Beliefs based on Conflicting and Uncertain Evidence
-
Dempster-Shafer Theory & Topology of Evidence

Aybüke Özgün

ILLC, University of Amsterdam

WoLLIC 2023
Dalhousie University
July 11-14, 2023

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Outline

Tutorial

- Introduction to topological modeling of evidence and belief.
- Preliminaries of Dempster-Shafer Theory.

Talk

- Our new model for measuring degrees of belief in detail.

Topological Models of Evidence

Motivation

What is the relationship between belief, evidence, and knowledge?

This is a very old question, but modern work in *epistemic and doxastic logic* offers new approaches and insights.

Moreover, *topology* provides a framework naturally suited to the representation of *evidence* and its relationship to knowledge and belief.

Topology & Epistemology

What can Topology do for Epistemic Logic?

Cf. [Kelly 1996, Özgün 2017, Baltag et al. 2022].

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We can employ topological tools to:

- represent various notions of evidence,
- distinguish current evidence from potential evidence,
- study knowledge, knowability, and belief.

Cf. [Kelly 1996, Özgün 2017, Baltag et al. 2022].

Topology & Epistemology

Today, I will focus on formalizing *justified, consistent beliefs* based on *possibly false* and *possible mutually contradictory evidence*.

Cf. [van Benthem & Pacuit 2011].

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- An agent gathers *evidence* about the actual world from different sources.
- These sources may or may not be reliable, the agent does not know which sources are reliable***.
- The evidence gathered may be mutually inconsistent.
- Despite the inconsistencies, this is all the agent has to form beliefs.

Cf. [van Benthem & Pacuit 2011].

Topological Evidence Models

A *topological evidence model* is a tuple $\mathcal{M} = (S, \boldsymbol{\mathcal{E}}, \boldsymbol{\tau_{\mathcal{E}}})$, where

- $S \neq \emptyset$ is the set of *possible worlds*;
- $\boldsymbol{\mathcal{E}} \subseteq \mathcal{P}(S)$ is the set of *basic evidence sets*;
- $\boldsymbol{\tau_{\mathcal{E}}}$ is the *evidential topology*, the topology generated by $\boldsymbol{\mathcal{E}}$.

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$E \in \boldsymbol{\mathcal{E}}$: a piece of *direct evidence*.

Topology as Information Structures

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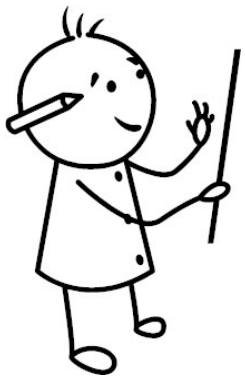

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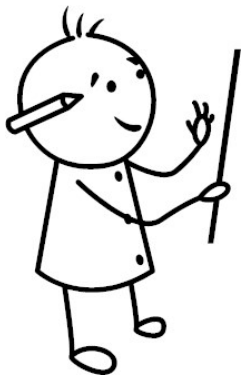

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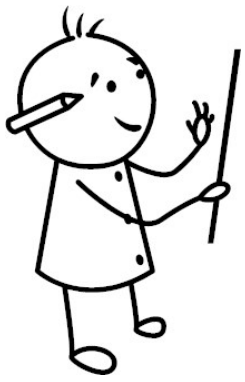

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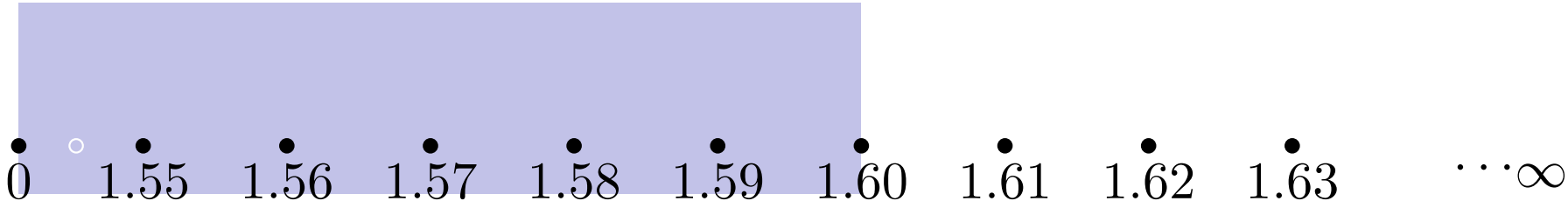

