Learning Context-Free Grammars from Positive Data and Membership Queries

(based on joint work with Ryo Yoshinaka)

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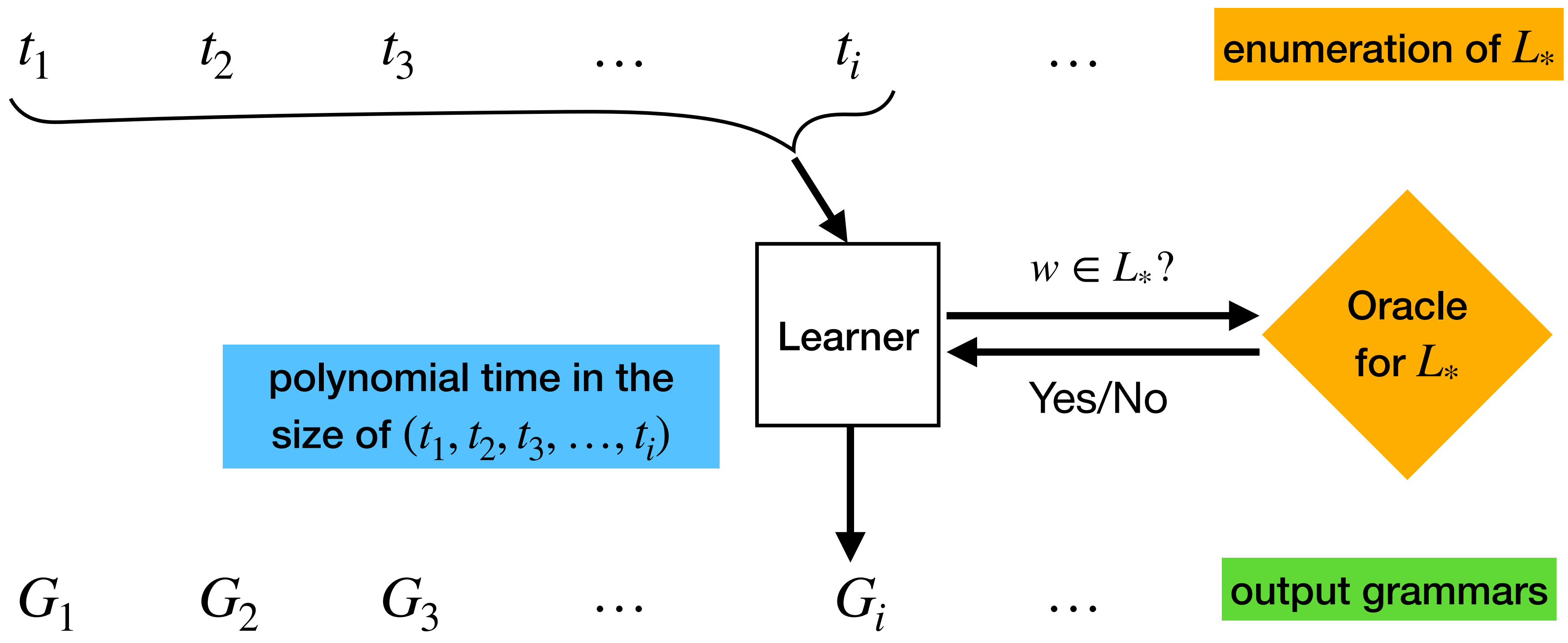
Background

- Algorithmic Learning Theory
  - ✳ **Grammatical Inference**
    - Finite automata

      → satisfactory learning algorithms
    - Context-free grammars / pushdown automata

      → special subclasses
      - deterministic one-counter machines
      - context-deterministic grammars

Learning from Positive Data and Membership Queries



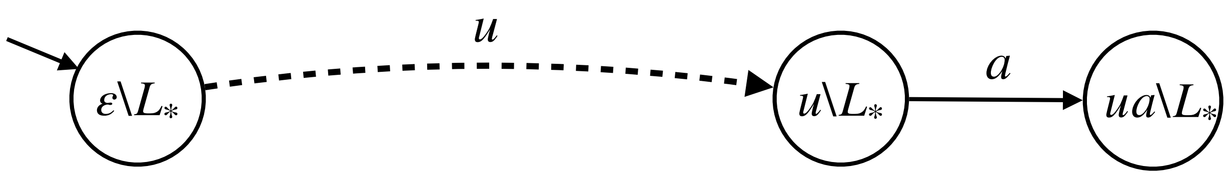
- There exists l such that $G_l = G_{l+1} = \cdots$ and $L(G_l) = L_*$.
- No "delaying trick".

Regular Languages

- **Left quotient** of a language $L \subseteq \Sigma^*$ by a string $u \in \Sigma^*$:

$$u \backslash L = \{ \, x \in \Sigma^* \mid ux \in L \, \}$$

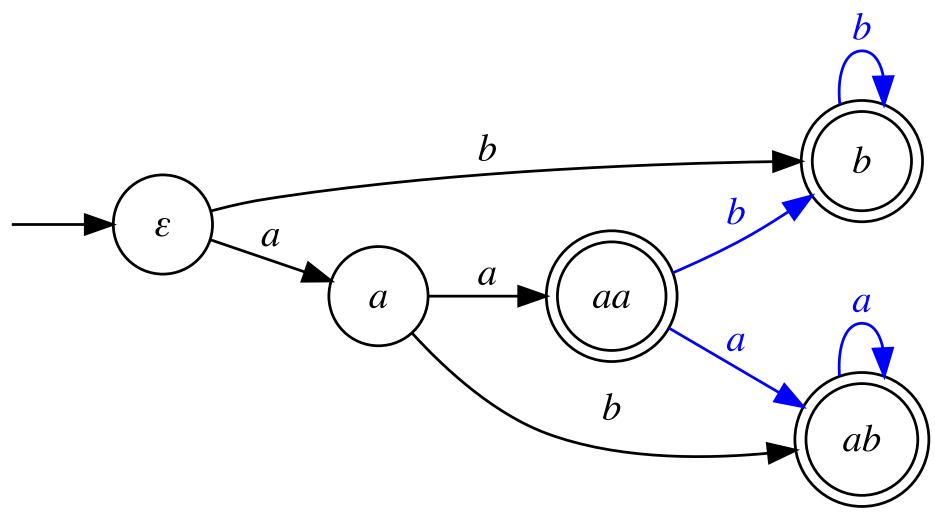
- A language L is regular if and only if $\{ \, u \backslash L \mid u \in \Sigma^* \, \}$ is finite.
- Every regular language has a canonical **minimal DFA**.



