Parallelism in Realizability Models

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Introduction

Realizability semantics $\approx$
Interpreting a formal system based on a computational model.

- computational models
  Turing machines/Type 2 machines/λ-calculus/$\cdots$
- formal systems
  Arithmetic/Analysis/(Higher-order) Logic/Programming language

The recent realizability theory expresses a computational model by a simple algebraic structure.
$\rightsquigarrow$ Partial Combinatory Algebra (PCA)

The interpretation itself has been given a categorical generalization.
$\rightsquigarrow$ The category of assemblies $\mathbf{Ass}(A)$, Realizability topos $\mathbf{RT}(A)$

Introduction

Realizability theory $\approx$ Study of such categorical models over PCAs

Question

How is the structure of $\mathbf{Ass}(A)$ affected by the choice of PCA A?

Parallel operation in A v.s. Predominance in $\mathbf{Ass}(A)$

Outline

Parallel operation in PCA A

Predominance and Σ-subset in $\mathbf{Ass}(A)$

Characterization theorems

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Definition (PCA)

A *partial combinatory algebra (PCA)* is a set A equipped with a partial binary operation $\cdot : A \times A \rightharpoonup A$ such that $\exists\, \mathrm{k}, \mathrm{s} \in A$ satisfying

$$\mathrm{k} \cdot a \downarrow, \quad (\mathrm{k} \cdot a) \cdot b = a$$

$$(\mathrm{s} \cdot a) \cdot b \downarrow, \quad ((\mathrm{s} \cdot a) \cdot b) \cdot c \cong (a \cdot c) \cdot (b \cdot c)$$

for any $a, b, c \in A$.
We often write abc instead of $((a \cdot b) \cdot c)$.

The closed λ-terms (modulo a λ-theory T) form a PCA Λ^0 by letting

$$\mathrm{k} \coloneqq \lambda xy.\, x, \qquad \mathrm{s} \coloneqq \lambda xyz.\, (xz)(yz).$$

This is the simplest example of a total PCA.

Conversely, every PCA can imitate "untyped λ-terms" by using k and s (combinatory completeness).

Examples of non-total PCAs

(i) **Kleene's first algebra** $\mathcal{K}_1$:

Let $[\![n]\!] : \mathbb{N} \rightharpoonup \mathbb{N}$ denote the n-th partial computable function.

Underlying set: the set of natural numbers $\mathbb{N}$

Application:

$$n \cdot m :\equiv [\![n]\!](m)$$

(ii) **Call-by-value λ-term model** Λ_v^0:

Consider the *(lazy) call-by-value reduction strategy*.

Underlying set: the set of closed values Λ_v^0

Application:

$$V_1 \cdot V_2 \; :\equiv W \; \text{ if } V_1 V_2 \twoheadrightarrow_{cbv} W \text{ for some } W \in \Lambda_v^0$$

Otherwise, $V_1 \cdot V_2$ is undefined.

Comparing $\mathcal{K}_1$ and Λ_v^0 as PCA, Λ_v^0 excludes Plotkin's parallel-or function, whereas $\mathcal{K}_1$ includes that.

Notation (Some useful combinators)

Hereafter, A denotes a PCA.

Identity $\mathrm{i} := \lambda^* x.\, x = \mathrm{skk}.$

Boolean $\mathrm{true} := \lambda^* xy.\, x, \quad \mathrm{false} := \lambda^* xy.\, y.$

Pairing $\langle x, y \rangle := \lambda^* z.\, zxy.$

Notation

Given subsets $S_0, S_1 \subseteq A$,

$$S_0 \times S_1 := \{ \langle a_0, a_1 \rangle \in A \mid a_0 \in S_0 \text{ and } a_1 \in S_1 \}$$

What is Parallelism?

