Lambek Calculus and its modal extensions

Michael Moortgat, Utrecht University

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Plan

Ieri Categorial modalities, then and now.

- Soft Linear Logic ! and its subexponential, multimodal refinements
- Residuated families $\diamondsuit_i, \Box_i$

Oggi Dependency and function-argument structure.

- Dependency roles (subj, obj, ...) demarcating locality domains
- Rethinking constituency

Domani The neurosymbolic turn.

- Training data for type inference; constructive supertagging
- neural proof nets for parsing

A landscape of logics

Lambek calculi Identity $A \longrightarrow A$, composition $A \longrightarrow C$ if $A \longrightarrow B$ and $B \longrightarrow C$

Residuation: $B \longrightarrow A \backslash C$ iff $A \bullet B \longrightarrow C$ iff $A \longrightarrow C/B$

Options: $\bullet$ associativity and/or commutativity; multiplicative unit

Substructural, sublinear a hierarchy of type logics reflecting different views on the structure of the assumptions Γ in sequent judgements $\Gamma \vdash A$.

LOGIC	Γ	ASS	COMM
LP	multiset	✓	✓
L	string	✓	-
NL	tree	-	-

- **(N)L**: syntactic types

  NL types assigned to phrases (bracketed strings); **L**: types assigned to strings

- **LP** (aka unit-free MILL): semantic types aka unit-free MILL

The need for control

- languages exhibit phenomena that seem to require some form of

  reordering, restructuring, copying

- global structural options are problematic

  too little (undergeneration), too much (overgeneration)

- extended type language with modalities for structural control:
  - licensing structural reasoning that is lacking by default
  - blocking structural reasoning that would otherwise be available

Global associativity ☹

Recall our relative clause example, derivable in **L** thanks to global associativity.

$$
\dfrac{\dfrac{\text{paper}}{n} \qquad \dfrac{\dfrac{\text{that}}{(n\backslash n)/(s/np)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\text{Bob}}{np} \qquad \dfrac{\dfrac{\text{rejected}}{(np\backslash s)/np} \qquad np \vdash np}{\text{rejected} \cdot np \vdash np\backslash s}\,/E}{\text{Bob} \cdot (\text{rejected} \cdot np) \vdash s}\,\backslash E}{(\text{Bob} \cdot \text{rejected}) \cdot np \vdash s}\,\mathsf{A}^r}{\text{Bob} \cdot \text{rejected} \vdash s/np}\,/I}{\text{that} \cdot (\text{Bob} \cdot \text{rejected}) \vdash n\backslash n}\,/E}{\text{paper} \cdot (\text{that} \cdot (\text{Bob} \cdot \text{rejected})) \vdash n}\,\backslash E
$$

- not enough restricted to peripheral gaps, but

  paper that Bob rejected __ immediately

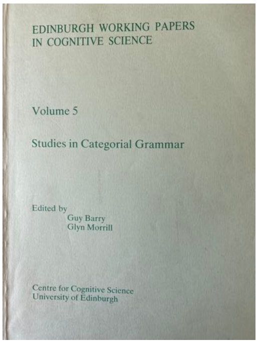
- too much insensitive to island constraints

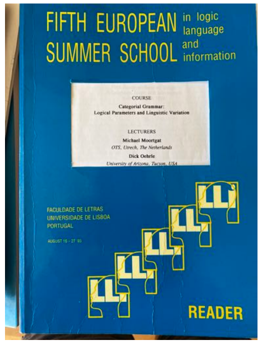
  paper that (Alice reviewed a thesis) and$_{(s\backslash s)/s}$ (B rejected __)

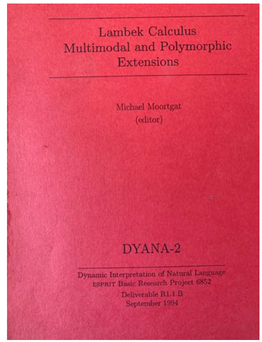
Vintage

The two views on modal extensions go back to the early 1990ies

- (Soft) Linear Logic ! and its subexponential, multimodal refinements
- Residuated families $\Diamond_i, \Box_i$







Morrill, Leslie, Hepple and Barry, 1990, Categorial Deductions and Structural Operations • MM & Oehrle, 1993, ESSLLI Lisbon Lecture Notes • MM ed 1994, DYANA Report, Residuation in mixed Lambek systems, Controlling resource management

Modalities I: decomposing !

$$\frac{!\,\Gamma \vdash A}{!\,\Gamma \vdash !\,A}\ !\,R \qquad \frac{\Gamma, A \vdash B}{\Gamma, !\,A \vdash B}\ !\,L \qquad \frac{\Gamma, A^n \vdash B}{\Gamma, !\,A \vdash B}\ M$$

$$\frac{\Gamma \vdash B}{\Gamma, !\,A \vdash B}\ W \qquad \frac{\Gamma, !\,A, !\,A \vdash B}{\Gamma, !\,A \vdash B}\ C \qquad \frac{\Gamma \vdash B}{!\,\Gamma \vdash !\,B}\ SP$$

Exponentials, multimodally Indexed $!_i$ for particular structural rules.

Cf Jacobs (1993,94) for syn/sem of $!_c$, $!_w$; fully generalized in Blaisdell et al 2022,23.

(Soft) linear logic ! Lafont 2004 terms: Baillot & Mogbil 2004

- Promotion $(!\,R)$ is replaced by soft promotion (SP) (i.e. $!\,A \nvdash !\,!\,A$); Dereliction $(!\,L)$, Contraction, Weakening are replaced by Multiplexing (M)
- Cut elim/normalization: P
- Moot/Retoré 2019: SLL enough expressivity to specify lexical lambda terms
- SLL for syntax: ingenuity required for compatibility with non-comm, non-ass

Modalities II: residuated pairs

- The type language is extended with a pair of unary connectives $\Diamond, \Box$ satisfying

$$\frac{\Diamond A \longrightarrow B}{A \longrightarrow \Box B}$$

- Logic: $\Diamond, \Box$ form a residuated pair. One easily shows

compositions: $\Diamond\Box A \longrightarrow A$ (interior) $\quad A \longrightarrow \Box\Diamond A$ (closure)

monotonicity: from $A \longrightarrow B$ infer $\Diamond A \longrightarrow \Diamond B$, $\Box A \longrightarrow \Box B$

- Structure: global rules $\rightsquigarrow$ $\Diamond$ controlled restricted versions, e.g.

$$\mathsf{A}^r_\diamond : \quad (A \bullet B) \bullet \Diamond C \longrightarrow A \bullet (B \bullet \Diamond C)$$

$$\mathsf{C}^r_\diamond : \quad (A \bullet B) \bullet \Diamond C \longrightarrow (A \bullet \Diamond C) \bullet B$$

Multimodal generalization families $\{\Diamond_i, \Box_i\}_{i \in I}$ for particular structural choices