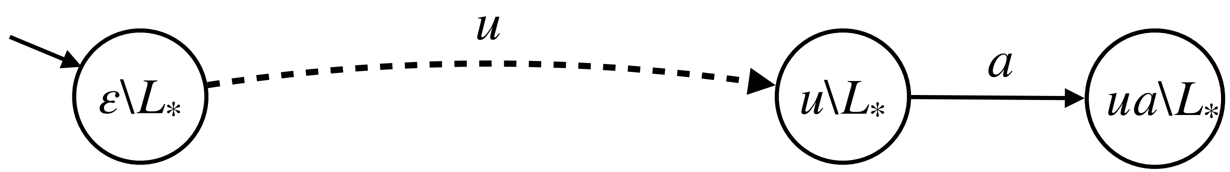
states = left quotients

Example

$$L_* = aba^* \cup bb^* \cup aa(a^* \cup b^*)$$



$$\varepsilon\backslash L_* = L_*$$
$$a\backslash L_* = ba^* \cup a(a^* \cup b^*)$$
$$b\backslash L_* = b^*$$
$$aa\backslash L_* = a^* \cup b^*$$
$$ab\backslash L_* = a^*$$



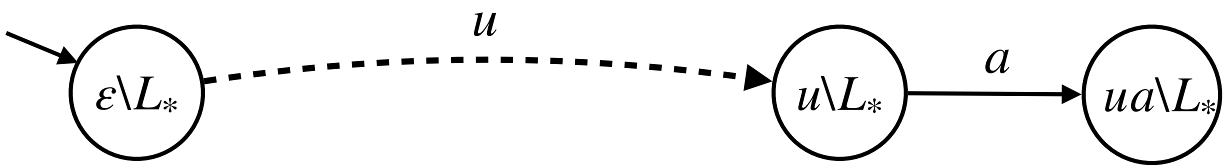
states = left quotients

$$bb\backslash L_* = b^* = b\backslash L_*$$
$$aaa\backslash L_* = a^* = ab\backslash L_*$$
$$aab\backslash L_* = b^* = b\backslash L_*$$
$$aba\backslash L_* = a^* = ab\backslash L_*$$

Right-Linear Grammars

- Right-linear CFG G_* corresponding to a minimal DFA for L_*:

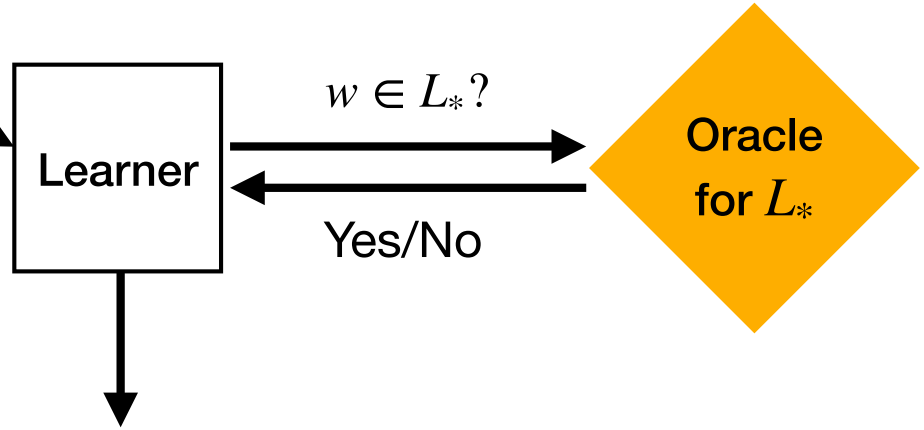
$$G_* = (N_*, \Sigma, P_*, S)$$
$$N_* = \{\ u \backslash L_* \mid u \in \Sigma^*\ \}$$
$$P_* = \{\ u \backslash L_* \to a\,(ua \backslash L_*) \mid u \in \Sigma^*, a \in \Sigma\ \} \cup \{\ u \backslash L_* \to \varepsilon \mid u \in L_*\ \}$$
$$S = L_*\ (= \varepsilon \backslash L_*)$$



states = nonterminals = left quotients

Inference of Regular Languages

$$T = \{t_1, \ldots, t_i\}$$



$\mathrm{Pref}(T) = \{\, u \mid uv \in T \,\}$
$\mathrm{Suff}(T) = \{\, v \mid uv \in T \,\}$

$$G = (N, \Sigma, P, \langle\!\langle \varepsilon \rangle\!\rangle)$$

represents $u \backslash L_*$

length-lexicographic order on Σ^*

$$N = \{\, \langle\!\langle u \rangle\!\rangle \mid u \in \mathrm{Pref}(T) \wedge \forall v \in \mathrm{Pref}(T)\,(v \prec u \to$$
$$(\{\varepsilon\} \cup \Sigma)\,\mathrm{Suff}(T) \cap (v \backslash L_*) \neq (\{\varepsilon\} \cup \Sigma)\,\mathrm{Suff}(T) \cap (u \backslash L_*)\,\}$$

approximates $v \backslash L_* \neq u \backslash L_*$

$$P = \{\, \langle\!\langle u \rangle\!\rangle \to a\, \langle\!\langle v \rangle\!\rangle \mid \mathrm{Suff}(T) \cap (ua \backslash L_*) = \mathrm{Suff}(T) \cap (v \backslash L_*)\,\} \cup$$
$$\{\, \langle\!\langle u \rangle\!\rangle \to \varepsilon \mid u \in L_*\,\}$$

approximates $ua \backslash L_* = v \backslash L_*$

$$N_* = \{\ u\backslash L_* \mid u \in \Sigma^*\ \}$$

represents $u\backslash L_*$

lexicographic order on Σ^*

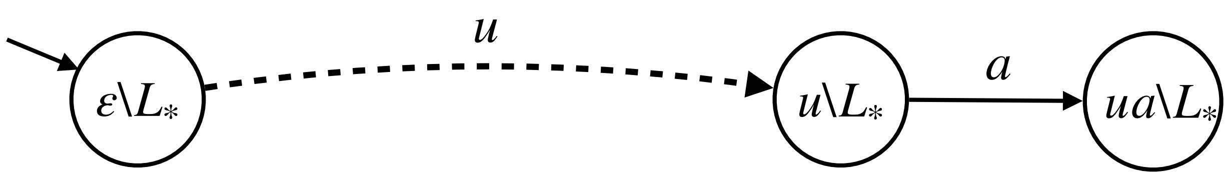
$$N = \{\ \langle\!\langle u \rangle\!\rangle \mid u \in \mathrm{Pref}(T) \wedge \forall v \in \mathrm{Pref}(T)\,(v \prec u \rightarrow$$
$$(\{\varepsilon\} \cup \Sigma)\,\mathrm{Suff}(T) \cap (v\backslash L_*) \neq (\{\varepsilon\} \cup \Sigma)\,\mathrm{Suff}(T) \cap (u\backslash L_*)\ \}$$

approximates $v\backslash L_* \neq u\backslash L_*$

$$P_* = \{\ u\backslash L_* \rightarrow a\,(ua\backslash L_*) \mid u \in \Sigma^*, a \in \Sigma\ \} \cup \{\ u\backslash L_* \rightarrow \varepsilon \mid u \in L_*\ \}$$
$$= \{\ u\backslash L_* \rightarrow a\,(v\backslash L_*) \mid ua\backslash L_* = v\backslash L_*\ \} \cup \{\ u\backslash L_* \rightarrow \varepsilon \mid u \in L_*\ \}$$