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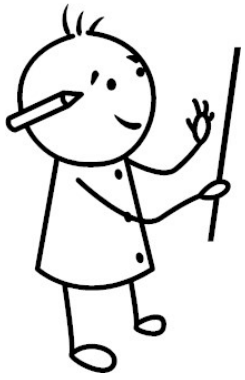

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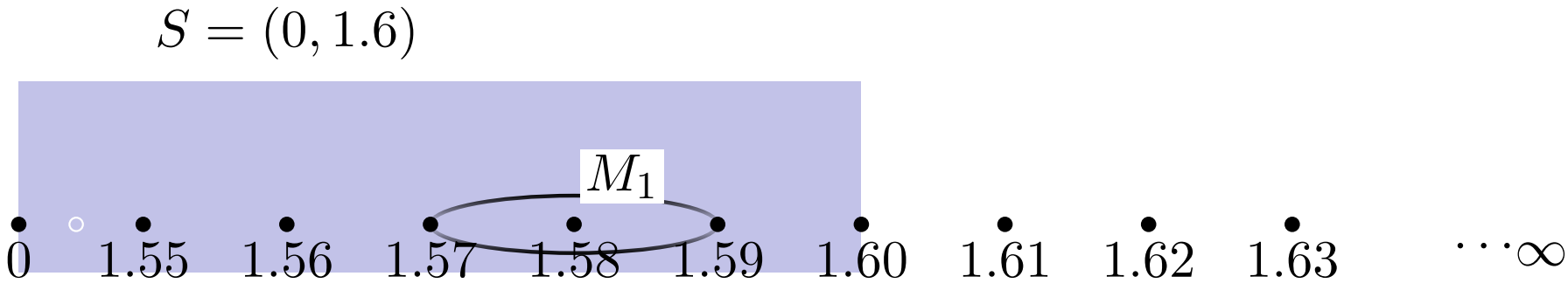

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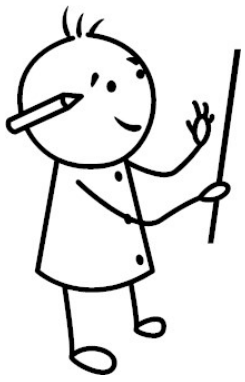

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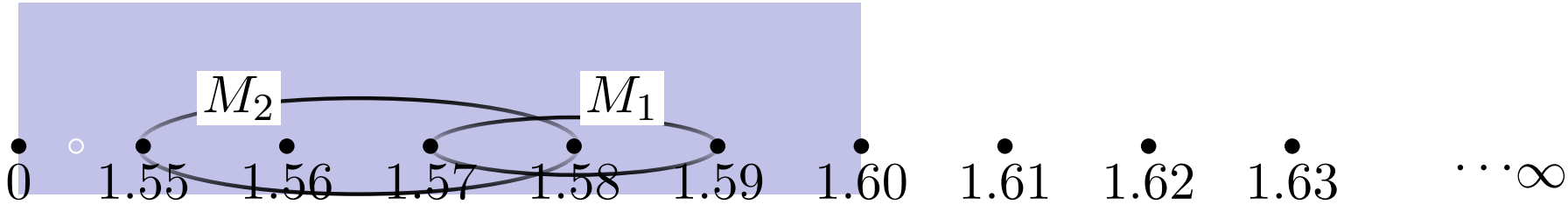

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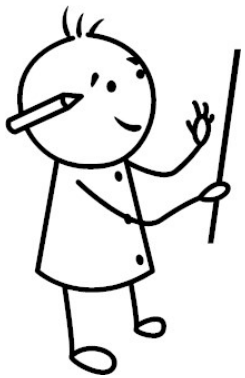