$\Diamond, \Box$ inverse duals

Relational semantics

Frames (W, R^2, R^3). Valuation v sends types to subsets of W,

$$
\begin{array}{rcl}
v(A \bullet B) & = & \{x \mid \exists yz.Rxyz \wedge y \in v(A) \wedge z \in v(B)\} \\
v(C/B) & = & \{y \mid \forall xz.(Rxyz \wedge z \in v(B)) \Rightarrow x \in v(C)\} \\
v(A\backslash C) & = & \{z \mid \forall xy.(Rxyz \wedge y \in v(A)) \Rightarrow x \in v(C)\} \\
v(\Diamond A) & = & \{x \mid \exists y.(Rxy \wedge y \in v(A)\} \\
v(\Box A) & = & \{y \mid \forall x.(Rxy \Rightarrow x \in v(A)\}
\end{array}
$$

Soundness/completeness Kurtonina 1995 generalizing Došen 1992 for **(N)L(P)**

Extensions of $\mathbf{NL}_\Diamond$ with weak Sahlqvist postulates are complete w.r.t. the class of 2/3-ary frames satisfying the corresponding 1st order constraint effectively computable by the Sahlqvist-van Benthem algorithm.

Weak Sahlqvist postulates $A \longrightarrow B$ such that A is built out of single-use atoms and connectives $\bullet, \Diamond$; B also is pure $\bullet, \Diamond$ frm containing at least one occurrence of $\bullet$ or $\Diamond$, with all atoms of B occurring in A.

Structural communication

Let $\mathcal{L}' = \mathcal{L} + P$ for some structural postulate P (Ass, Comm).

Kurtonina & MM 1997: two types of modal translation to relate $\mathcal{L}, \mathcal{L}'$:

- $\mathcal{L}_{/,\bullet,\backslash} \vdash A \longrightarrow B$ iff $\mathcal{L}'_{\Diamond,\Box,/,\bullet,\backslash} \vdash A^\flat \longrightarrow B^\flat$

  inhibiting $\cdot^\flat$ blocks applicability of structural option P

- $\mathcal{L}'_{/,\bullet,\backslash} \vdash A \longrightarrow B$ iff $\mathcal{L}_{\Diamond,\Box,/,\bullet,\backslash} + P_\diamond \vdash A^\sharp \longrightarrow B^\sharp$

  licensing $\cdot^\sharp$ provides access to a controlled version of P

The $\cdot^\sharp$ direction cf obtaining IL within MILL via ! exponential ($A \rightarrow B = {!}A \multimap B$).

We illustrate with **NL** vs **L**.

Controlling Associativity

One schema serves for the licensing/inhibiting directions:

$$\begin{array}{rcl} p^\natural & = & p \\ (A \bullet B)^\natural & = & \Diamond(A^\natural \bullet B^\natural) \\ (A/B)^\natural & = & \Box A^\natural / B^\natural \\ (B\backslash A)^\natural & = & B^\natural \backslash \Box A^\natural \end{array}$$

- expressing **NL** in **L**: $\Diamond$ blocks applicability of Ass, e.g.

$$\not\vdash ((a\backslash b) \bullet (b\backslash c))^\flat \longrightarrow (a\backslash c)^\flat$$

- expressing **L** in **NL**: $\Diamond$ provides access to controlled Ass

$$\Diamond(\Diamond(A \bullet B) \bullet C) \longleftrightarrow \Diamond(A \bullet \Diamond(B \bullet C)) \quad (A^\diamond) = (A)^\sharp$$

N.D. Proofs and terms: syntactic calculi (N)L$_{/,\backslash}$

Types, terms p atomic

$$A, B ::= p \mid A\backslash B \mid B/A \qquad M, N ::= x \mid \lambda^r x.M \mid \lambda^l x.M \mid (M \ltimes N) \mid (N \rtimes M)$$

Wansing, 1990, Formulas-as-types for a Hierarchy of Sublogics of Int Prop Logic

Typing rules Axiom $x : A \vdash x : A$ var Γ, Δ all distinct

$$\frac{\Gamma \cdot x : A \vdash M : B}{\Gamma \vdash \lambda^r x.M : B/A}\ I/ \qquad \frac{x : A \cdot \Gamma \vdash M : B}{\Gamma \vdash \lambda^l x.M : A\backslash B}\ I\backslash$$

$$\frac{\Gamma \vdash M : B/A \quad \Delta \vdash N : A}{\Gamma \cdot \Delta \vdash (M \ltimes N) : B}\ E/ \qquad \frac{\Gamma \vdash N : A \quad \Delta \vdash M : A\backslash B}{\Gamma \cdot \Delta \vdash (N \rtimes M) : B}\ E\backslash$$

Compare: LP$_{\multimap}$ **L** extended with product commutativity, a.k.a. MILL, Multiplicative Intuitionistic Linear Logic. In MILL, the slashes $/, \backslash$ collapse to linear implication $\multimap$.

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \multimap B}\ (\multimap I) \qquad \frac{\Gamma \vdash M : A \multimap B \quad \Delta \vdash N : A}{\Gamma, \Delta \vdash M\ N : B}\ (\multimap E)$$

Control operators: N.D. rules, terms

Structures Unary $\langle\rangle$ structural counterpart of $\Diamond$: $\Gamma, \Delta ::= A \mid \langle\Gamma\rangle \mid \Gamma \cdot \Delta$

$$\frac{\langle\Gamma\rangle \vdash A}{\Gamma \vdash \Box A}\ \Box I \qquad \frac{\Gamma \vdash \Box A}{\langle\Gamma\rangle \vdash A}\ \Box E$$

$$\frac{\Gamma \vdash A}{\langle\Gamma\rangle \vdash \Diamond A}\ \Diamond I \qquad \frac{\Delta \vdash \Diamond A \quad \Gamma[\langle A\rangle] \vdash B}{\Gamma[\Delta] \vdash B}\ \Diamond E \qquad \frac{\Gamma[\langle A\rangle] \vdash B}{\Gamma[\Diamond A] \vdash B}\ \Diamond E'$$

shorthand ($\Diamond E'$) if left premise of ($\Diamond E$) is an axiom

Control operators: terms Terms: $M, N ::= x \mid \ldots \mid \triangledown\, M \mid \vartriangle M \mid \blacktriangledown M \mid \blacktriangle M$

$$\frac{\langle\Gamma\rangle \vdash M : A}{\Gamma \vdash \blacktriangle M : \Box A}\ \Box I \qquad \frac{\Gamma \vdash M : \Box A}{\langle\Gamma\rangle \vdash \blacktriangledown M : A}\ \Box E$$

$$\frac{\Gamma \vdash M : A}{\langle\Gamma\rangle \vdash \vartriangle M : \Diamond A}\ \Diamond I \qquad \frac{\Delta \vdash M : \Diamond A \quad \Gamma[\langle x : A\rangle] \vdash N : B}{\Gamma[\Delta] \vdash N[\triangledown\, M/x] : B}\ \Diamond E$$