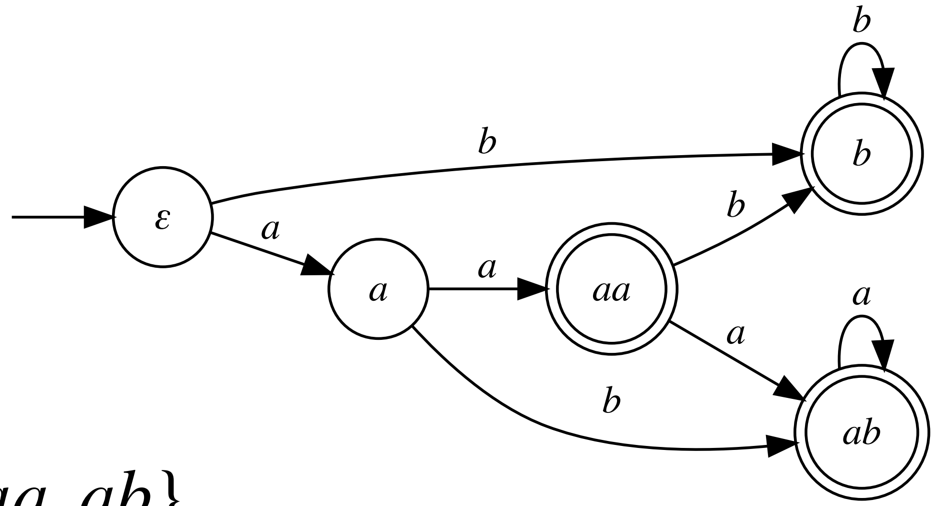
$$P = \{\ \langle\!\langle u \rangle\!\rangle \rightarrow a\,\langle\!\langle v \rangle\!\rangle \mid \mathrm{Suff}(T) \cap (ua\backslash L_*) = \mathrm{Suff}(T) \cap (v\backslash L_*)\ \} \cup$$
$$\{\ \langle\!\langle u \rangle\!\rangle \rightarrow \varepsilon \mid u \in L_*\ \}$$

approximates $ua\backslash L_* = v\backslash L_*$



Example

$$L_* = aba^* \cup bb^* \cup aa(a^* \cup b^*)$$



$$T = \{b, aa, ab\}$$

$$\mathrm{Pref}(T) = \{\varepsilon, a, b, aa, ab\}$$

$$\mathrm{Suff}(T) = \{\varepsilon, a, b, aa, ab\}$$

$$\{\varepsilon, a, b\}\,\mathrm{Suff}(T) \cap (\varepsilon \backslash L_*) = \{b, aa, ab, bb, aaa, aab\}$$

$$\{\varepsilon, a, b\}\,\mathrm{Suff}(T) \cap (a \backslash L_*) = \{a, b, aa, ab, ba, aaa, baa\}$$

$$\{\varepsilon, a, b\}\,\mathrm{Suff}(T) \cap (b \backslash L_*) = \{\varepsilon, b, bb\}$$

$$\{\varepsilon, a, b\}\,\mathrm{Suff}(T) \cap (aa \backslash L_*) = \{\varepsilon, a, b, aa, aaa, bb\}$$

$$\{\varepsilon, a, b\}\,\mathrm{Suff}(T) \cap (ab \backslash L_*) = \{\varepsilon, a, aa, aaa\}$$

represents $u \backslash L_*$

lexicographic order on Σ^*

$$N = \{\, \langle\!\langle u \rangle\!\rangle \mid u \in \mathrm{Pref}(T) \wedge \forall v \in \mathrm{Pref}(T)\, (v \prec u \to (\{\varepsilon\} \cup \Sigma)\, \mathrm{Suff}(T) \cap (v \backslash L_*) \neq (\{\varepsilon\} \cup \Sigma)\, \mathrm{Suff}(T) \cap (u \backslash L_*)\,) \,\}$$

approximates $v \backslash L_* \neq u \backslash L_*$

$$P = \{\, \langle\!\langle u \rangle\!\rangle \to a\, \langle\!\langle v \rangle\!\rangle \mid \mathrm{Suff}(T) \cap (ua \backslash L_*) = \mathrm{Suff}(T) \cap (v \backslash L_*) \,\} \cup \{\, \langle\!\langle u \rangle\!\rangle \to \varepsilon \mid u \in L_* \,\}$$

approximates $ua \backslash L_* = v \backslash L_*$

$$x \in (\{\varepsilon\} \cup \Sigma)\, \mathrm{Suff}(T) \cap (u \backslash L_*) \iff x \in (\{\varepsilon\} \cup \Sigma)\, \mathrm{Suff}(T) \wedge ux \in L_*$$

- N and P are determined by $\mathrm{Pref}(T)\, (\{\varepsilon\} \cup \Sigma)\, \mathrm{Suff}(T) \cap L_*$.
- $O(n^4)$ queries to the oracle for L_* suffice.

Context-Free Grammars

$$G = (N, \Sigma, P, S)$$

- What do nonterminals correspond to?

$$\frac{\text{left quotients}}{\text{regular languages}} = \frac{??}{\text{context-free languages}}$$

- The set of terminal strings derived from a nonterminal is included in some **quotient** of the language of the grammar:

$$S \Rightarrow_G^* uAv \quad \text{implies} \quad L_G(A) \subseteq u\backslash L(G)/v = \{\, x \in \Sigma^* \mid uxv \in L(G) \,\}$$

$$L_G(A) = \{\, x \in \Sigma^* \mid A \Rightarrow_G^* x \,\}$$
$$L(G) = L_G(S)$$

- It seems there's nothing further that can be said in general.

Simple Case: Grammars with Just One Nonterminal

Dyck language

$$\boxed{\begin{array}{l} S \to \varepsilon \\ S \to aSbS \end{array}} \qquad D_1 = \{\, x \in \{a, b\}^* \mid |x|_a = |x|_b \wedge \forall uv(uv = x \to |u|_a \geq |u|_b)\,\}$$

$$\pi: \qquad S \to w_0\, S\, w_1\ \ldots\ S\, w_k \qquad\qquad (w_i \in \Sigma^*)$$

When should π be in the hypothesized grammar?

$$\pi \text{ is } \textbf{valid} \overset{\text{def}}{\iff} L_* \supseteq w_0\, L_*\, w_1\ \ldots\ L_*\, w_k$$

Why is this reasonable?

$$\pi: \quad S \to w_0\, S\, w_1\ \ldots\ S\, w_k \qquad (w_i \in \Sigma^*)$$

$$\pi \text{ is } \textbf{valid} \overset{\text{def}}{\iff} L_* \supseteq w_0\, L_*\, w_1\ \ldots\ L_*\, w_k$$

- If π is not valid, π can't be in a correct grammar for L_*.

  If $x_1, \ldots, x_k \in L_*$, $w_0 x_1 w_1 \ldots x_k w_k \notin L_*$, and $L_* \subseteq L(G)$, then

$$\begin{aligned} S &\Rightarrow w_0\, S\, w_1\ \ldots\ S\, w_k \\ &\Rightarrow^* w_0\, x_1\, w_1\ \ldots\ x_k\, w_k \end{aligned}$$

- If all productions in G are valid, then $L(G) \subseteq L_*$.
- All productions in G_* are valid.

