An Axiom System for Basic Hybrid Logic with Propositional Quantification

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July 13, 2023

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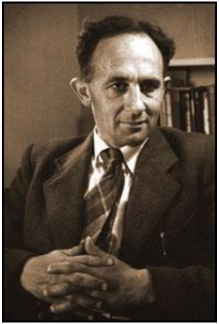
Plan of talk

- **I** **Basic hybrid logic**
- **II** **Propositional quantifiers**
- **III** **Arthur Prior's Q operator**
- **IV** **An axiom system**
- **V** **Prior's two views on nominals**
- **VI** **Concluding remarks**

Part I

Basic hybrid logic

Hybrid logic was invented by Arthur Prior (1914-1969)



- Prior's aim was to solve a problem in the philosophy of time
- Technically, he increased the expressive power of ordinary modal logic
- First key idea in hybrid logic:
  add nominals to the modal language, propositional symbols true at precisely one world/time/person/state/location: for example patrick and julie
- Second key idea in hybrid logic:
  build satisfaction statements, formulas like $@_{patrick}\,philosopher$ and $@_{julie}\,physicist$

Standard nominals

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It is 12:30 November 22nd 1963

which is true at a particular time, but false at all other times

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a *It is 12:30 November 22nd 1963*

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Remedy:
Add new propositional symbols a, b, c, ... called *nominals*

A nominal a is true at exactly one time, so it refers to a time

Satisfaction operators

Again, ordinary tense logic cannot formalize statements involving reference to particular times, e.g.

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$@_a p$ *At 12:30 Nov. 22nd 1963, J.F. Kennedy is shot*

which is about what happens at a particular time

Remedy: For each nominal a add a *satisfaction operator* $@_a$

The satisfaction operator $@_a$ moves the time of evaluation to the time referred to by the nominal a

Thus, a formula $@_a\phi$ is true iff ϕ is true at the time a refers to

Important distinction for later use

Nominals are used in two syntactically distinct ways:

- If the nominal a appears as a subscript to @, then we say it occurs in *operator position*
- If the nominal a occurs as an atomic formula, then we say it occurs in *formula position*

Part II

Propositional quantifiers

Examples of first-order/second-order quantifiers

Compare the following two arguments:

All logicians are mortal	All logicians are humble
Arthur is a logician	Arthur is all a logician is
Therefore Arthur is mortal	Therefore Arthur is humble

Here they are in first-order and second-order notation:

$$\frac{\forall x(Lx \to Mx) \quad La}{\therefore\ Ma} \qquad \frac{\forall x(Lx \to Hx) \quad \forall P(\forall x(Lx \to Px) \to Pa)}{\therefore\ Ha}$$

Note that on the right, quantifiers can bind predicate positions

The first-order/second-order divide is not innocent!

First-order logic is axiomatisable, but second-order logic is not!

In 1950, Leon Henkin showed how to 'tame' second-order logic:

Instead of interpreting second-order quantifiers as ranging over **all** subsets of the domain of quantification, view them as ranging over **a pre-selected set** of admissible subsets

Hence, instead of working with models dictated by set theory (all the subsets), work with deliberately pre-structured models

The pre-structured models have to satisfy certain intuitive constraints (closure properties)

Frames and closure properties

Definition

A *general frame* is a triple $\langle W, R, \Pi \rangle$ where W is a non-empty set (worlds), R is a binary relation on W (the accessibility relation) and Π is a non-empty collection of subsets of W (the admissible subsets) closed under the following operations:

- relative complement: if $X \in \Pi$, then $W - X \in \Pi$
- intersection: if $X, Y \in \Pi$, then $X \cap Y \in \Pi$
- modal projection: if $X \in \Pi$, then $\{w \in W : \forall v(wRv \rightarrow v \in X)\} \in \Pi$

Note: If $\Pi = \mathscr{P}(W)$, then we are working with a standard frame.

Closure properties guarantees that interpretations of booleans and modalities are admissible sets

But note that interpretations of nominals (singletons) aren't necessarily admissible

Models and truth-conditions

Definition

A *general model* $\mathfrak{M}$ based on a general frame $\langle W, R, \Pi \rangle$ is a tuple $\langle W, R, \Pi, N, V \rangle$ where $N : NOM \to W$ and $V : PROP \to \Pi$.
The truth-conditions are as follows:

$$
\begin{array}{rcl}
\mathfrak{M}, w \models p & \textit{iff} & w \in V(p) \textit{ where } p \in PROP \\
\mathfrak{M}, w \models i & \textit{iff} & w = N(i) \\
\mathfrak{M}, w \models \neg\varphi & \textit{iff} & \textit{it is not the case that } \mathfrak{M}, w \models \varphi \\
\mathfrak{M}, w \models \varphi \wedge \psi & \textit{iff} & \mathfrak{M}, w \models \varphi \textit{ and } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models \Box\varphi & \textit{iff} & \textit{for all } v \in W \textit{ such that } wRv, \textit{we have } \mathfrak{M}, v \models \varphi \\
\mathfrak{M}, w \models @_i\varphi & \textit{iff} & \mathfrak{M}, N(i) \models \varphi \\
\mathfrak{M}, w \models \forall p\varphi & \textit{iff} & \textit{for all } \mathfrak{M}' = \langle W, R, \Pi, N, V' \rangle \\
& & \textit{such that } V'(q) = V(q) \textit{ whenever } q \neq p, \\
& & \textit{we have } \mathfrak{M}', w \models \varphi
\end{array}
$$

In connection with Prior's Q operator, we consider general frames where $R = W \times W$.