$\Diamond E$ officially: case $\triangledown\, M$ of x in N

Controlled associativity/commutativity ☺

$\Diamond\Box np$: ‘moveable’ np; key-and-lock: contract $\Diamond\Box np$ to np, once in place.

$$\frac{\text{rejected}}{(np\backslash s)/np} \qquad \frac{\Box np \vdash \Box np}{\langle \Box np \rangle \vdash np}\ \Box E \qquad \frac{\text{immediately}}{(np\backslash s)\backslash(np\backslash s)} \qquad \frac{\text{Bob}}{np} \qquad \frac{\text{that}}{(n\backslash n)/(s/\Diamond\Box np)} \qquad \frac{\text{paper}}{n}$$

$$\frac{(np\backslash s)/np \qquad \langle \Box np \rangle \vdash np}{\text{rejected} \cdot \langle \Box np \rangle \vdash np\backslash s}\ /E$$

$$\frac{\text{rejected} \cdot \langle \Box np \rangle \vdash np\backslash s \qquad (np\backslash s)\backslash(np\backslash s)}{(\text{rejected} \cdot \langle \Box np \rangle) \cdot \text{immediately} \vdash np\backslash s}\ \backslash E$$

$$\frac{np \qquad (\text{rejected} \cdot \langle \Box np \rangle) \cdot \text{immediately} \vdash np\backslash s}{\text{Bob} \cdot ((\text{rejected} \cdot \langle \Box np \rangle) \cdot \text{immediately}) \vdash s}\ \backslash E$$

$$\frac{\text{Bob} \cdot ((\text{rejected} \cdot \langle \Box np \rangle) \cdot \text{immediately}) \vdash s}{\text{Bob} \cdot ((\text{rejected} \cdot \text{immediately}) \cdot \langle \Box np \rangle) \vdash s}\ \mathsf{C}^r_\Diamond$$

$$\frac{\text{Bob} \cdot ((\text{rejected} \cdot \text{immediately}) \cdot \langle \Box np \rangle) \vdash s}{(\text{Bob} \cdot (\text{rejected} \cdot \text{immediately})) \cdot \langle \Box np \rangle \vdash s}\ \mathsf{A}^r_\Diamond$$

$$\frac{(\text{Bob} \cdot (\text{rejected} \cdot \text{immediately})) \cdot \langle \Box np \rangle \vdash s}{(\text{Bob} \cdot (\text{rejected} \cdot \text{immediately})) \cdot \Diamond\Box np \vdash s}\ \Diamond E'$$

$$\frac{(\text{Bob} \cdot (\text{rejected} \cdot \text{immediately})) \cdot \Diamond\Box np \vdash s}{\text{Bob} \cdot (\text{rejected} \cdot \text{immediately}) \vdash s/\Diamond\Box np}\ /I$$

$$\frac{(n\backslash n)/(s/\Diamond\Box np) \qquad \text{Bob} \cdot (\text{rejected} \cdot \text{immediately}) \vdash s/\Diamond\Box np}{\text{that} \cdot (\text{Bob} \cdot (\text{rejected} \cdot \text{immediately})) \vdash n\backslash n}\ /E$$

$$\frac{n \qquad \text{that} \cdot (\text{Bob} \cdot (\text{rejected} \cdot \text{immediately})) \vdash n\backslash n}{\text{paper} \cdot (\text{that} \cdot (\text{Bob} \cdot (\text{rejected} \cdot \text{immediately}))) \vdash n}\ \backslash E$$

$$\mathsf{A}^r_\Diamond : \quad (A \bullet B) \bullet \Diamond C \longrightarrow A \bullet (B \bullet \Diamond C) \qquad\qquad \mathsf{C}^r_\Diamond : \quad (A \bullet B) \bullet \Diamond C \longrightarrow (A \bullet \Diamond C) \bullet B$$

Proofs and terms

Adjusted lexical meaning recipe for the relative pronoun, $(n\backslash n)/(s/\Diamond\Box np)$

$$\lceil \text{that} \rceil^{lex} = \lambda v \lambda w \lambda z.((w\ (\blacktriangledown\ \triangledown\ z)) \wedge (v\ z))$$

- v of type $\lceil s/\Diamond\Box np \rceil^{lex} = \Diamond\Box e \rightarrow t$; w of type $\lceil n \rceil^{lex} = e \rightarrow t$
- z reusable $\Diamond\Box e$ variable distributed over the $\wedge$ conjuncts

Proof term M, derivational $\lceil M \rceil^{der}$ and lexical $\lceil M \rceil^{lex}$ translations:

$$\begin{array}{rcl} M & = & \text{paper} \rtimes (\text{that} \ltimes \lambda^r x.(\text{Bob} \rtimes ((\text{rejected} \ltimes (\blacktriangledown\ \triangledown\ x))) \rtimes \text{immediately})) : n \\ \lceil M \rceil^{der} & = & (\lceil \text{that} \rceil\ \lambda x.((\lceil \text{immediately} \rceil\ (\lceil \text{rejected} \rceil\ (\blacktriangledown\ \triangledown\ x)))\ \lceil \text{Bob} \rceil))\ \lceil \text{paper} \rceil : e \multimap t \\ \lceil M \rceil^{lex} & = & \lambda z.((\textsc{paper}\ (\blacktriangledown\ \triangledown\ z)) \wedge ((\textsc{immediately}\ (\textsc{rejected}\ (\blacktriangledown\ \triangledown\ z)))\ \textsc{bob})) : \Diamond\Box e \rightarrow t \end{array}$$

From postulates to structural rules

Linearity general form of linear structural rules: Moot 2002

$$\frac{\Gamma[\Xi[\Delta_1, \ldots, \Delta_n]] \vdash A}{\Gamma[\Xi'[\Delta_{\pi_1}, \ldots, \Delta_{\pi_n}]] \vdash A}\ R$$

- $\Xi[], \Xi'[]$ generalized contexts of arity n: $\mathcal{C} ::= [] \mid \langle\mathcal{C}\rangle \mid \mathcal{C} \cdot \mathcal{C}$ arity: # holes
- $\Xi[\Gamma_1, \ldots, \Gamma_n]$ structure obtained by substitution of $\Gamma_1, \ldots, \Gamma_n$ in $\Xi[]$ of arity n