Context-Free Grammars

abbreviates three productions:
$S \to aD_1bS$, $S \to aA$, $S \to bU$

$$\begin{array}{|rl|} \hline S & \to aD_1bS \mid aA \mid bU \\ D_1 & \to \varepsilon \mid aD_1bD_1 \\ A & \to \varepsilon \mid aD_1bA \mid aA \\ U & \to \varepsilon \mid Ua \mid Ub \\ \hline \end{array}$$

$$\overline{D_1} = \{\, x \in \{a,b\}^* \mid \\ |x|_a \neq |x|_b \vee \exists uv(uv = x \wedge |u|_a < |u|_b) \,\}$$

$$L_G(A) = \{\, x \in \Sigma^* \mid A \Rightarrow_G^* x \,\}$$

$(L_G(S), L_G(D_1), L_G(A), L_G(U))$ is the **least fixed point** of the operator $\Phi_G\colon (\mathscr{P}(\{a,b\}^*))^4 \to (\mathscr{P}(\{a,b\}^*))^4$:

$$\Phi_G(X_S, X_{D_1}, X_A, X_U) =$$
$$(aX_{D_1}bX_S \cup aX_A \cup bX_U,\ \varepsilon \cup aX_{D_1}bX_{D_1},\ \varepsilon \cup aX_{D_1}bX_A \cup aX_A,\ \varepsilon \cup X_Ua \cup X_Ub)$$

Pre-fixed Points of Context-Free Grammars

$$G = (N, \Sigma, P, S)$$
Φ_G: associated operator

- $(X_B)_{B \in N}$ is a **pre-fixed point** of $\Phi_G \overset{\text{def}}{\iff}$ $\Phi_G((X_B)_{B\in N}) \subseteq (X_B)_{B \in N}$

  componentwise inclusion

- $(L_G(B))_{B\in N}$ is the least pre-fixed point of Φ_G.

- $(X_B)_{B \in N}$ is a pre-fixed point of Φ_G if and only if for every production $A \to w_0\, B_1\, w_1\, \dots\, B_k\, w_k$ in P,

$$X_A \supseteq w_0\, X_{B_1}\, w_1\, \dots\, X_{B_k}\, w_k.$$

Fixed Points

$$\boxed{\begin{aligned} S &\to aD_1bS \mid aA \mid bU \\ D_1 &\to \varepsilon \mid aD_1bD_1 \\ A &\to \varepsilon \mid aD_1bA \mid aA \\ U &\to \varepsilon \mid Ua \mid Ub \end{aligned}}$$

$$\begin{aligned} X_S &= aX_{D_1}bX_S \cup aX_A \cup bX_U \\ X_{D_1} &= \varepsilon \cup aX_{D_1}bX_{D_1} \\ X_A &= \varepsilon \cup aX_{D_1}bX_A \cup aX_A \\ X_U &= \varepsilon \cup X_Ua \cup X_Ub \end{aligned}$$

Pre-fixed Points

$$\boxed{\begin{aligned} S &\to aD_1bS \mid aA \mid bU \\ D_1 &\to \varepsilon \mid aD_1bD_1 \\ A &\to \varepsilon \mid aD_1bA \mid aA \\ U &\to \varepsilon \mid Ua \mid Ub \end{aligned}}$$

$$\begin{aligned} X_S &\supseteq aX_{D_1}bX_S \cup aX_A \cup bX_U \\ X_{D_1} &\supseteq \varepsilon \cup aX_{D_1}bX_{D_1} \\ X_A &\supseteq \varepsilon \cup aX_{D_1}bX_A \cup aX_A \\ X_U &\supseteq \varepsilon \cup X_Ua \cup X_Ub \end{aligned}$$

Pre-fixed Points

$$
\begin{aligned}
S &\rightarrow aD_1bS \mid aA \mid bU \\
D_1 &\rightarrow \varepsilon \mid aD_1bD_1 \\
A &\rightarrow \varepsilon \mid aD_1bA \mid aA \\
U &\rightarrow \varepsilon \mid Ua \mid Ub
\end{aligned}
$$

$$
\begin{aligned}
S &\rightarrow aD_1bS & X_S &\supseteq aX_{D_1}bX_S \\
S &\rightarrow aA & X_S &\supseteq aX_A \\
S &\rightarrow bU & X_S &\supseteq bX_U \\
D_1 &\rightarrow \varepsilon & X_{D_1} &\supseteq \varepsilon \\
D_1 &\rightarrow aD_1bD_1 & X_{D_1} &\supseteq aX_{D_1}bX_{D_1} \\
A &\rightarrow \varepsilon & X_A &\supseteq \varepsilon \\
A &\rightarrow aD_1bA & X_A &\supseteq aX_{D_1}bX_A \\
A &\rightarrow aA & X_A &\supseteq aX_A \\
U &\rightarrow \varepsilon & X_U &\supseteq \varepsilon \\
U &\rightarrow Ua & X_U &\supseteq X_Ua \\
U &\rightarrow Ub & X_U &\supseteq X_Ub
\end{aligned}
$$

Simple Case: Grammars with Just One Nonterminal

$$\pi: \quad S \to w_0\, S\, w_1 \;\ldots\; S\, w_k \qquad (w_i \in \Sigma^*)$$

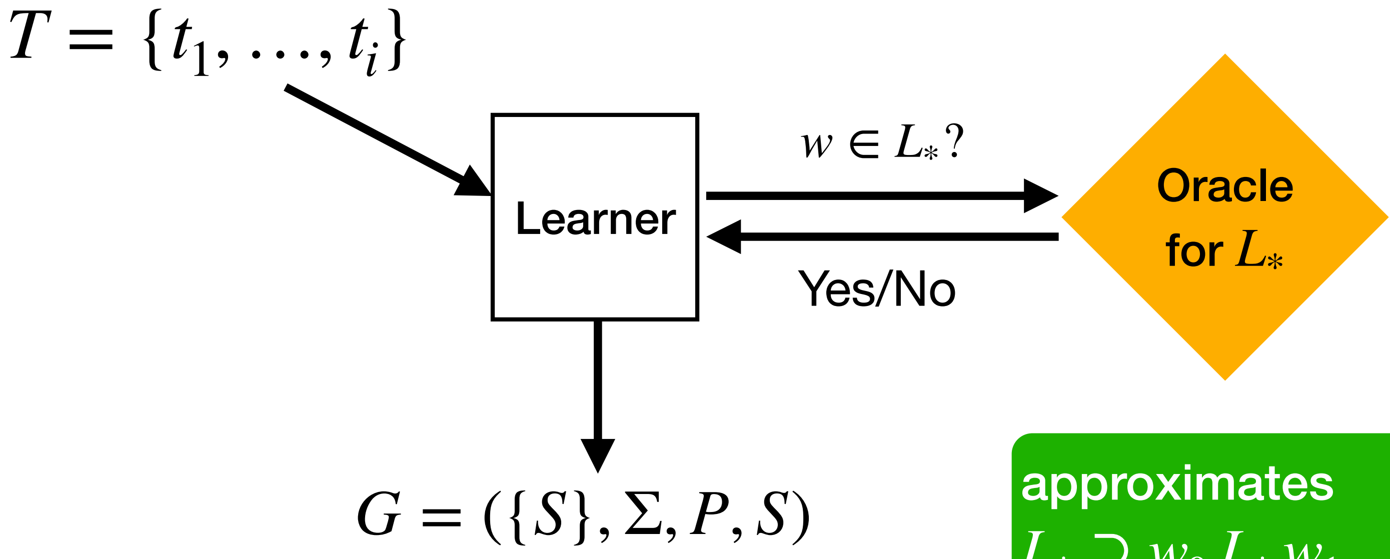
$$\pi \text{ is } \textbf{valid} \overset{\text{def}}{\iff} L_* \supseteq w_0\, L_*\, w_1 \;\ldots\; L_*\, w_k$$

- If π is not valid, π can't be in a correct grammar for L_*.
- If all productions in G are valid, then $L(G) \subseteq L_*$.