Recall the original parallel-or function $\mathrm{por}^{\mathrm{p}}$ [Plotkin 77].

$$\begin{array}{ll} \mathrm{por}^{\mathrm{p}} MN \Downarrow \mathrm{true} & \text{if } M \Downarrow \mathrm{true} \text{ or } N \Downarrow \mathrm{true} \\ \mathrm{por}^{\mathrm{p}} MN \Downarrow \mathrm{false} & \text{if } M \Downarrow \mathrm{false} \text{ and } N \Downarrow \mathrm{false} \\ \mathrm{por}^{\mathrm{p}} MN \Uparrow & \text{otherwise} \end{array}$$

(M, N are λ-terms and $M \Downarrow V$ means that M evaluates to a value V). You cannot evaluate M and N one by one because the evaluation may diverge. Thus they must be evaluated "in parallel".

In fact, it is known that sequential programming languages do not admit parallel-or.

- "Parallel-or" cannot be defined in PCF [well-known].
- cf: sequentiality theorem for PCA $\Lambda^0/T_{\mathcal{BT}}$ [Berry 78].

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Let's now identify false with divergence.

$$\mathrm{por} MN \Downarrow \quad \text{iff} \quad M \Downarrow \text{ or } N \Downarrow .$$

Parallel combinators: formal definition

Definition (Parallel combinators relative to Σ)

Let $\Sigma = (T, F)$ be a pair of nonempty subsets of A.
An element $\mathrm{or}_\Sigma \in A$ is called a Σ-*or combinator* if it satisfies

$$\mathrm{or}_\Sigma(T \times T) \subseteq T, \quad \mathrm{or}_\Sigma(T \times F) \subseteq T,$$
$$\mathrm{or}_\Sigma(F \times T) \subseteq T, \quad \mathrm{or}_\Sigma(F \times F) \subseteq F.$$

Dually, we define a Σ-*and combinator* and_Σ.

These are defined relative to each "abstract truth value" $\Sigma = (T, F)$.

In particular, whenever $T \cup F = A$,

$$\mathrm{or}_\Sigma \cdot \langle f, g\rangle \in T \quad \text{iff} \quad f \in T \text{ or } g \in T.$$

$$\mathrm{and}_\Sigma \cdot \langle f, g\rangle \in T \quad \text{iff} \quad f \in T \text{ and } g \in T.$$

Definition (Mulry 1982)

For a non-total PCA A, define $\Sigma_{\mathrm{sd}} \coloneqq (T_{\mathrm{sd}}, F_{\mathrm{sd}})$ by

$$T_{\mathrm{sd}} \coloneqq \{\, a \in A \mid a \cdot \mathrm{i} \downarrow \},\quad F_{\mathrm{sd}} \coloneqq \{\, a \in A \mid a \cdot \mathrm{i} \uparrow \}.$$

"sd" means semi-decidable.

Then a Σ_{sd}-or combinator $\mathrm{or}_{\Sigma_{\mathrm{sd}}}$ and a Σ_{sd}-and combinator $\mathrm{and}_{\Sigma_{\mathrm{sd}}}$ behave as

$$\begin{aligned} \mathrm{or}_{\Sigma_{\mathrm{sd}}}\langle f, g\rangle \cdot \mathrm{i}\downarrow \quad &\text{iff} \quad f \cdot \mathrm{i}\downarrow \text{ or } g \cdot \mathrm{i}\downarrow, \\ \mathrm{and}_{\Sigma_{\mathrm{sd}}}\langle f, g\rangle \cdot \mathrm{i}\downarrow \quad &\text{iff} \quad f \cdot \mathrm{i}\downarrow \text{ and } g \cdot \mathrm{i}\downarrow, \end{aligned}$$

for every $f, g \in A$.
Thus we simply call them parallel-or and parallel-and combinators.

Theorem

1. Every non-total PCA admits parallel-and ($\mathrm{and}_{\Sigma_{\mathrm{sd}}}$).
2. $\mathcal{K}_1$ admits parallel-or ($\mathrm{or}_{\Sigma_{\mathrm{sd}}}$).
3. Λ_v^0 does not admit parallel-or ($\mathrm{or}_{\Sigma_{\mathrm{sd}}}$).

Another perspective of parallel combinators

Such a parallel operation is also important in elementary recursion theory, implicitly. Indeed, parallel-or seems to be necessary to prove the following easy fact:

Consequence of Σ_{sd}-or in $\mathcal{K}_1$

If $U, V \subseteq \mathbb{N}$ are semi-decidable subsets, then so is the union $U \cup V$.