Example controlled associativity/commutativity postulates in rule form

$\mathsf{A}^r_\diamond : \quad (A \bullet B) \bullet \Diamond C \longrightarrow A \bullet (B \bullet \Diamond C) \qquad \mathsf{C}^r_\diamond : \quad (A \bullet B) \bullet \Diamond C \longrightarrow (A \bullet \Diamond C) \bullet B$

$$\frac{\Gamma[\Delta \cdot (\Delta' \cdot \langle\Delta''\rangle)] \vdash A}{\Gamma[(\Delta \cdot \Delta') \cdot \langle\Delta''\rangle] \vdash A}\ \mathsf{A}^r_\diamond \qquad \frac{\Gamma[(\Delta \cdot \langle\Delta''\rangle) \cdot \Delta'] \vdash A}{\Gamma[(\Delta \cdot \Delta') \cdot \langle\Delta''\rangle] \vdash A}\ \mathsf{C}^r_\diamond$$

$\rightsquigarrow$ replace formula vars by structure vars, $\Diamond, \bullet$ by their structural counterparts

Terms the linear structural rules leave the proof term unchanged

From postulates to structural rules (cont'd)

$$\frac{\Gamma[\Xi[\Delta_1,\ldots,\Delta_n]] \vdash A}{\Gamma[\Xi'[\Delta_{\pi_1},\ldots,\Delta_{\pi_n}]] \vdash A}\ R$$

Linear, non-increasing R is non-increasing if $|\Xi'[]| \leq |\Xi[]|$

- number of unary $\langle\rangle$ in conclusion $\leq$ in number of $\langle\rangle$ premise
- compare: $\Diamond(A \bullet B) \longrightarrow \Diamond A \bullet \Diamond B$ ✓; but not $\Diamond A \bullet \Diamond B \longrightarrow \Diamond(A \bullet B)$

Complexity, expressivity (Moot 2002) $\mathbf{NL}_\Diamond$ + linear, non-increasing structural rules:

- decidable
- PSPACE complete
- recognizes the context-sensitive languages

Mildly CS fragments? Moot 2008, simulating TAGs $\simeq$ 2-MCFG$_{wn}$

Controlling copying: lexicon or syntax?

Parasitic gaps felicitous only in the context of a primary gap, compare c, d

a	papers that Bob rejected ␣ (immediately)	gap
b	Bob left the room without closing the window	
c	*window that Bob left the room without closing ␣	island
d	papers that reviewers rejected ␣ without reading ␣ (carefully)	pg: adjunct
e	security breach that a report about ␣ in the NYT made public ␣	

Reduction to lexical polymorphism MM, Sadrzadeh, Wijnholds 2019

$$\text{without}^{b,c} \quad :: \quad \Box(X\backslash X)/Z, X = iv, Z = gp \text{ (gerund)}$$
$$\text{without}^{d} \quad :: \quad \Box((X/\Diamond\Box np)\backslash(X/\Diamond\Box np))/(Z/\Diamond\Box np)$$

Semantically, with $\lceil np\backslash s\rceil = \lceil gp\rceil = \mathsf{N}^* \otimes \mathsf{S}$, $\lceil\Diamond\Box np\rceil = \mathsf{N}$, without^d reduces to transitive verb coordination, i.e. $\lceil$rejected$\rceil \odot \neg\lceil$reading$\rceil$

$$(\mathsf{N} \otimes \mathsf{S}^* \otimes \mathsf{N}) \otimes (\mathsf{N}^* \otimes \mathsf{S} \otimes \mathsf{N}^*) \otimes (\mathsf{N} \otimes \mathsf{S}^* \otimes \mathsf{N})$$

Alternative: controlled contraction in syntax

Recall the postulates for regular gaps (no copying involved): controlled associativity $A_\diamond$, controlled commutativity $C_\diamond$ allowing non-peripheral gaps.

$$\begin{array}{ll} A_\diamond : & (A \bullet B) \bullet \Diamond C \longrightarrow A \bullet (B \bullet \Diamond C) \\ C_\diamond : & (A \bullet B) \bullet \Diamond C \longrightarrow (A \bullet \Diamond C) \bullet B \end{array}$$

We now add variants of $A_\diamond$, $C_\diamond$ for the cases of extraction that involve copying:

$$\begin{array}{ll} A^!_\diamond : & (\Diamond A \bullet B) \bullet \Diamond C \longrightarrow \Diamond(A \bullet \Diamond C) \bullet (B \bullet \Diamond C) \\ C^!_\diamond : & (A \bullet \Diamond B) \bullet \Diamond C \longrightarrow (A \bullet \Diamond C) \bullet \Diamond(B \bullet \Diamond C) \end{array}$$

- In addition to the principal gap, $A^!_\diamond$ and $C^!_\diamond$ drop a secondary gap in an island phrase ($\Diamond$ marked) that would be inaccessible without the principal gap.
- $A^!_\diamond$: pg precedes principal gap
- $C^!_\diamond$: pg follows principal gap

Illustration

$$\frac{\text{reading}}{gp/np} \qquad \frac{\Box np \vdash \Box np}{\langle \Box np \rangle \vdash np}\,\Box E$$

$$\frac{gp/np \qquad \langle \Box np \rangle \vdash np}{\text{reading} \cdot \langle \Box np \rangle \vdash gp}\,/E$$

$$\frac{\text{without}}{\Box(s\backslash s)/gp}$$

$$\frac{\Box(s\backslash s)/gp \qquad \text{reading} \cdot \langle \Box np \rangle \vdash gp}{\text{without} \cdot (\text{reading} \cdot \langle \Box np \rangle) \vdash \Box(s\backslash s)}\,/E$$

$$\frac{\text{without} \cdot (\text{reading} \cdot \langle \Box np \rangle) \vdash \Box(s\backslash s)}{\langle \text{without} \cdot (\text{reading} \cdot \langle \Box np \rangle)\rangle \vdash s\backslash s}\,\Box E$$

$$\frac{\text{rejected}}{(np\backslash s)/np} \qquad \frac{\Box np \vdash \Box np}{\langle \Box np \rangle \vdash np}\,\Box E$$

$$\frac{(np\backslash s)/np \qquad \langle \Box np \rangle \vdash np}{\text{rejected} \cdot \langle \Box np \rangle \vdash np\backslash s}\,/E$$

$$\frac{\dfrac{\text{bob}}{np} \qquad \text{rejected} \cdot \langle \Box np \rangle \vdash np\backslash s}{\text{bob} \cdot (\text{rejected} \cdot \langle \Box np \rangle) \vdash s}\,\backslash E$$

$$\frac{\text{bob} \cdot (\text{rejected} \cdot \langle \Box np \rangle) \vdash s \qquad \langle \text{without} \cdot (\text{reading} \cdot \langle \Box np \rangle)\rangle \vdash s\backslash s}{(\text{bob} \cdot (\text{rejected} \cdot \langle \Box np \rangle)) \cdot \langle \text{without} \cdot (\text{reading} \cdot \langle \Box np \rangle)\rangle \vdash s}\,\backslash E$$