  $\because$ L_* is a pre-fixed point of G.
- All productions in G_* are valid.

  $\because$ $L_* = L(G_*)$ is the least pre-fixed point of G_*.

Inference of Context-Free Grammars with Just One Nonterminal



approximates $L_* \supseteq w_0 \, L_* \, w_1 \ \ldots \ L_* \, w_k$

$$P = \{\, S \to w_0 \, S \, w_1 \ \ldots \ S \, w_k \mid L_* \supseteq w_0 \, T \, w_1 \ \ldots \ T \, w_k, \; k \leq r, |w_i| \leq s \ (0 \leq i \leq k) \,\}$$

- Need a constant bound r on k, since deciding $L_* \supseteq w_0 \, T \, w_1 \ \ldots \ T \, w_k$ requires $|T|^k$ queries to the oracle for L_*.
- Placing a constant bound s on $|w_i|$ makes the set of possible productions finite.

Learning Context-Free Grammars (with More Than One Nonterminal)

$$\pi : \quad A \to w_0\, B_1\, w_1\, \ldots\, B_k\, w_k$$

$$\pi \text{ is } \textbf{valid} \overset{\text{def}}{\iff} [\![A]\!]^{L_*} \supseteq w_0\, [\![B_1]\!]^{L_*}\, w_1\, \ldots\, [\![B_k]\!]^{L_*}\, w_k$$

- A hypothesized nonterminal B "denotes" a set $[\![B]\!]^{L_*}$ relative to the target language L_*, independently of the rest of the hypothesized grammar.
- In particular, it is not necessarily the case that $L_G(B) = [\![B]\!]^{L_*}$ (even in the limit).
- Membership in $[\![B]\!]^{L_*}$ **reduces in polynomial time** to membership in L_*.
- This reduction is uniform across different target languages.

Simplest Class: Nonterminals Denote Quotients

- The learner uses pairs of strings as nonterminals.

$$\begin{aligned} [\![\langle\!\langle u, v\rangle\!\rangle]\!]^{L_*} &= u \backslash L_* / v \\ &= \{\, x \in \Sigma^* \mid uxv \in L_* \,\} \end{aligned}$$

$$\begin{aligned} P = \{\, A \to w_0\, B_1\, w_1\ \ldots\ B_k\, w_k \mid\ & \\ & [\![A]\!]^{L_*} \supseteq w_0\, (\mathrm{Sub}(T) \cap [\![B_1]\!]^{L_*})\, w_1\ \ldots\ (\mathrm{Sub}(T) \cap [\![B_k]\!]^{L_*})\, w_k, \\ & k \leq r, |w_i| \leq s\ (0 \leq i \leq k) \,\} \end{aligned}$$

approximates $[\![A]\!]^{L_*} \supseteq w_0\, [\![B_1]\!]^{L_*}\, w_1\ \ldots\ [\![B_k]\!]^{L_*}\, w_k$

$\mathrm{Sub}(T) = \{\, x \mid uxv \in T \,\}$

$$S = \langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle$$

- L_* has infinitely many quotients unless it is regular, so the learner must stop creating new nonterminals.

Algorithm 1

$T_0 := \varnothing; E_0 := \varnothing; J_0 := \varnothing; G_0 := (\{\langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle\}, \Sigma, \varnothing, \langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle);$

for $i = 1, 2, 3, \ldots$ **do**

$T_i := T_{i-1} \cup \{t_i\};$

if $T_i \subseteq L(G_{i-1})$ **then**

$J_i := J_{i-1};$

expand J_i only when $T_i \nsubseteq L(G_{i-1})$

else

$\mathrm{Con}(T) = \{ (u, v) \mid uwv \in T \}$

$J_i := \mathrm{Con}(T_i);$

end

lexicographic product of $\prec$ with itself

$E = \Sigma^{\leq s} (\mathrm{Sub}(T_i)\, \Sigma^{\leq s})^{\leq r}$

$N_i := \{ \langle\!\langle u, v \rangle\!\rangle \mid (u, v) \in J_i \wedge$
$\forall (u', v') \in J_i ((u', v') \prec_2 (u, v) \to E \cap (u' \backslash L_* / v') \neq E \cap (u \backslash L_* / v) \};$

$P_i := \{ A \to w_0 B_1 w_1 \ldots B_k w_k \mid A, B_1, \ldots, B_k \in N_i,$
$[\![A]\!]^{L_*} \supseteq w_0 (\mathrm{Sub}(T_i) \cap [\![B_1]\!]^{L_*}) w_1 \ldots (\mathrm{Sub}(T_i) \cap [\![B_k]\!]^{L_*}) w_k,$
$k \leq r, |w_j| \leq s \ (1 \leq j \leq k) \}$

output $G_i := (N_i, \Sigma, P_i, \langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle);$

end

CFGs with the Quotient Property

- $G = (N, \Sigma, P, S)$ has the **quotient property** $\stackrel{\text{def}}{\iff}$ G has a pre-fixed point $(X_B)_{B \in N}$ with $X_S = L(G)$ such that $X_B \in \mathcal{Q}(L(G))$ for all $B \in N$.

$\mathcal{Q}(L) = \{ u \backslash L / v \mid u, v \in \Sigma^* \}$

Theorem.

- Algorithm 1 successfully learns L_* if and only if L_* has a grammar with the quotient property.
- If Algorithm 1 converges to G, then G has the quotient property.