Because

- $\mathrm{dom}(f) := \{ a \in \mathcal{K}_1 \mid f \cdot a \downarrow \}$.
- $U \subseteq \mathbb{N}$ is semi-decidable $\iff \exists f_U \in \mathcal{K}_1$ s.t. $U = \mathrm{dom}(f_U)$.
- Let $g := \lambda^* a.(\mathrm{or}_{\Sigma_{\mathrm{sd}}} \langle \lambda^* x.(f_U a), \lambda^* x.(f_V a) \rangle \cdot \mathrm{i})$.
  Then $U \cup V = \mathrm{dom}(g)$, so $U \cup V$ is also semi-decidable. □

Our goal: generalize this phenomenon to other PCAs.
$\rightsquigarrow$ categorical realizability

Outline

Parallel operation in PCA A

Predominance and Σ-subset in $\mathbf{Ass}(A)$

Characterization theorems

We here focus on the category of assemblies, because it is sufficiently rich as semantics of many formal systems.

Definition (The category of assemblies $\mathbf{Ass}(A)$)

Object: $X = (|X|, ||\cdot||_X)$
where $|X|$ is a set and $||\cdot||_X : |X| \to \mathcal{P}(A) \setminus \{\emptyset\}$ (a "coding" function). We call such a pair X an *assembly over* A.

Morphism: $f : X \to Y$
where f is a function from $|X|$ to $|Y|$ which has a *realizer* $r_f \in A$ satisfying

$$\forall a \in ||x||_X \quad r_f \cdot a \in ||f(x)||_Y .$$

Its internal logic is precisely equal to realizability interpretation:

$$\mathbf{Ass}(\mathcal{K}_1) \models \varphi \iff \varphi : \text{Kleene-realizable}.$$

Properties of $\mathbf{Ass}(A)$

Theorem

- $\mathbf{Ass}(A)$ is a finitely complete and locally cartesian-closed category with a natural number object.
- $\mathbf{Ass}(A)$ does not have a subobject classifier unless A is trivial.

Nevertheless, there is a "restricted subobject classifier" in $\mathbf{Ass}(A)$.

Let $\mathcal{C}$ be a finitely complete category.

Definition (Subobject classifier)

A morphism $t : 1 \rightarrowtail \Sigma$ in $\mathcal{C}$ is a *subobject classifier* if for every subobject $m : U \rightarrowtail X$ there is exactly one morphism $\chi_m : X \to \Sigma$ which gives a pullback diagram

$$\begin{array}{ccc} U & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle m}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle t} \\ X & \xrightarrow[\chi_m]{} & \Sigma. \end{array}$$

Let $\mathcal{C}$ be a finitely complete category.

Definition (Predominance, Σ-subset; Rosolini 86)

A morphism $t : 1 \rightarrowtail \Sigma$ in $\mathcal{C}$ is a *predominace* if for every subobject $m : U \rightarrowtail X$ there is at most one morphism $\chi_m : X \to \Sigma$ which gives a pullback diagram

$$\begin{array}{ccc} U & \xrightarrow{\ !\ } & 1 \\ {\scriptstyle m}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle t} \\ X & \xrightarrow[\chi_m]{} & \Sigma. \end{array}$$

A subobject $m : U \rightarrowtail X$ is called Σ*-subset of* X and written $U \subseteq_\Sigma X$ if m arises as a pullback of $t : 1 \rightarrowtail \Sigma$.
Let $\mathrm{Sub}_\Sigma(X)$ denote the set of Σ-subsets of X.

Notice that the Σ-subset relation may not transitive.

- Every predominance in $\mathbf{Ass}(A)$ can be induced by $\Sigma = (T, F)$ [Longley 94]:

$$|\Sigma| := \{ t, f \} \quad ||t||_\Sigma := T, \quad ||f||_\Sigma := F.$$

  Thus we call $\Sigma = (T, F)$ a predominance on A.