$$\frac{(\text{bob} \cdot (\text{rejected} \cdot \langle \Box np \rangle)) \cdot \langle \text{without} \cdot (\text{reading} \cdot \langle \Box np \rangle)\rangle \vdash s}{((\text{bob} \cdot \text{rejected}) \cdot \langle \Box np \rangle) \cdot \langle (\text{without} \cdot \text{reading}) \cdot \langle \Box np \rangle\rangle \vdash s}\,A^{\diamond}, A^{\diamond}$$

$$\frac{((\text{bob} \cdot \text{rejected}) \cdot \langle \Box np \rangle) \cdot \langle (\text{without} \cdot \text{reading}) \cdot \langle \Box np \rangle\rangle \vdash s}{((\text{bob} \cdot \text{rejected}) \cdot \langle \text{without} \cdot \text{reading}\rangle) \cdot \langle \Box np \rangle \vdash s}\,C^{!}_{\diamond}$$

$$\frac{((\text{bob} \cdot \text{rejected}) \cdot \langle \text{without} \cdot \text{reading}\rangle) \cdot \langle \Box np \rangle \vdash s}{((\text{bob} \cdot \text{rejected}) \cdot \langle \text{without} \cdot \text{reading}\rangle) \cdot \Diamond\Box np \vdash s}\,\Diamond E, Ax$$

$$\frac{((\text{bob} \cdot \text{rejected}) \cdot \langle \text{without} \cdot \text{reading}\rangle) \cdot \Diamond\Box np \vdash s}{((\text{bob} \cdot \text{rejected}) \cdot \langle \text{without} \cdot \text{reading}\rangle) \vdash s/\Diamond\Box np}\,/I$$

$$\frac{\dfrac{\text{that}}{(n\backslash n)/(s/\Diamond\Box np)} \qquad ((\text{bob} \cdot \text{rejected}) \cdot \langle \text{without} \cdot \text{reading}\rangle) \vdash s/\Diamond\Box np}{\text{that} \cdot \ldots \vdash n\backslash n}\,/E$$

$$\frac{\dfrac{\text{paper}}{n} \qquad \text{that} \cdot \ldots \vdash n\backslash n}{\text{paper} \cdot (\text{that} \cdot \ldots) \vdash n}\,\backslash E$$

Blocking structural rules

Recall the island violations caused by (global or controlled!) associativity:

paper that (Alice reviewed a thesis) but$_{(s\backslash s)/s}$ (Bob rejected __)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\ldots}{s} \quad \dfrac{\dfrac{\text{but}}{(s\backslash\Box s)/s} \quad \dfrac{\dfrac{\text{B}}{np} \quad \dfrac{\dfrac{\text{rejected}}{(np\backslash s)/np} \quad \dfrac{\Box np \vdash \Box np}{\langle \Box np\rangle \vdash np}}{\text{rejected}\cdot\langle\Box np\rangle \vdash np\backslash s}\,/E}{\text{B}\cdot(\text{rejected}\cdot\langle\Box np\rangle)\vdash s}\,\backslash E}{\text{but}\cdot(\text{B}\cdot(\text{rejected}\cdot\langle\Box np\rangle))\vdash s\backslash\Box s}\,/E}{\ldots\cdot(\text{but}\cdot(\text{B}\cdot(\text{rejected}\cdot\langle\Box np\rangle)))\vdash \Box s}\,\backslash E}{\langle\ldots\cdot(\text{but}\cdot(\text{B}\cdot(\text{rejected}\cdot\langle\Box np\rangle)))\rangle\vdash s}\,\Box E}{\langle\ldots\cdot(\text{but}\cdot(\text{B}\cdot\text{rejected}))\rangle\cdot\langle\Box np\rangle\vdash s}\,\lightning\lightning\lightning
$$

$\diamond$ **as an obstacle** a modified type assignment imposes the desired island constraint:

- but :: $(s\backslash\Box s)/s$ Morrill 1994
- $\Box$ Elim seals off the conjunction as an island from which $\langle\Box np\rangle$ cannot escape

We will generalize this idea to demarcate dependency domains ...

Comparing RES and BANG

Correspondences Similarities more striking than differences, reading $!_i$ as $\Diamond_i \Box_i$

Simulating $!_i$ properties as combinations of $\Diamond, \Box$ logical and structural rules, e.g.

$$\frac{\Gamma \vdash B}{!\,\Gamma \vdash !\,B}\ SP \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash B}{\langle \Box \rangle \Gamma \vdash B}\ \Box L}{\langle \Box \Gamma \rangle \vdash B}\ K}{\Box \Gamma \vdash \Box B}\ \Box R$$

MM 1996

Differences some features of RES not shared by BANG

- licensing and blocking uses of modalities share same logical rules
- components $\Diamond$ and $\Box$ have individual uses, cf the dependency annotation

Resolution? Multitype approach, Palmigiano c.s., arguing that ! cannot be seen as primitive, but must be deconstructed in heterogeneous adjoint pair $\Diamond\blacksquare$

Dependency modalities

Heads vs dependents

Dependency roles articulate the linguistic material on the basis of two oppositions:

- head - complement relations
  - verbal domain: subj, (in)direct object, ...
  - nominal domain: prepositional object, ...
- adjunct - head relations
  - verbal domain: (time, manner, ...) adverbial
  - nominal domain: adjectival, numeral, determiner, ...

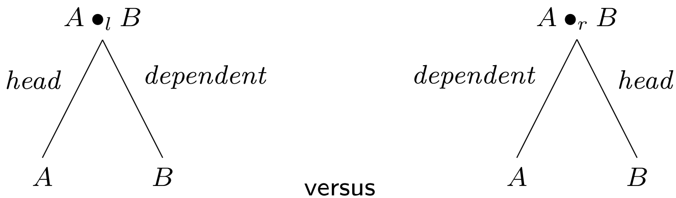
Compare: fa-structure: function vs argument

Orthogonality The fa and the dependency articulation are in general not aligned. This asks for a multidimensional type logic.

E.g. Determiner. Semantically, characteristic function of $(\llbracket N \rrbracket, \llbracket VP \rrbracket)$ relation; morphologically, dependent on head noun.

DNL

Bimodal NL Moortgat & Morrill, 1991, Heads and phrases. Type calculus for dependency and constituent structure. Ms UU



$$A \longrightarrow C/_l B \quad \text{iff} \quad A \bullet_l B \longrightarrow C \quad \text{iff} \quad B \longrightarrow A\backslash_l C$$
$$A \longrightarrow C/_r B \quad \text{iff} \quad A \bullet_r B \longrightarrow C \quad \text{iff} \quad B \longrightarrow A\backslash_r C$$

- left vs right-headed $\bullet$
- heads: $C/_l B$, $A\backslash_r C$; dependents: $C/_r B$, $A\backslash_l C$
- models: prosodic prominence, morphosyntactic government/rection, . . .

