Two-layered logics for paraconsistent probabilities

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Classical vs paraconsistent probabilities

Measures

$$\mu\left(\bigcup_{i\in I}E_i\right)=\sum_{i\in I}\mu(E_i) \qquad (\forall i,j\in I: i\neq j\Rightarrow E_i\cap E_j=\varnothing)$$

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- However, if we interpret measures as our degrees of certainty in a given event based on the information at hand, this is not the case.
  - Different trusted sources can contradict one another.
  - Sources can give contradictory accounts or give no account at all.

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We need a (paraconsistent) probability theory that can accommodate this.

1. A logic for non-classical event description

2. Probabilities over BD

3. Two-layered logics for paraconsistent probabilities

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What do we need?

A propositional logic with $\neg$, $\wedge$, and $\vee$ that

- does not make contradictions unsatisfiable (otherwise, the measure of a contradictory event is 0);
- allows the instances of the excluded middle to be non-true (otherwise, the measure of $p \vee \neg p$ is always 1 even if there is no information on p);
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We don't need implication!

- Classically, $\phi \rightarrow \chi \equiv \neg\phi \vee \chi$.
- Intuitively, conditional statements *do not describe events*.

Belnap–Dunn logic

$$\mathscr{L}_{\mathsf{BD}} \ni \phi \coloneqq p \in \mathtt{Prop} \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi)$$

The idea

- Classical intuitions of $\neg$, $\wedge$, $\vee$ remain.
- Truth and falsity become independent to model contradictory and incomplete information.

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Frame semantics of BD

For a model $\mathfrak{M} = \langle W, v^+, v^- \rangle$ with $v^+, v^- : \mathtt{Prop} \rightarrow 2^W$, define $w \vDash^+ \phi$ and $w \vDash^- \phi$.

$w \vDash^+ p$	iff	$w \in v^+(p)$	$w \vDash^- p$	iff	$w \in v^-(p)$
$w \vDash^+ \neg\phi$	iff	$w \vDash^- \phi$	$w \vDash^- \neg\phi$	iff	$w \vDash^+ \phi$
$w \vDash^+ \phi \wedge \phi'$	iff	$w \vDash^+ \phi$ and $w \vDash^+ \phi'$	$w \vDash^- \phi \wedge \phi'$	iff	w$\vDash^- \phi$ or $w \vDash^- \phi'$
$w \vDash^+ \phi \vee \phi'$	iff	$w \vDash^+ \phi$ or $w \vDash^+ \phi'$	$w \vDash^- \phi \vee \phi'$	iff	$w \vDash^- \phi$ and $w \vDash^- \phi'$

Associating $\mathscr{L}_{\mathsf{BD}}$-formulas to events

Recall that *in the classical logic*, each formula ϕ corresponds to its *extension* $\|\phi\| = \{w : w \vDash \phi\}$. If $\|\phi\| = W$, ϕ can be considered *true*, and if ϕ is always true, it is *valid*.

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Extensions of $\mathscr{L}_{\mathsf{BD}}$ formulas

Every $\mathscr{L}_{\mathsf{BD}}$ formula has *positive* and *negative extensions*:

$$|\phi|^+ := \{w \in W \mid w \vDash^+ \phi\} \qquad |\phi|^- := \{w \in W \mid w \vDash^- \phi\}$$

Additionally, we define *pure belief*, *pure disbelief*, *conflict*, and *uncertainty in* ϕ:

$$|\phi|^{\mathsf{b}} = |\phi|^+ \setminus |\phi|^- \quad |\phi|^{\mathsf{d}} = |\phi|^- \setminus |\phi|^+ \quad |\phi|^{\mathsf{c}} = |\phi|^+ \cap |\phi|^- \quad |\phi|^{\mathsf{u}} = W \setminus (|\phi|^+ \cup |\phi|^-)$$

Validity in BD

$\phi \vdash \chi$ is *satisfied on* $\mathfrak{M} = \langle W, v^+, v^- \rangle$ ($\mathfrak{M} \models [\phi \vdash \chi]$) iff $|\phi|^+ \subseteq |\chi|^+$ and $|\chi|^- \subseteq |\phi|^-$.
$\phi \vdash \chi$ is BD-*valid* ($\phi \models_{\mathsf{BD}} \chi$) iff it is satisfied on every model.