- Considering $\Sigma_{\mathrm{sd}} := (T_{\mathrm{sd}}, F_{\mathrm{sd}})$ in $\mathcal{K}_1$, Σ_{sd}-subsets of NNO N are in bijective correspondence with semi-decidable subsets!

$$\mathrm{Sub}_{\Sigma_{\mathrm{sd}}}(N) \simeq \{ U \subseteq \mathbb{N} \mid U \text{ is semi-decidable } \}$$

- A dominance is a predominance Σ such that $\subseteq_\Sigma$ is transitive. Dominance is one of the necessary pieces to construct a subcategory of "abstract domains" (Synthetic domain theory).

Outline

Under a natural assumption on predominance Σ,
the parallel operations in our sense and the order structure of Σ-subsets correspond perfectly.

Definition (Rice partition)

Let a, b be elements of A.

- $a \cong b$ means that $a \cdot x \cong b \cdot x$ for every $x \in A$.
- A predominance $\Sigma = (T, F)$ is a *Rice partition* if T is closed under $\cong$ and $F = A \setminus T$.

$\Sigma_{\mathrm{sd}} = (T_{\mathrm{sd}}, F_{\mathrm{sd}})$ is always a Rice partition in any non-total PCA.

$$T_{\mathrm{sd}} \coloneqq \{\, a \in A \mid a \cdot \mathrm{i} \downarrow \}, \quad F_{\mathrm{sd}} \coloneqq \{\, a \in A \mid a \cdot \mathrm{i} \uparrow \}$$

Role of Σ-and combinator

Then we obtain the first characterization theorem under this assumption.

Theorem (N.)

Let $\Sigma = (T, F)$ be a Rice partition of A. Then the following are equivalent:

1. A admits Σ-and combinator.
2. Σ is a dominance.
3. $(\mathrm{Sub}_\Sigma(X), \subseteq_\Sigma)$ is a poset for every $X \in \mathbf{Ass}(A)$.
4. $(\mathrm{Sub}_\Sigma(X), \subseteq_\Sigma, \cap)$ is a meet-semilattice for every $X \in \mathbf{Ass}(A)$.

($2 \iff 3 \iff 4$ due to [Rosolini 86], [Hyland 91])

Role of Σ-or combinator

Recall that existence of Σ_{sd}-or in $\mathcal{K}_1$ implies that the semi-decidable sets are closed under union.
This fact can be generalized and refined as follows.

Theorem (N.)

Let $\Sigma = (T, F)$ be a Rice partition of A. Then the following are equivalent:

1. A admits Σ-or combinator.
2. $\mathrm{Sub}_{\Sigma}(X)$ is closed under union $\cup$ for every $X \in \mathbf{Ass}(A)$.

Since the λ-term model Λ_v^0 does not admit Σ_{sd}-or, $\mathrm{Sub}_{\Sigma_{\mathrm{sd}}}(X)$ is not closed under $\cup$ for some $X \in \mathrm{Ass}(\Lambda_v^0)$.

Summary

Combining the two characterization theorems, we obtain

Theorem

Suppose that $\Sigma = (T, F)$ is a Rice partition of A. Then A admits both Σ-and and Σ-or if and only if $(\mathrm{Sub}_\Sigma(X), \subseteq_\Sigma, \cap, \cup)$ forms a lattice for every $X \in \mathbf{Ass}(A)$.

Corollary

Let A be a non-total PCA. Then A admits parallel-or $\mathrm{or}_{\Sigma_{\mathrm{sd}}}$ if and only if $(\mathrm{Sub}_{\Sigma_{\mathrm{sd}}}(X), \subseteq_{\Sigma_{\mathrm{sd}}}, \cap, \cup)$ forms a lattice for every $X \in \mathbf{Ass}(A)$.

Conclusion and Future work

Conclusion:

- We defined Σ-or and Σ-and combinators on PCA. In an appropriate Σ, Σ-or seems to be parallel-or.
- We have studied the relationship between the existence of parallel operations in A and the order structure of Σ-subsets in $\mathbf{Ass}(A)$.

Future work:

- A new logical model based on $\mathrm{Sub}_\Sigma(X)$
- Extension to $\mathbf{RT}(A)$: $\mathbf{Ass}(A) \subseteq \mathbf{RT}(A)$
- Relationship between models of PCF + por and this work

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