Defining headed products

Left/right headed $\bullet$ as composition of regular $\bullet$ and modal marking of the dependent:

$$\text{left headed} := A \bullet \Diamond B \qquad \text{right headed} := \Diamond A \bullet B$$

Residuation: translation of the slashes

recall: $\Diamond A \longrightarrow B$ iff $A \longrightarrow \Box B$

$$\begin{array}{c} A \longrightarrow C/\Diamond B \\ \hline\hline A \bullet \Diamond B \longrightarrow C \\ \hline\hline \Diamond B \longrightarrow A\backslash C \\ \hline\hline B \longrightarrow \Box(A\backslash C) \end{array} \qquad \begin{array}{c} A \longrightarrow \Box(C/B) \\ \hline\hline \Diamond A \longrightarrow C/B \\ \hline\hline \Diamond A \bullet B \longrightarrow C \\ \hline\hline B \longrightarrow \Diamond A\backslash C \end{array}$$

Multimodal generalization families $\{\Diamond_d, \Box_d\}_{d \in DepLabel}$

- $\Diamond_d A\backslash C$, $C/\Diamond_d B$ head functor assigning dependency role d to its complement
- $\Box_d(A\backslash C)$, $\Box_d(C/B)$ dependent functor projecting adjunct role d

Dependency structure

Dependency-enhanced types:

$$\dfrac{\dfrac{\mathsf{the}}{\Box_{det}(np/n)}}{\langle \mathsf{the}\rangle^{det} \vdash np/n}\ {\color{red}\Box E} \qquad \dfrac{\mathsf{skies}}{n}$$

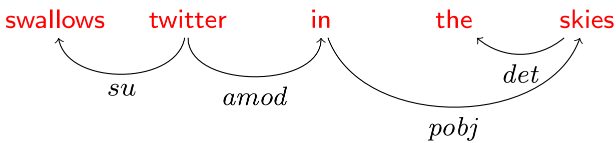
$$\dfrac{\langle \mathsf{the}\rangle^{det} \vdash np/n \qquad n}{\langle \mathsf{the}\rangle^{det} \cdot \mathsf{skies} \vdash np}\ /E \qquad \dfrac{\langle \mathsf{the}\rangle^{det} \cdot \mathsf{skies} \vdash np}{\langle\langle \mathsf{the}\rangle^{det} \cdot \mathsf{skies}\rangle^{pobj} \vdash \Diamond_{pobj} np}\ {\color{red}\Diamond I}$$

$$\dfrac{\dfrac{\mathsf{in}}{\Box_{amod}(s\backslash s)/\Diamond_{pobj} np} \qquad \langle\langle \mathsf{the}\rangle^{det} \cdot \mathsf{skies}\rangle^{pobj} \vdash \Diamond_{pobj} np}{\dfrac{\mathsf{in} \cdot \langle\langle \mathsf{the}\rangle^{det} \cdot \mathsf{skies}\rangle^{pobj} \vdash \Box_{amod}(s\backslash s)}{\langle \mathsf{in} \cdot \langle\langle \mathsf{the}\rangle^{det} \cdot \mathsf{skies}\rangle^{pobj}\rangle^{amod} \vdash s\backslash s}\ {\color{red}\Box E}}\ /E$$

$$\dfrac{\dfrac{\dfrac{\mathsf{swallows}}{np}}{\langle \mathsf{swallows}\rangle^{su} \vdash \Diamond_{su} np}\ {\color{red}\Diamond I} \qquad \dfrac{\mathsf{twitter}}{\Diamond_{su} np\backslash s}}{\langle \mathsf{swallows}\rangle^{su} \cdot \mathsf{twitter} \vdash s}\ \backslash E$$

$$\dfrac{\langle \mathsf{swallows}\rangle^{su} \cdot \mathsf{twitter} \vdash s \qquad \langle \mathsf{in} \cdot \langle\langle \mathsf{the}\rangle^{det} \cdot \mathsf{skies}\rangle^{pobj}\rangle^{amod} \vdash s\backslash s}{(\langle \mathsf{swallows}\rangle^{su} \cdot \mathsf{twitter}) \cdot \langle \mathsf{in} \cdot \langle\langle \mathsf{the}\rangle^{det} \cdot \mathsf{skies}\rangle^{pobj}\rangle^{amod} \vdash s}\ \backslash E$$

Induced dependency structure:



$\rightsquigarrow$ within dependency domain, outgoing arcs from head to (head of) dependents

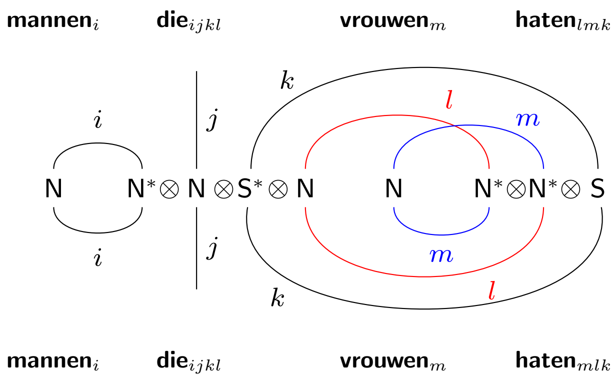
Extraction revisited

NL Relatives Dutch left-branch extraction via controlled associativity, commutativity

$$\Diamond_x A \bullet (B \bullet C) \longrightarrow (\Diamond_x A \bullet B) \bullet C \qquad \Diamond_x A \bullet (B \bullet C) \longrightarrow B \bullet (\Diamond_x A \bullet C)$$

Relative pronoun: die :: $(n\backslash n)/(!_x\, np\backslash s)$ $\qquad !_x\, A \triangleq \Diamond_x \Box_x A$

ambiguous between subj/obj relativization: $\qquad s$ subordinate clause, head-final



MM & Wijnholds 2017

Extraction revisited (cont'd)

Dependency refinement derivational ambiguity is traded in for lexical ambiguity, to be resolved in the supertagging phase.

- NL is head-final: transitive verb type:

$$\text{haten} \quad :: \quad \Diamond_{obj}\, np \backslash (\Diamond_{subj}\, np \backslash s)$$

- two relative pronoun types: subject vs object relativization

$$\text{die} \quad :: \quad \Box_{mod}(n \backslash n) / \Diamond_{body}(\Diamond_{subj}\, np \backslash s)$$

$$\text{die} \quad :: \quad \Box_{mod}(n \backslash n) / \Diamond_{body}(\Diamond_{obj}\, np \backslash s)$$

Rethinking constituency

Associativity head + $\diamond_d$ demarcated dependents constitutes dependency domain; within these domains $\bullet$ associativity freely available.

Down the rabbit hole The above relpro types restrict access to immediate dependents of the rel clause body. die :: $\Box_{mod}(n\backslash n)/\diamond_{body}(!_x\diamond_{subj|obj}np\backslash s)$ reaches more deeply embedded hypotheses.