Part III

Arthur Prior's Q operator

Definition of nominals using the Q operator

Instead of introducing nominals as a second sort of propositional symbol, Prior sometimes defined them using his Q operator.

> *For 'p is an individual' (or an instant, or a possible total world-state) we write Qp. If we have propositional quantifiers, we can define Qp thus:*

$$Qp = \Diamond p \wedge \forall q(\Box(p \to q) \vee \Box(p \to \neg q))$$

Here $\Box$ means *true at all worlds* and $\Diamond$ means *true at some world* (universal modalities)

So Qp says that p is *possible* and *maximal*: p is true *somewhere* and p strictly implies *every* proposition q or its negation
(Read $\Box(p \to q)$ as "p is included in q" etc.)

Interpretation of the $\forall q$ quantifier in the Q operator

If $\forall q$ ranges over **all** subsets of worlds, then Qp says that p is a singleton set, in other words, a standard nominal

But if $\forall q$ is given a general (Henkin) semantics where it ranges over **a pre-selected set** of subsets of worlds[2] then Qp says that p is an atom, that is, a non-empty minimal preselected subset

So the general semantics gives a non-standard 'species' of nominals

[2]With closure properties giving 'enough logical structure' cf. earlier.

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To make the differences between our two species of nominals concrete, it will help to have a proof-system

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Part IV

An axiom system

Axioms and rules for the propositional quantifier

First a definition: A *soft-QF* formula has no nominals in formula position ("*soft*") and no propositional quantifiers ("*QF*")

We then extend an axiom system for basic hybrid logic with the following axioms

Q1: $\forall p(\varphi \to \psi) \to (\varphi \to \forall p\psi)$
where φ contains no free occurrences of p

Q2-sqf: $\forall p\varphi \to \varphi[\psi/p]$
where $\varphi[\psi/p]$ is a soft-QF substitution

Barcan$_{@}$: $\forall p @_i\varphi \leftrightarrow @_i\forall p\varphi$

and the following rule

Gen$_\forall$: If $\vdash \varphi$ then $\vdash \forall p\varphi$

The side-condition on *Q2-sqf* is where the distinction between the standard and non-standard nominals becomes important

Why only substitute soft-QF formulas?

The axiom *Q2-sqf* *without* the restriction to soft-QF formulas is sound wrt. the standard semantics

But *Q2-sqf* *without* the restriction to soft formulas is *not* sound wrt. the general semantics

However, our axiom system with the restricted version of *Q2-sqf* is sound and complete wrt. the general semantics

Thus, we have found a proof-system that can deal with the new species of nominals!

Prior's two views on nominals

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Our two species of nominals allign well with Prior's two views, expressed in a single sentence:

> *We might . . . equate the instant a with a conjunction of all those propositions which would ordinarily be said to be true at that instant, or we might equate it with some proposition which would ordinarily be said to be true at that instant only, and so could serve as an index of it.*

Prior's 'index' view of nominals matches Qp under the standard interpretation of propositional quantifiiers

Prior's 'content' view fits well with Qp under the general interpretation (involving sets of relevant propositions)

Nominals: Index or content?

Under the index view: to assert $@_i p$ is to assert that p holds at some world and that this world is called i (that is, you name the world i and stipulate that p holds there)

Under the content view: to assert $@_i p$ is to assert that p is true at the world named i since i implicitly embodies the information that p (and much else)

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> *'Possible worlds' are* stipulated, *not* discovered *by powerful telescopes*

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We have not tried to decide on this issue, we merely wished to make the distinction clearer!

VI

Concluding remarks

Summing up

If the standard semantics is chosen for the propositional quantifiers, the Q operator gives standard nominals

But if the general semantics is chosen, the Q operator gives a new species of nominals

We have provided an axiom system that is sound and complete with respect to the general semantics

More information on hybrid logic

Carlos Areces and Balder ten Cate's chapter on hybrid logic in *Handbook of Modal Logic*, Elsevier, 2007

Torben Braüner's entry on hybrid logic in *Stanford Encyclopaedia of Philosophy*

Torben Braüner's book *Hybrid Logic and its Proof-Theory*, Springer, 2011

Patrick Blackburn's paper Arthur Prior and Hybrid Logic, *Synthese*, volume 150, 2006