Examples of CFGs with the Quotient Property

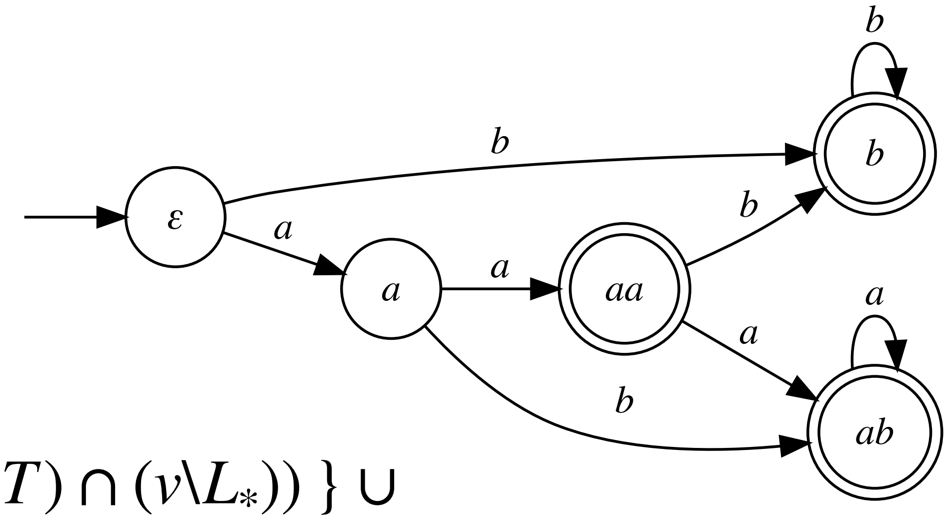
- CFGs with just one nonterminal.
- Right-linear grammars corresponding to minimal DFAs of regular languages.
- $L = \{\, a^n b^n \mid n \geq 0 \,\}\{\, a^n b^n \mid n \geq 0 \,\}$

$$S \to AA$$
$$A \to \varepsilon \mid aAb$$

$$S = \varepsilon \backslash L / \varepsilon \ (= L),$$
$$A = a \backslash L / bab$$

An Analogous Algorithm for Regular Languages

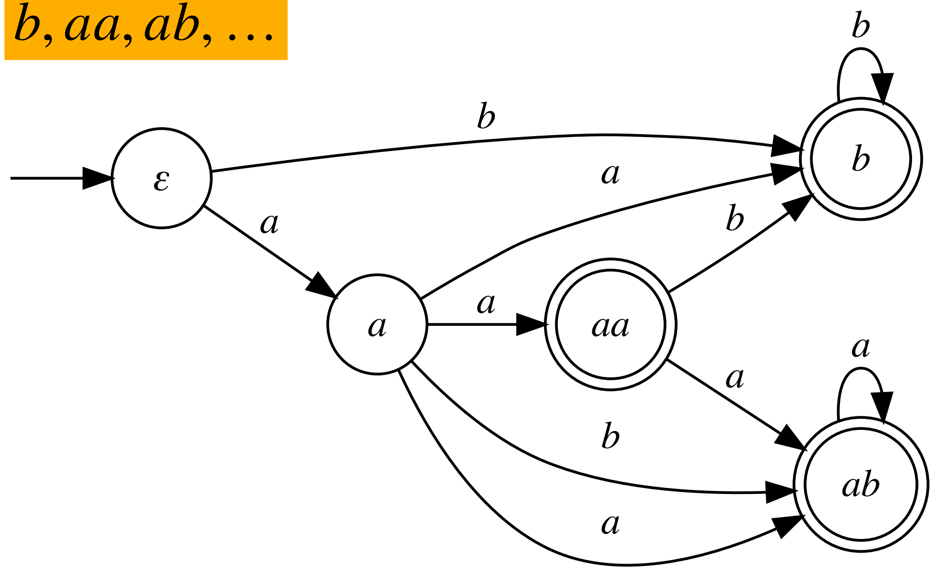
$$L_* = aba^* \cup bb^* \cup aa(a^* \cup b^*)$$



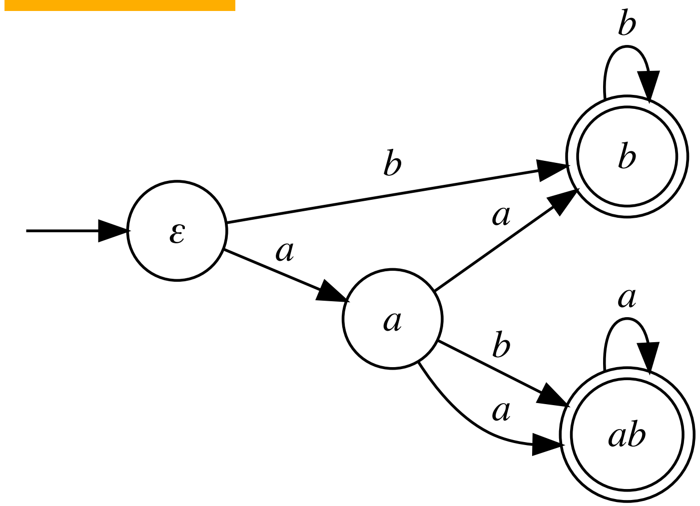
$$P = \{ \langle\!\langle u \rangle\!\rangle \to a\,\langle\!\langle v \rangle\!\rangle \mid u\backslash L_* \supseteq a\,(\mathrm{Suff}(T) \cap (v\backslash L_*)) \} \cup$$
$$\{ \langle\!\langle u \rangle\!\rangle \to \varepsilon \mid u \in L_* \}$$

approximates $u\backslash L_* \supseteq a\,(v\backslash L_*)$ $(\iff ua\backslash L_* \supseteq v\backslash L_*)$

$b, aa, ab, \ldots$



$b, aba, \ldots$



Γ-closure

- Γ: finite set of operations on $\mathscr{P}(\Sigma^*)$ (of variable arity)
- For $\mathscr{L} \subseteq \mathscr{P}(\Sigma^*)$,

$$\Gamma(\mathscr{L}) = \{ f(L_1, \ldots, L_m) \mid f\colon (\mathscr{P}(\Sigma^*))^m \to \mathscr{P}(\Sigma^*), f \in \Gamma, L_1, \ldots, L_m \in \mathscr{L} \}$$

$$\Gamma^0(\mathscr{L}) = \mathscr{L}$$

$$\Gamma^{n+1}(\mathscr{L}) = \mathscr{L} \cup \Gamma(\Gamma^n(\mathscr{L}))$$

Γ-closure of $\mathscr{Q}(L)$

- Sets in $\bigcup_{t \geq 0} \Gamma^t(\mathscr{Q}(L))$ can be represented by expressions built from $\langle\!\langle u, v \rangle\!\rangle$ and symbols for operations in Γ.

Extended Regular Closure

$$\Gamma = \{\cap\,,\ \overline{\ \cdot\ }\,,\cup\,\} \cup \{\varnothing, \varepsilon\} \cup \Sigma \cup \{\text{concatenation}, *\,\}$$

extended regular expression over query atoms

$$\llbracket \langle\!\langle aa, bb \rangle\!\rangle \cap \overline{(\langle\!\langle a, b \rangle\!\rangle\, \overline{\varnothing})\,(a \cup b)} \rrbracket^{L}$$