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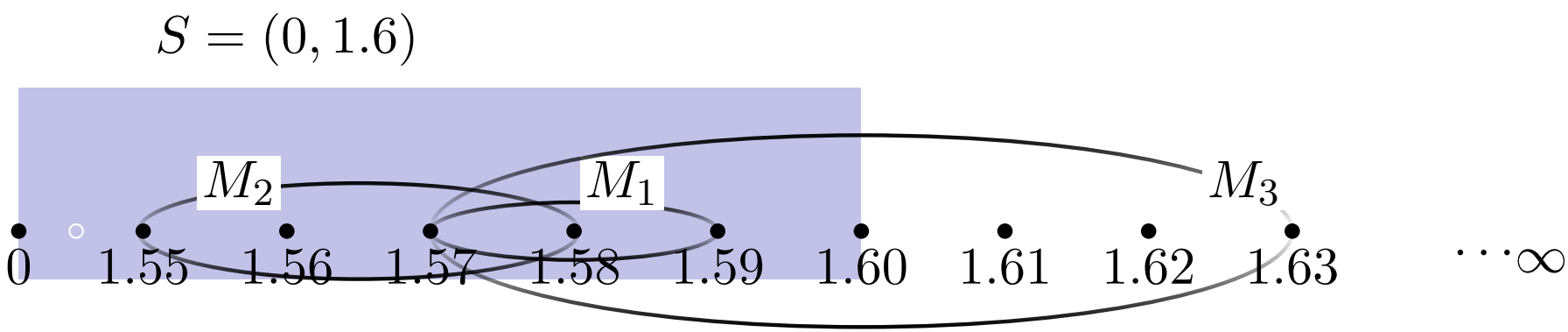

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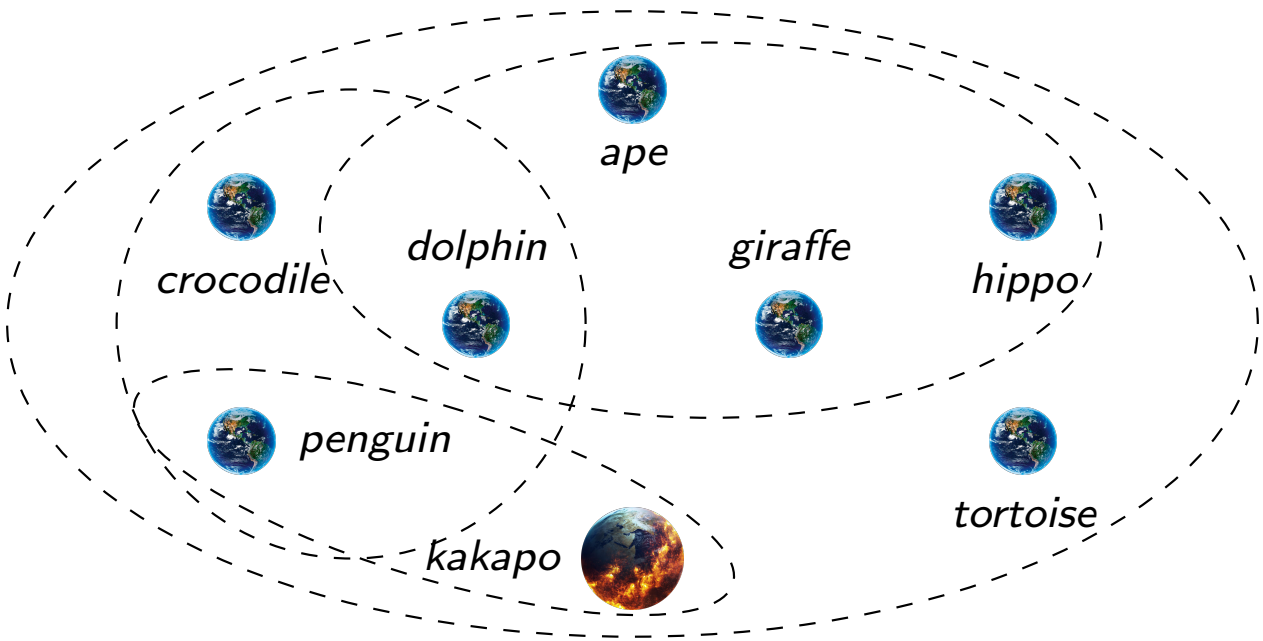