Probabilities on BD models: two ways

Definition (BD models with ±-probabilities: $\mathfrak{M}_\mu = \langle \mathfrak{M}, \mu \rangle$, $\mu : 2^W \rightarrow [0,1]$)

mon: if $X \subseteq Y$, then $\mu(X) \leq \mu(Y)$;

neg: $\mu(|\phi|^-) = \mu(|\neg\phi|^+)$;

ex: $\mu(|\phi \vee \chi|^+) = \mu(|\phi|^+) + \mu(|\chi|^+) - \mu(|\phi \wedge \chi|^+)$.

Definition (BD models with 4-probabilities: $\mathfrak{M}_{\mathbf{4}} = \langle \mathfrak{M}, \mu_{\mathbf{4}} \rangle$, $\mu_{\mathbf{4}} : 2^W \rightarrow [0,1]$)

part: $\mu_{\mathbf{4}}(|\phi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{d}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{u}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}}) = 1$;

neg: $\mu_{\mathbf{4}}(|\neg\phi|^{\mathsf{b}}) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{d}})$, $\mu_{\mathbf{4}}(|\neg\phi|^{\mathsf{c}}) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}})$;

contr: $\mu_{\mathbf{4}}(|\phi \wedge \neg\phi|^{\mathsf{b}}) = 0$, $\mu_{\mathbf{4}}(|\phi \wedge \neg\phi|^{\mathsf{c}}) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}})$;

BCmon: if $\mathfrak{M} \models [\phi \vdash \chi]$, then $\mu_{\mathbf{4}}(|\phi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}}) \leq \mu_{\mathbf{4}}(|\chi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\chi|^{\mathsf{c}})$;

BCex: $\mu_{\mathbf{4}}(|\phi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}}) + \mu_{\mathbf{4}}(|\psi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\psi|^{\mathsf{c}}) = \mu_{\mathbf{4}}(|\phi \wedge \psi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi \wedge \psi|^{\mathsf{c}}) + \mu_{\mathbf{4}}(|\phi \vee \psi|^{\mathsf{b}}) + \mu_{\mathbf{4}}(|\phi \vee \psi|^{\mathsf{c}})$.

Probabilities in BD: paraconsistency

Theorem (Klein, Majer, Raffie Rad; 2021)

For every BD *model with a* $\pm$*-probability* $\langle W, v^+, v^-, \mu\rangle$ *(resp.,* BD *model with* **4***-probability* $\langle W, v^+, v^-, \mu_{\mathbf{4}}\rangle$*), there is a* BD *model* $\langle W', v'^+, v'^-, \pi\rangle$ *with a classical probability measure* π *s.t.* $\pi(|\phi|^+) = \mu(|\phi|^+)$ *(resp.,* $\pi(|\phi|^{\mathsf{x}}) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{x}})$ *for* $\mathsf{x} \in \{\mathsf{b}, \mathsf{d}, \mathsf{c}, \mathsf{u}\}$*)*

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Example (Non-classical events)

Consider the following BD model.

$$w_0 : p^\pm, \not{q} \qquad\qquad w_1 : p^-, q^-$$

Let $\mu = \mu_\mathbf{4}$ be defined as follows: $\mu(\{w_0\}) = \frac{2}{3}$, $\mu(\{w_1\}) = \frac{1}{3}$, $\mu(W) = 1$, $\mu(\varnothing) = 0$. We have

$$\mu(|p \wedge \neg p|^+) = \frac{2}{3} \qquad \mu(|q \vee \neg q|^+) = \frac{1}{3} \qquad \mu(|p \wedge \neg q|^\mathsf{d}) = 1 \qquad \mu(|p \vee q|^\mathsf{u}) = 0$$

Two-layered logics

We want to formalise reasoning about $\pm$- and **4**-probabilities. Thus, we need logics that can express addition and subtraction and incorporate event descriptions in BD.

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Two-layered logics — the idea

- Use BD to describe events.
- Use modal formulas $\mathsf{M}\phi$ to stand for the measure of the event corresponding to ϕ.
- Use (an expansion of) Łukasiewicz logic to reason about these measures and encode their axioms.