The *xleft* (derived) inference rule now has $\Gamma[\ ]$ traversing unary+binary structure:

$$\frac{\Gamma[A\cdot\Delta]\vdash B}{\Gamma[\Delta]\vdash\,!_x\,A\backslash B}\ xleft$$

$\leadsto$ requires extra postulate allowing $\diamond_x$ to commute with dependency modalities $\diamond_d$ for (all | some) $d \in \mathit{DepLabel}$:

$$\diamond_x A\bullet\diamond_d B\longrightarrow\diamond_d(\diamond_x A\bullet B)$$

$$\frac{\Gamma[\langle\langle\Delta\rangle^x\cdot\Delta'\rangle^d]\vdash A}{\Gamma[\langle\Delta\rangle^x\cdot\langle\Delta'\rangle^d]\vdash A}\ xleft'$$

A neurosymbolic perspective

Challenges

Recall we write $L(G, B)$ for the strings of type B recognized by grammar G.

$w_1 \cdots w_n \in L(G, B)$ if the following hold:

- $(w_i, A_i) \in$ Lex for $1 \leq i \leq n$;
- $\Gamma_{[A_1,\ldots,A_n]} \vdash B$, for Γ an antecedent structure with yield $A_1, \ldots, A_n$

- type ambiguity: what is the right type for w_i given its context?

  $\rightsquigarrow$ supertagging

- structural ambiguity: what is the proper structure for Γ to derive B

  $\rightsquigarrow$ parsing

Training data: NL types in the wild

Type lexicon + derivations/λ terms extracted from Lassy Small, gold standard treebank of written Dutch. 68782 samples.

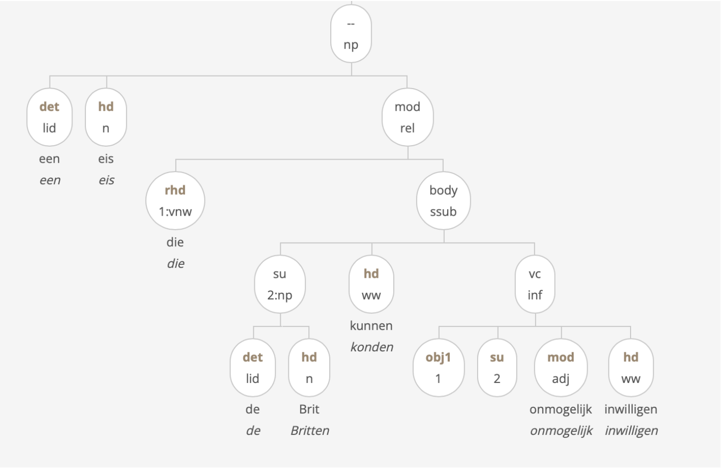
- Lassy annotation: DAGs, nodes: categories, edges: dependency relations
- Re-entrancy: gaps, coordination, ('understood subjects' of non-finite verb forms)
- traditional dependency roles; can be mapped to UD Bouma & vNoord 2017

Lassy2Æthel extraction

- non-directional syntax types: alignment with surface string left to neural parser
- modalities: dependency marking; structural control (extraction)
- finegrained result — compare CCG: categories 5292/1323, slashes($+\Diamond, \Box$) 29/2

Ref Kogkalidis, MM & Moot, 2020
Æthel: Automatically Extracted Typelogical Derivations for Dutch. LREC.
`https://github.com/konstantinosKokos/aethel`

A demand that the British couldn't possibly grant: Lassy



- tree display format, avoiding crossing edges — word order: position indices
- re-entrancy relpro 'die' ∼ obj1: gap hypothesis
- re-entrancy su 'de Britten' ∼ su: understood subject infinitive

A demand that the British couldn't possibly grant

Sample WR-P-E-I-0000015007.p.1.s.51.xml(27) ()

```
In [26]: sample = aethel[37628]
```

```
In [28]: sample.sentence
```

```
Out[28]: 'een eis die de Britten onmogelijk konden inwilligen .'
```

```
In [27]: list(sample.lexical_phrases)
```

```
Out[27]: [LexicalPhrase(string=een, type=□det(N⟶NP), len=1),
          LexicalPhrase(string=eis, type=N, len=1),
          LexicalPhrase(string=die, type=(◇relcl(◇x(□x(◇obj1(VNW)))⟶SSUB))⟶□mod(NP⟶NP), len=1),
          LexicalPhrase(string=de, type=□det(N⟶NP), len=1),
          LexicalPhrase(string=Britten, type=N, len=1),
          LexicalPhrase(string=onmogelijk, type=□mod(INF⟶INF), len=1),
          LexicalPhrase(string=konden, type=◇vc(INF)⟶◇su(NP)⟶SSUB, len=1),
          LexicalPhrase(string=inwilligen, type=◇obj1(VNW)⟶INF, len=1),
          LexicalPhrase(string=., type=PUNCT, len=1)]
```

```
In [22]: proof=sample.proof
```

```
In [23]: print(proof)
```

```
⟨c2, ⟨c6, ⟨ ⟨c5⟩ mod, c7⟩ vc, ⟨ ⟨c3⟩ det, c4⟩ su⟩ relcl⟩ mod, ⟨c0⟩ det, c1 ⊢ ▾mod(c2 ▵relcl((λx0.case ▿x(x0) of x1
in (c6 ▵vc(▾mod(c5) (c7 ▾x(x1))) ▵su(▾det(c3) c4))))) (▾det(c0) c1) : NP
```

$\blacktriangledown^{mod}$(die $\triangle^{relcl}$ (λx_0.**case** $\triangledown^{x} x_0$ **of** x_1 **in** (konden $\triangle^{vc}$ ($\blacktriangledown^{mod}$onmogelijk (inwilligen $\blacktriangledown^{x} x_1$)) $\triangle^{su}$ ($\blacktriangledown^{det}$de Britten)))) ($\blacktriangledown^{det}$een eis)

Going neural

PhD project Konstantinos Kogkalidis

- Kogkalidis, 2023, Dependency as Modality, Parsing as Permutation.
  Phd Thesis, Utrecht University. url
- Kogkalidis & MM, 2022, arXiv
  Geometry-Aware Supertagging with Heterogeneous Dynamic Convolutions
- Kogkalidis, MM & Moot, 2020
  Neural Proof Nets. CoNLL url

Code: `https://github.com/konstantinosKokos/spindle`

Integrating supertagging and neural parsing

Neural proof nets The parsing method uses LL proof nets. Proof net construction can be seen as a staged process:

- proof frame: forest of formula decomposition trees — supertagging ☺
- proof structure: p frame plus pairwise linking of in/out atoms
- proof net: p structure with successful traversal

  $\mathrm{MILL}_{\multimap}^{\Diamond,\Box}$ lambda term as byproduct of traversal