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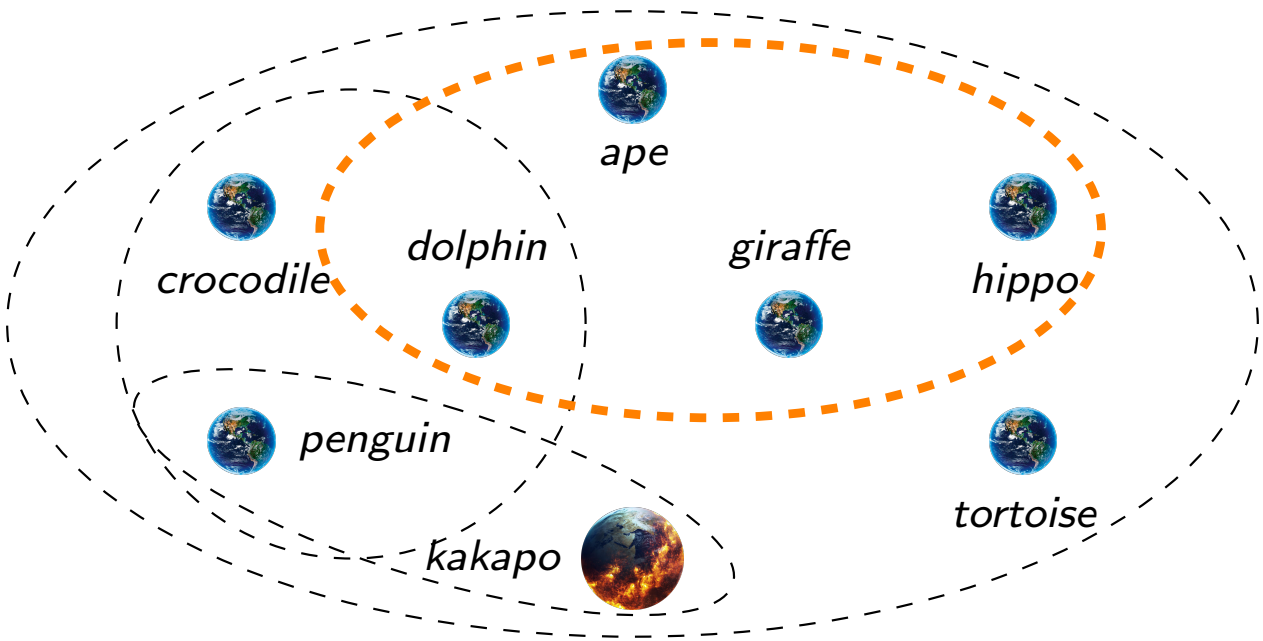

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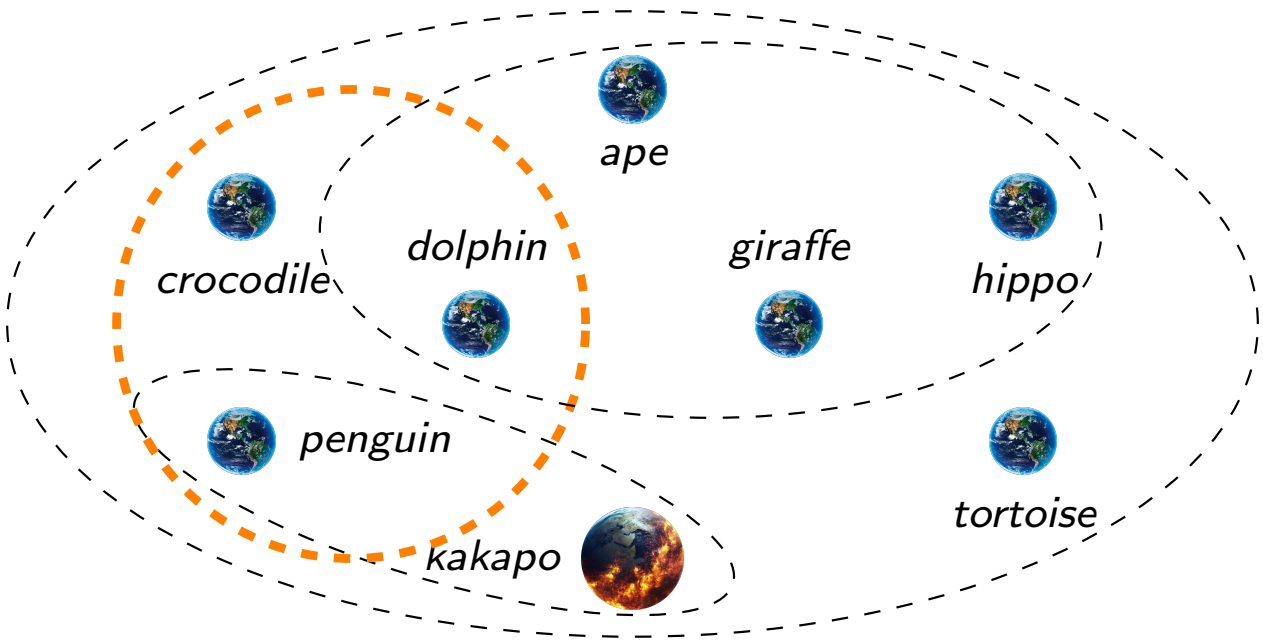