The logic of **4**-probabilities

$$\mathscr{L}_{\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}\triangle}} \ni \alpha \coloneqq \mathsf{Bl}\phi \mid \mathsf{Db}\phi \mid \mathsf{Cf}\phi \mid \mathsf{Uc}\phi \mid {\sim}\alpha \mid \triangle\alpha \mid (\alpha \rightarrow \alpha) \qquad (\phi \in \mathscr{L}_{\mathsf{BD}})$$

A $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}\triangle}$ model is a tuple $\mathbb{M} = \langle \mathfrak{M}, \mu_{\mathbf{4}}, e \rangle$ with $\langle \mathfrak{M}, \mu_{\mathbf{4}} \rangle$ being a BD model with **4**-probability s.t. $e(\mathsf{Bl}\phi) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{b}})$, $e(\mathsf{Db}\phi) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{d}})$, $e(\mathsf{Cf}\phi) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{c}})$, $e(\mathsf{Uc}\phi) = \mu_{\mathbf{4}}(|\phi|^{\mathsf{u}})$. The values of complex formulas are computed as follows:

$$e({\sim}\alpha) = 1 - e(\alpha) \quad e(\alpha \rightarrow \beta) = \min(1, 1 - e(\alpha) + e(\beta)) \quad e(\triangle\alpha) = \begin{cases} 1 & \text{if } e(\alpha) = 1 \\ 0 & \text{otherwise} \end{cases}$$

We say that α is $\mathbf{4}\mathsf{Pr}^{\mathsf{Ł}\triangle}$ *valid* iff $e(\alpha) = 1$ in every model.

The logic of $\pm$-probabilities

$$\mathscr{L}_{\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}} \ni \alpha \coloneqq \mathsf{Pr}\phi \mid {\sim}\alpha \mid \neg\alpha \mid \triangle\alpha \mid (\alpha \rightarrow \alpha) \qquad (\phi \in \mathscr{L}_{\mathsf{BD}})$$

A $\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}$ model is a tuple $\mathbb{M} = \langle \mathfrak{M}, \mu, e_1, e_2 \rangle$ with $\langle \mathfrak{M}, \mu \rangle$ being a BD model with $\pm$-probability and $e_1, e_2 : \mathscr{L}_{\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}} \rightarrow [0,1]$ s.t. $e_1(\mathsf{Pr}\phi) = \mu(|\phi|^+)$, $e_2(\mathsf{Pr}\phi) = \mu(|\phi|^-)$. The values of complex formulas are computed as follows:

$$e_1(\neg\alpha) = e_2(\alpha) \qquad e_2(\neg\alpha) = e_1(\alpha)$$

$$e_1({\sim}\alpha) = 1 - e_1(\alpha) \qquad e_2({\sim}\alpha) = 1 - e_2(\alpha)$$

$$e_1(\triangle\alpha) = \begin{cases} 1 & \text{if } e_1(\alpha) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad e_2(\triangle\alpha) = \begin{cases} 1 & \text{if } e_2(\alpha) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$e_1(\alpha \rightarrow \beta) = \min(1, 1 - e_1(\alpha) + e_1(\beta)) \qquad e_2(\alpha \rightarrow \beta) = \max(0, e_2(\beta) - e_2(\alpha))$$

We say that α is $\mathsf{Pr}^{\mathsf{Ł}^2}_{\triangle}$ *valid* iff $e(\alpha) = (1,0)$ in every model.

From events to two-layered formulas

Example (Non-classical events)

Recall our BD model.

$$w_0 : p^{\pm}, \not{q} \qquad\qquad w_1 : p^{-}, q^{-}$$

Let $\mu = \mu_4$ be defined as follows: $\mu(\{w_0\}) = \frac{2}{3}$, $\mu(\{w_1\}) = \frac{1}{3}$, $\mu(W) = 1$, $\mu(\varnothing) = 0$. We have

$$\mu(|p \wedge \neg p|^{+}) = \frac{2}{3} \qquad \mu(|p \wedge \neg p|^{-}) = 1 \qquad \mu(|p \wedge \neg q|^{\mathsf{d}}) = 1 \qquad \mu(|p \vee q|^{\mathsf{u}}) = 0$$

Thus, the values are: $e(\mathsf{Pr}(p \wedge \neg p)) = \left(\frac{2}{3}, 1\right)$, $e(\mathsf{Db}(p \wedge \neg q)) = 1$, $e(\mathsf{Uc}(p \vee q)) = 0$.

Equivalence

$\pm$-probabilities and **4**-probabilities are equivalent. How to show that their corresponding logics are equivalent as well?