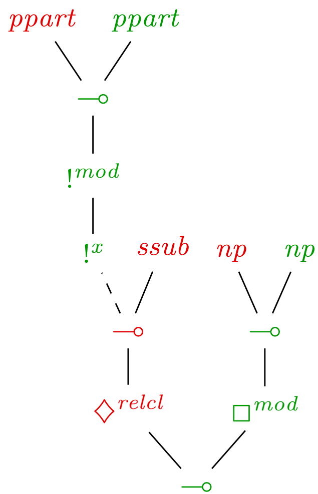
Key neural methods

- supertagging: parallel tree decoding with dynamic graph convolutions
- axiom linking: Sinkhorn iterative method to approach double stochastic matrix
- verification: Lamarche traversal method Lamarche 2008

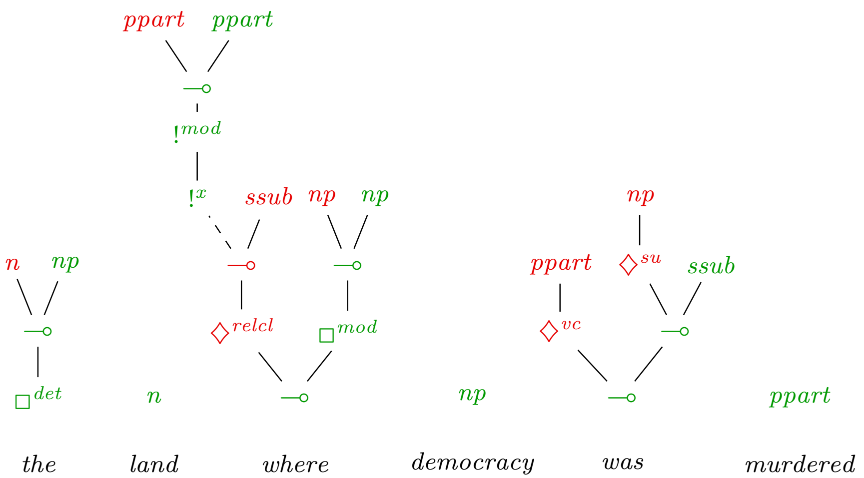
Supertag = polarized formula decomposition tree

Example the land where democracy was murdered

Polarities green: given, input; red: to prove, output

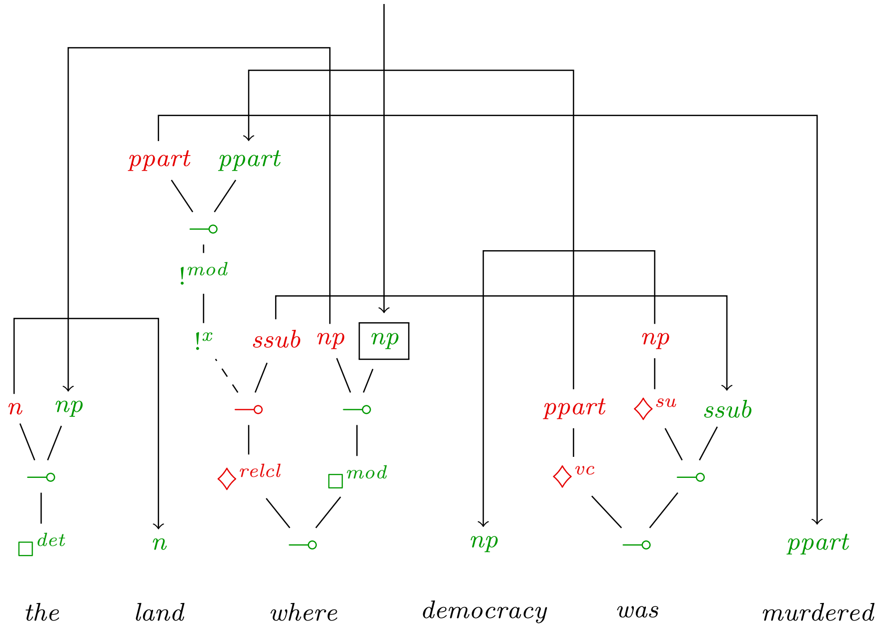


Proof frame



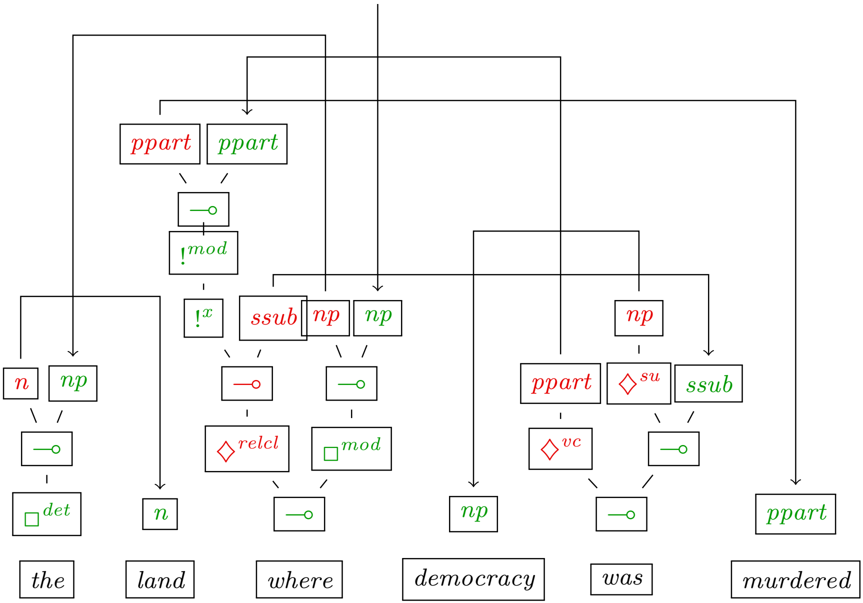
ppart
ppart
!mod
!x
ssub
np
np
np
n
np
ppart
su
ssub
relcl
mod
vc
det
n
np
ppart
the
land
where
democracy
was
murdered

Proof frame $\rightsquigarrow$ proof structure



Proof structure ⇝ proof net

the ESSLLI2022 movie is here



$\blacktriangledown^{mod}$(where $\vartriangle^{relcl}$ (λx_0.(was $\vartriangle^{vc}$ ($\blacktriangledown^{mod}\triangledown^{mod}\blacktriangledown^{x}\triangledown^{x} x_0$ murdered) $\vartriangle^{su}$ democracy))) ($\blacktriangledown^{det}$the land)

Conclusions

Some key themes of the talk:

- Logic: a modally enhanced multi-dimensional type logic
  - dependency structure $\perp$ function-argument structure
  - linear$^{\Diamond,\Box}$ lambda terms as general-purpose recipes for meaning composition
  - where possible, confine non-linearity to lexical meaning recipes
- NLP: end-to-end compositionality:
  - obtain elementary word embeddings from data, and additionally
  - their types and their internal composition
  - neural parsing (grounded in/informed by) data-driven word representations

$\Box$

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