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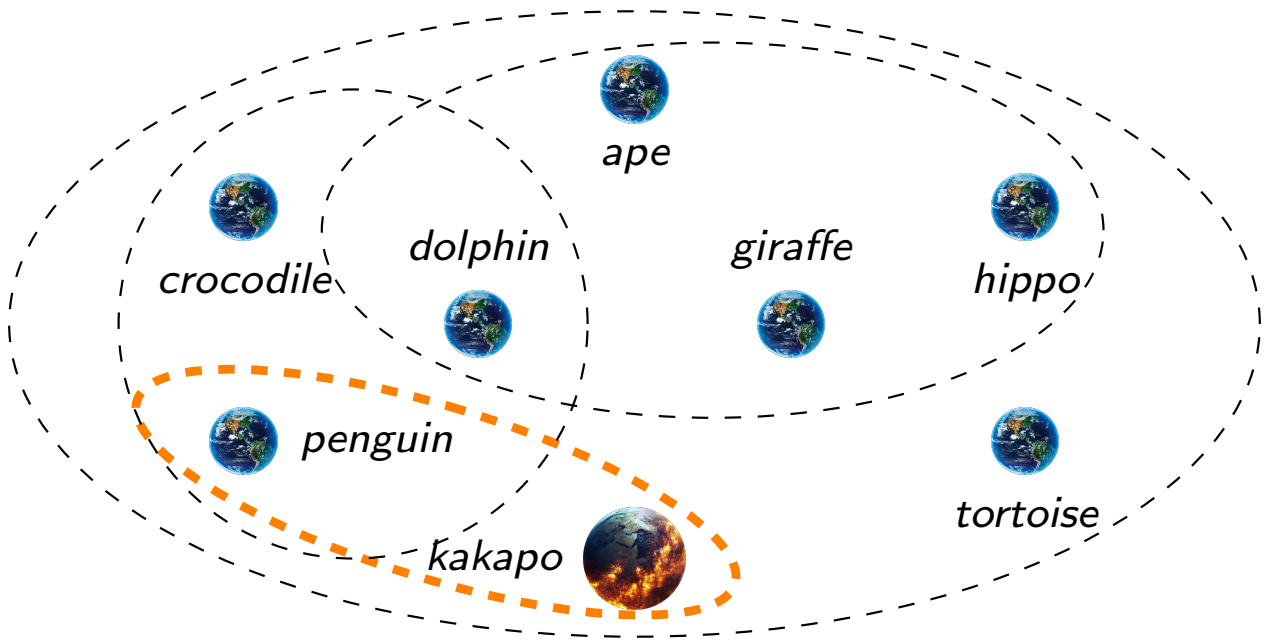

How can Joey form consistent beliefs based on their evidence?

Topology as Information Structures

An autonomous car detects an object at the road crossing in front of it. It receives several pieces of evidence from its various sensors.

