

Finally, observe that $(\widehat{Y}_{c-1}, \widehat{R}_{c-1})$ is the frame $(\widehat{X}, \widehat{R})$. ◻

Theorem 3. *For each* $k < \omega$*, the logics* $\mathrm{KB}_{\neq} + \{\textsc{Con}, \Diamond\top\}$ *and* $\mathrm{KB}_{\neq} + \{\chi_k^{>}, \textsc{Con}, \Diamond\top\}$ *have the exponential finite model property and are decidable.*

Proof. Similar to the proof of Theorem 1. Let φ be a non-theorem of one these logics, M a point-generated submodel of the canonical model of the logic where φ is falsified. Consider the frame F of the minimal filtration of M via the subformulas of φ. We only need to check that F validates $\Diamond\top$ and Con (validity of other axioms was checked in the proof of Theorem 1). That $\Diamond\top$ is valid is trivial. The validity of Con follows from Lemma 2 and Proposition 7. ◻

Theorem 4. *Let* $\mathcal{C}$ *be the class of connected non-singleton graphs,* $\mathcal{C}^{>k}$ *the class of non-k-colorable graphs in* $\mathcal{C}$*. Then* $\mathrm{Log}\,\mathcal{C}_{\neq} = \mathrm{KB}_{\neq} + \{\textsc{Con}, \Diamond\top\}$*, and* $\mathrm{Log}\,\mathcal{C}^{>k}_{\neq} = \mathrm{KB}_{\neq} + \{\chi_k^{>}, \textsc{Con}, \Diamond\top\}$.

Proof. Similar to the proof of Theorem 2. Completeness of $\mathrm{KB}_{\neq} + \{\textsc{Con}, \Diamond\top\}$ and $\mathrm{KB}_{\neq} + \{\chi_k^{>}, \textsc{Con}, \Diamond\top\}$ with respect to their finite point-generated frames follows from Theorem 3.

Assume that $F = (X, R, D)$ is a point-generated $\mathrm{KB}_{\neq}$-frame, and (X, R) is connected and validates $\Diamond\top$. Consider the repairing $F^{(\neq)} = (Y, S, \neq_Y)$ of F. Clearly, $\Diamond\top$ is valid in $F^{(\neq)}$. Let (x, i) and (y, j) be in Y. First, assume that $x \neq y$. Since (X, R) is connected, there is a path between x and y in (X, R), which induces a path between (x, i) and (y, j) in (Y, S) by the definition of S. Now consider two distinct points (x, i) and (x, j) in Y. Since $\Diamond\top$ is valid in F, we have xRy for some y in F. Then we have $(x, i)S(y, 0)$ and $(x, j)S(y, 0)$. It follows that (Y, S) is connected and so $F^{(\neq)}$ validates $\textsc{Con}$ by Proposition 7.

That other axioms hold in $(Y, S, \neq_Y)$ was shown in Theorem 2. Now the theorem follows from the fact that F is a p-morphic image of $F^{(\neq)}$. □

5 Corollaries

Lemmas 1 and 2 were stated in a more general way than it was required for the proofs of Theorems 1 and 3. The aim of using these, more technical, statements is the following.

Definition 3. A logic L *admits (rooted) definable filtration*, if for any (point-generated) model M with $M \vDash L$, and for any finite Sub-closed set of formulas Γ, there exists a finite model $\widehat{M}$ with $\widehat{M} \vDash L$ that is a definable Γ-filtration of M.

In [KSZ14,KSZ20], it was shown that if a modal logic L admits definable filtration, then its enrichments with modalities for the transitive closure and converse relations also admit definable filtration.

Notice that if $L = \mathrm{K}_2 + \varphi$, where K_2 is the smallest bimodal logic and φ is a bimodal formula, then $M \vDash L$ iff $M \vDash [\varphi]$. In particular, the logics $\mathrm{K}_2 + \chi_k^{>}$ admit definable filtration by Lemma 1. This fact immediately extends to any bimodal logic $L + \chi_k^{>}$, whenever L admits definable filtration.

Corollary 1. *If a bimodal logic L admits definable filtration, then all $L + \chi_k^{>}$ admit definable filtration, and consequently have the finite model property.*

Applying Lemmas 1 and 2 to the case of point-generated models, we obtain the following version of Theorems 1 and 3.

Corollary 2. *Assume that a bimodal logic L admits rooted definable filtration, $k < \omega$. Then $L + \chi_k^{>}$ has the finite model property. If also L extends $\mathrm{KB}_{\neq}$, then $L + \{\chi_k^{>}, \textsc{Con}\}$ has the finite model property.*

6 Discussion

We have shown that modal logics of different classes of non-k-colorable graphs are decidable. It is of definite interest to consider logics of certain graphs, for which the chromatic number is unknown.

Let $F = (\mathbb{R}^2, R_{=1})$ be the unit distance graph of the real plane. It is a long-standing open problem what is $\chi(F)$ (Hadwiger–Nelson problem). It is known that $5 \leq \chi(F) \leq 7$ [DG18],[EI20].

Let $L_{=1}$ be the bimodal logic of the frame $(\mathbb{R}^2, R_{=1}, \neq_{\mathbb{R}^2})$. In modal terms, the problem asks whether $\chi_5^>, \chi_6^>$ belong to $L_{=1}$. We know that $L_{=1}$ extends $L = \mathrm{KB}_{\neq} + \{\chi_4^>, \textsc{Con}, \Diamond\top, \Diamond p \to \langle\neq\rangle p\}$ (it is an easy corollary of the above results that L is decidable). However, $L_{=1}$ contains extra formulas. For example, consider the formulas

$$\mathrm{P}(k,m,n) = \bigwedge_{i<k} \Diamond^m \Box^n p_i \to \bigvee_{i \neq j < k} \Diamond^m (p_i \wedge p_j).$$

For various k, m, n, $\mathrm{P}(k,m,n)$ is in $L_{=1}$ (and not in L); this can be obtained from known solutions for problems of packing equal circles in a circle.

Problem 1. Is $L_{=1}$ decidable? Finitely axiomatizable? Recursively enumerable? Does it have the finite model property?

Notice that instead of considering the difference auxiliary modality, one can consider the logic with the universal modality: this logic is a fragment of $L_{=1}$, but still can express formulas $\chi_k^>$.

Let $V_r \subseteq \mathbb{R}^2$ be a disk of radius r. It follows from de Bruijn–Erdős theorem, that if $\chi(F) > k$, then $\chi(V_r, R_{=1}) > k$ for some r.

Let $L_{=1,r}$ be the unimodal logic of the frame $(V_r, R_{=1})$. If $r > 1$, then the universal modality is expressible, and so are the formulas $\chi_k^>$. Hence, it is of interest to consider axiomatization problems and algorithmic problems for these logics.

Problem 2. To analyze the unimodal logics $L_{=1,r}$.

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