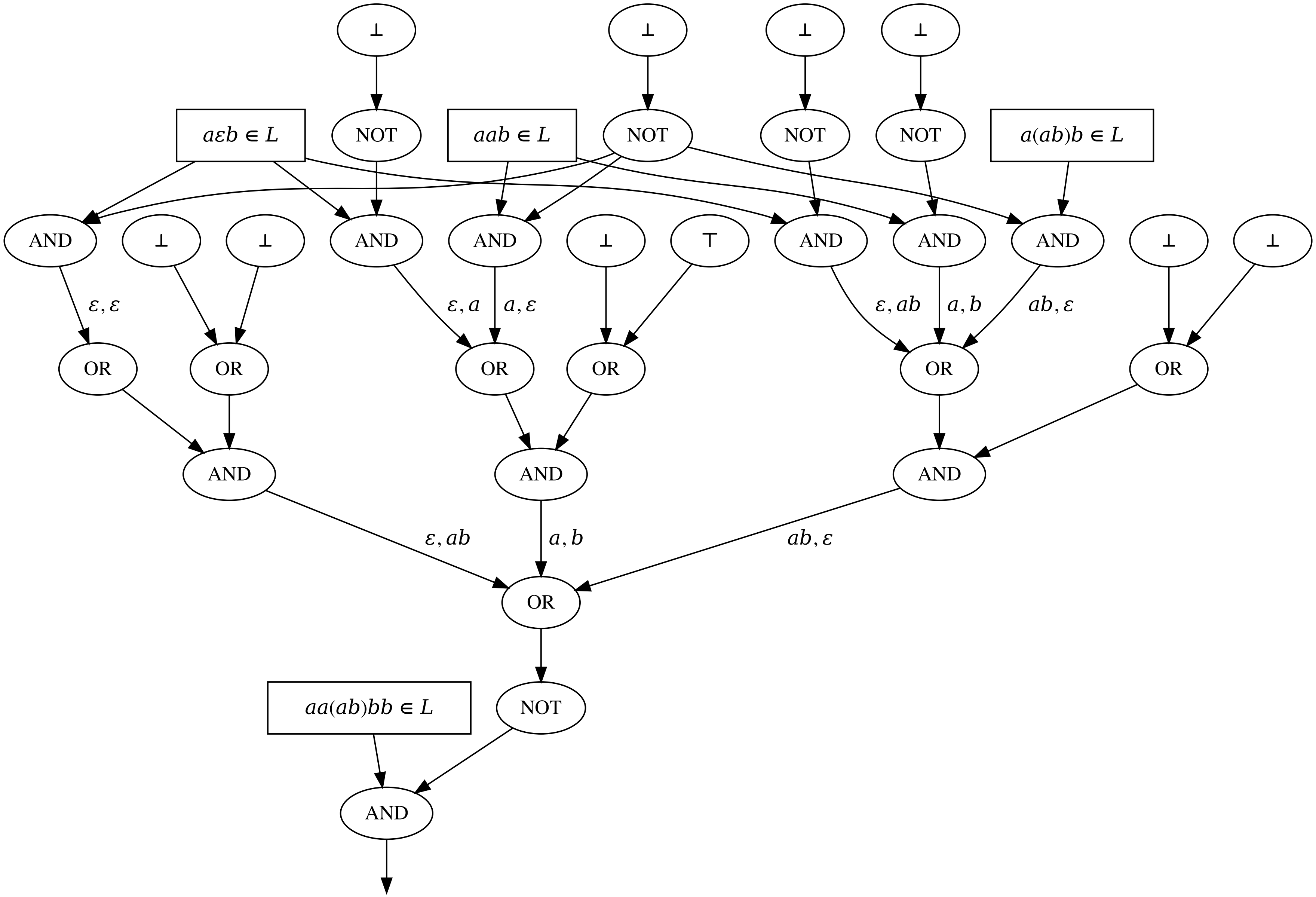
$$= (aa\backslash L/bb) \cap (\{a,b\}^* - ((a\backslash L/b)(\{a,b\}^* - \varnothing))(\{a\} \cup \{b\}))$$

$$= (aa\backslash L/bb) \cap (\{a,b\}^* - (a\backslash L/b)\{a,b\}^*\{a,b\})$$

$$= \{\, x \mid x \in aa\backslash L/bb \wedge \text{no proper prefix of } x \text{ is in } a\backslash L/b \,\}$$

- If e is an extended regular expression over query atoms, then $\llbracket e \rrbracket^L$ **reduces in polynomial time** to L.

$$\llbracket e \rrbracket^L \leq^P_{tt} L$$



$$ab \in \left[\!\left[\langle\!\langle aa, bb \rangle\!\rangle \cap \overline{(\langle\!\langle a, b \rangle\!\rangle\, \overline{\varnothing})\,(a \cup b)} \right]\!\right]^L$$

Γ-closure Property

- A CFG $G = (N, \Sigma, P, S)$ has the **Γ^t-property** $\overset{\text{def}}{\iff}$ G has a pre-fixed point $(X_B)_{B \in N}$ with $X_S = L(G)$ such that $X_B \in \Gamma^t(\mathcal{Q}(L(G)))$ for all $B \in N$.

- G has the **Γ-closure property** $\overset{\text{def}}{\iff}$ G has the Γ^t-property for some t.

Learning CFGs with the Extended Regular Closure Property

- The learner uses extended regular expressions over query atoms as nonterminals.

$$P = \{\, A \to w_0\, B_1\, w_1\, \ldots\, B_k\, w_k \mid$$

approximates $[\![A]\!]^{L_*} \supseteq w_0\, [\![B_1]\!]^{L_*}\, w_1\, \ldots\, [\![B_k]\!]^{L_*}\, w_k$

$$[\![A]\!]^{L_*} \supseteq w_0\, (\mathrm{Sub}(T) \cap [\![B_1]\!]^{L_*})\, w_1\, \ldots\, (\mathrm{Sub}(T) \cap [\![B_k]\!]^{L_*})\, w_k,$$

$$k \le r, |w_i| \le s\ (0 \le i \le k)\, \}$$

$$S = \langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle$$

Algorithm 2

$T_0 := \varnothing; E_0 := \varnothing; J_0 := \varnothing; G_0 := (\{\langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle\}, \Sigma, \varnothing, \langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle);$
for $i = 1, 2, 3, \ldots$ **do**
  $T_i := T_{i-1} \cup \{t_i\};$
  if $T_i \subseteq L(G_{i-1})$ **then**
    $J_i := J_{i-1};$
  else
    $J_i := \mathrm{Con}(T_i);$
  end
  $Q_i := \{ \langle\!\langle u, v \rangle\!\rangle \mid (u, v) \in J_i \wedge$
    $\forall (u', v') \in J_i ((u', v') \prec_2 (u, v) \rightarrow E \cap (u' \backslash L_* / v') \neq E \cap (u \backslash L_* / v) \};$
  $N_i :=$ the set of Γ^t-expressions over Q_i;
  $P_i := \{ A \rightarrow w_0 B_1 w_1 \ \ldots \ B_k w_k \mid A, B_1, \ldots, B_k \in N_i,$
    $[\![A]\!]^{L_*} \supseteq w_0 (\mathrm{Sub}(T_i) \cap [\![B_1]\!]^{L_*}) w_1 \ \ldots \ (\mathrm{Sub}(T_i) \cap [\![B_k]\!]^{L_*}) w_k,$
    $k \leq r, |w_j| \leq s \ (1 \leq j \leq k) \}$
  output $G_i := (N_i, \Sigma, P_i, \langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle);$
end

expand J_i only when $T_i \nsubseteq L(G_{i-1})$

$\mathrm{Con}(T) = \{ (u, v) \mid uwv \in T \}$

lexicographic product of $\prec$ with itself

$E = \Sigma^{\leq s} (\mathrm{Sub}(T_i) \Sigma^{\leq s})^{\leq r}$

$$\{\cap\} \subseteq \Gamma \subseteq \left\{\cap, \overline{\,\cdot\,}, \cup\right\} \cup \{\varnothing, \varepsilon\} \cup \Sigma \cup \{\text{concatenation}, *\}$$

Theorem.