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From $\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$ to $\mathbf{4}\mathsf{Pr}^{\text{Ł}_{\triangle}}$

Note that $\neg\mathsf{Pr}\phi$ is equivalent to $\mathsf{Pr}\neg\phi$ and that $\mathscr{L}_{\mathsf{Pr}^{\text{Ł}^2}_{\triangle}}$-formulas admit $\neg$ NNFs. Thus, we can eliminate $\neg$'s. After that, we apply the following translation.

$$(\mathsf{Pr}\phi)^{\mathbf{4}} = \mathsf{Bl}\phi \oplus \mathsf{Cf}\phi \quad (\sim\alpha)^{\mathbf{4}} = \sim\alpha^{\mathbf{4}} \quad (\triangle\alpha)^{\mathbf{4}} = \triangle\alpha^{\mathbf{4}} \quad (\alpha \rightarrow \alpha')^{\mathbf{4}} = \alpha^{\mathbf{4}} \rightarrow \alpha'^{\mathbf{4}}$$

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From $\mathbf{4}\mathsf{Pr}^{\text{Ł}_\triangle}$ to $\mathsf{Pr}^{\text{Ł}^2}_{\triangle}$

$$(\mathsf{Bl}\phi)^{\pm} = \mathsf{Pr}\phi \ominus \mathsf{Pr}(\phi \wedge \neg\phi) \qquad (\mathsf{Cf}\phi)^{\pm} = \mathsf{Pr}(\phi \wedge \neg\phi) \qquad (\mathsf{Uc}\phi)^{\pm} = {\sim}\mathsf{Pr}(\phi \vee \neg\phi)$$

$$(\mathsf{Db}\phi)^{\pm} = \mathsf{Pr}\neg\phi \ominus \mathsf{Pr}(\phi \wedge \neg\phi) \qquad ({\sim}\beta)^{\pm} = {\sim}\beta^{\pm} \qquad (\triangle\beta)^{\pm} = \triangle\beta^{\pm}$$

$$(\beta \rightarrow \beta')^{\pm} = \beta^{\pm} \rightarrow \beta'^{\pm}$$

Axiomatisation of 4-probabilities

To produce the Hilbert-style axiomatisation, we just need to translate the conditions on $\mu_{\mathbf{4}}$ into formulas. For example, $\mu_{\mathbf{4}}(|\phi|^{\mathsf{b}})+\mu_{\mathbf{4}}(|\phi|^{\mathsf{d}})+\mu_{\mathbf{4}}(|\phi|^{\mathsf{u}})+\mu_{\mathbf{4}}(|\phi|^{\mathsf{c}})=1$ is going to be represented as follows:

$$\mathsf{Bl}\phi\oplus\mathsf{Db}\phi\oplus\mathsf{Cf}\phi\oplus\mathsf{Uc}\phi$$

$$((\mathsf{X}_1\phi\oplus\mathsf{X}_2\phi\oplus\mathsf{X}_3\phi\oplus\mathsf{X}_4\phi)\ominus\mathsf{X}_4\phi)\leftrightarrow(\mathsf{X}_1\phi\oplus\mathsf{X}_2\phi\oplus\mathsf{X}_3\phi)$$

$$(\mathsf{X}_i\neq\mathsf{X}_j,\mathsf{X}_i\in\{\mathsf{Bl},\mathsf{Db},\mathsf{Cf},\mathsf{Uc}\})$$

To prove the completeness of the calculus, we use the completeness of $Ł_{\triangle}$ w.r.t. finite theories and encode the properties of the measure with probabilistic axioms.

Complexity

Reduction of probabilistic proofs to $Ł_{\triangle}$-proofs immediately gives us decidability (and NP-hardness) of $\mathsf{Pr}^{Ł^2}_{\triangle}$ and $\mathbf{4}\mathsf{Pr}^{Ł_{\triangle}}$. We can establish NP-completeness by using constraint tableaux for $Ł_{\triangle}$ and reducing $\mathbf{4}\mathsf{Pr}^{Ł_{\triangle}}$-formulas to instances of the bounded Mixed-Integer Problem.

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- plausibilities,
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Stronger languages for event descriptions:

- add implication,
- add ‘p is (non-)classical’,
- add ‘p is true’,
- …

Thank you for your attention!