- Algorithm 2 successfully learns L_* if and only if L_* has a grammar with the Γ^t-property.
- If Algorithm 2 converges to G, then G has the Γ^t-property.

Extended Regular Closure Property

$$\boxed{\begin{aligned} S &\to aD_1bS \mid aA \mid bU \\ D_1 &\to \varepsilon \mid aD_1bD_1 \\ A &\to \varepsilon \mid aD_1bA \mid aA \\ U &\to \varepsilon \mid Ua \mid Ub \end{aligned}}$$

$$\begin{aligned} \overline{D_1} &= \{\, x \in \{a,b\}^* \mid \\ &\qquad |x|_a \neq |x|_b \vee \exists uv(uv = x \wedge |u|_a < |u|_b) \,\} \\ &= \{\, x \in \{a,b\}^* \mid \\ &\qquad |x|_a > |x|_b \vee \exists uv(uv = x \wedge |u|_a < |u|_b) \,\} \end{aligned}$$

$$\begin{aligned} A &= \{\, x \in \{a,b\}^* \mid \forall uv(x = uv \to |u|_a \geq |u|_b) \,\} \\ &= \{\, x \in \{a,b\}^* \mid \text{no prefix of } x \text{ is in } D_1b \,\} \\ &= \overline{D_1b\{a,b\}^*} \\ &= [\![\overline{\overline{\langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle}\, b\, (a \cup b)^*}]\!]^{\overline{D_1}} \end{aligned}$$

Boolean Closure Property

$$\boxed{\begin{array}{l} S \to D_1 b D_1 \mid D_1 a D_1 \mid D_1 b S \mid S a D_1 \\ D_1 \to \varepsilon \mid a D_1 b D_1 \end{array}}$$

$$\overline{D_1} = \{\, x \in \{a, b\}^* \mid \exists mn(m + n > 0 \wedge \mathrm{nf}(x) = b^m a^n)\,\}$$

normal form of x under the rewriting $ab \to \varepsilon$

$$S \Rightarrow^* D_1\, b\ \ldots\ D_1\, b\, D_1\, a\, D_1\ \ldots\ a\, D_1$$
$$\Rightarrow^* x_1\, b\ \ldots\ x_m\, b\, y\, a\, z_1\ \ldots\ a\, z_n$$

$$D_1 = [\![\overline{\langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle}]\!]^{\overline{D_1}}$$

Extended Regular Closure Property

$$\boxed{\begin{aligned} S &\to aA \mid aD_1bS \mid bB \mid bD_1^RaS \\ A &\to \varepsilon \mid aA \mid aD_1bA \\ D_1 &\to \varepsilon \mid aD_1bD_1 \\ B &\to \varepsilon \mid bB \mid bD_1^RaB \\ D_1^R &\to \varepsilon \mid bD_1^RaD_1^R \end{aligned}}$$

$$\overline{O_1} = \{\, x \in \{a,b\}^* \mid |x|_a \neq |x|_b \,\}$$

$$\begin{aligned} A &= \{\, x \in \{a,b\}^* \mid \forall uv(x = uv \to |u|_a \geq |u|_b) \,\} \\ &= \{\, x \in \{a,b\}^* \mid \neg\exists uv(x = uv \wedge |u|_a + 1 = |u|_b) \,\} \\ &= \overline{O_1 b\{a,b\}^*} \\ &= [\![\overline{\overline{\langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle}\, b\,(a \cup b)^*}]\!]^{\overline{O_1}} \\ D_1 &= A \cap O_1 \\ &= [\![\overline{\overline{\langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle}\, b\,(a \cup b)^*} \cap \overline{\langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle}]\!]^{\overline{O_1}} \end{aligned}$$

Theorem. $\overline{O_1}$ does not have a grammar with the Boolean closure property.

The pumping lemma applied to a long string a^p gives

$$\begin{aligned} S &\Rightarrow^* a^{l_1} A a^{r_1} \\ A &\Rightarrow^* a^{l_2} A a^{r_2} \qquad\qquad (l_2 + r_2 > 0) \\ A &\Rightarrow^* a^m \end{aligned}$$

If $(X_B)_{B \in N}$ is a pre-fixed point with $X_S = \overline{O_1}$, then

$$\{\, a^{nl_2 + m + nr_2} \mid n \geq 0 \,\} \subseteq X_A \subseteq \bigcap_{n \geq 0} a^{l_1 + nl_2} \backslash \overline{O_1} / a^{nr_2 + r_1}$$

It follows that $\{\, |x|_a - |x|_b \mid x \in X_A \,\}$ is both infinite and co-infinite.

But $\{\, |x|_a - |x|_b \mid x \in u \backslash \overline{O_1} / v \,\} = \mathbb{Z} - \{\, -(|uv|_a - |uv|_b)\}$ is a co-finite set.

CFLs That Have No Grammar with the Extended Regular Closure Property

$$L = \{\, a^l b^m a^n b^q \mid l, m, n, q > 0 \wedge (l = n \vee m > q) \,\}$$

- L is **inherently ambiguous**.
- L does not have a grammar with the extended regular closure property.

Question. Are there any CFLs that are not inherently ambiguous that have no grammar with the extended regular closure property?

Star-Free Closure Property

$$
\boxed{\begin{aligned}
S &\to aA \mid aD_1bS \mid bB \mid bD_1^RaS \\
A &\to \varepsilon \mid aA \mid aD_1bA \\
D_1 &\to \varepsilon \mid aD_1bD_1 \\
B &\to \varepsilon \mid bB \mid bD_1^RaB \\
D_1^R &\to \varepsilon \mid bD_1^RaD_1^R
\end{aligned}}
$$

$$\overline{O_1} = \{\, x \in \{a,b\}^* \mid |x|_a \neq |x|_b \,\}$$

$$
\begin{aligned}
A &= \{\, x \in \{a,b\}^* \mid \forall uv(x = uv \to |u|_a \geq |u|_b) \,\} \\
&= \{\, x \in \{a,b\}^* \mid \neg\exists uv(x = uv \wedge |u|_a + 1 = |u|_b) \,\} \\
&= \overline{O_1 b\{a,b\}^*} \\
&= [\![\overline{\overline{\langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle}\, b\, (a \cup b)^*}]\!]^{\overline{O_1}} \\
&= [\![\overline{\overline{\langle\!\langle \varepsilon, \varepsilon \rangle\!\rangle}\, b\, \overline{\varnothing}}]\!]^{\overline{O_1}}
\end{aligned}
$$

star-free expression over query atoms

Extended Regular Closure vs. Star-Free Closure

Question. Are there any CFLs that have a grammar with the extended regular closure property but have no grammar with the star-free closure property?

$$L = \{\ a^n b^m c^l \mid (n \text{ is odd} \wedge n > m) \vee (n \text{ is even} \wedge n > l)\ \}$$