E_1: it is dynamic (d)
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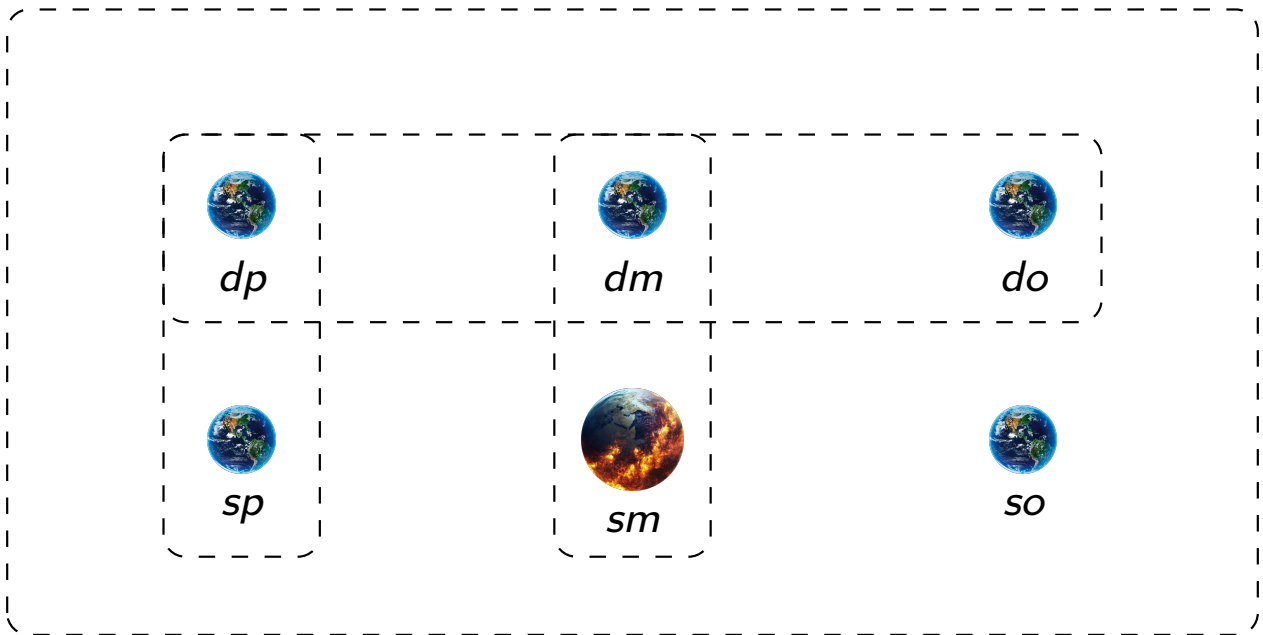

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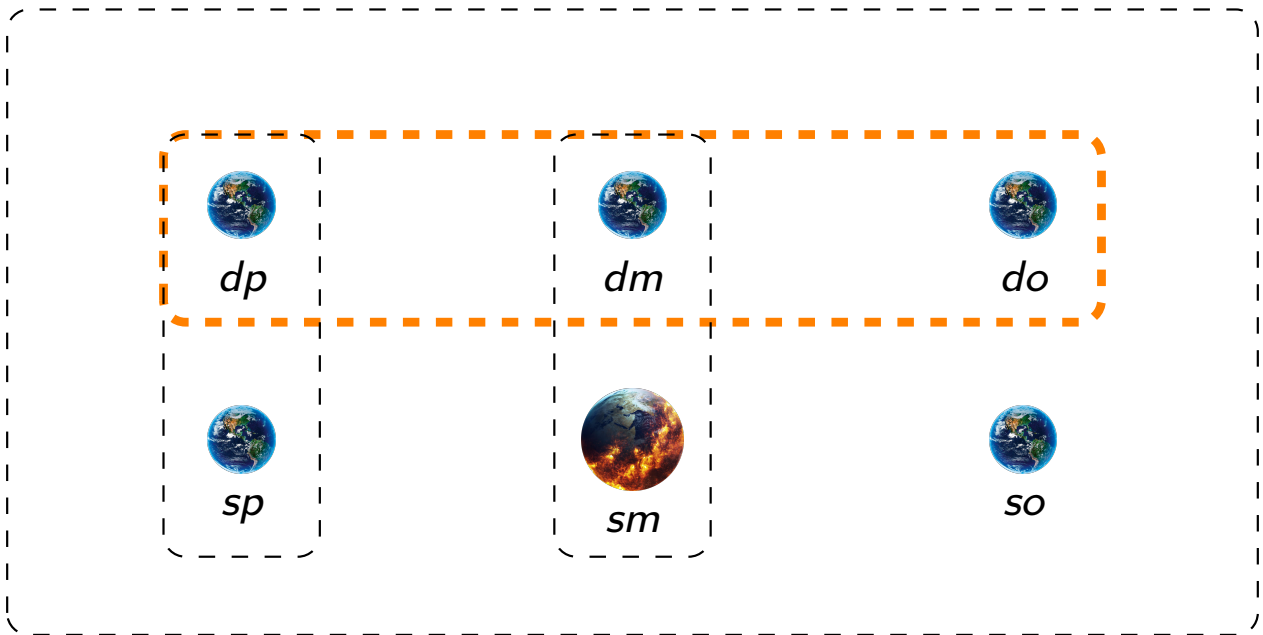

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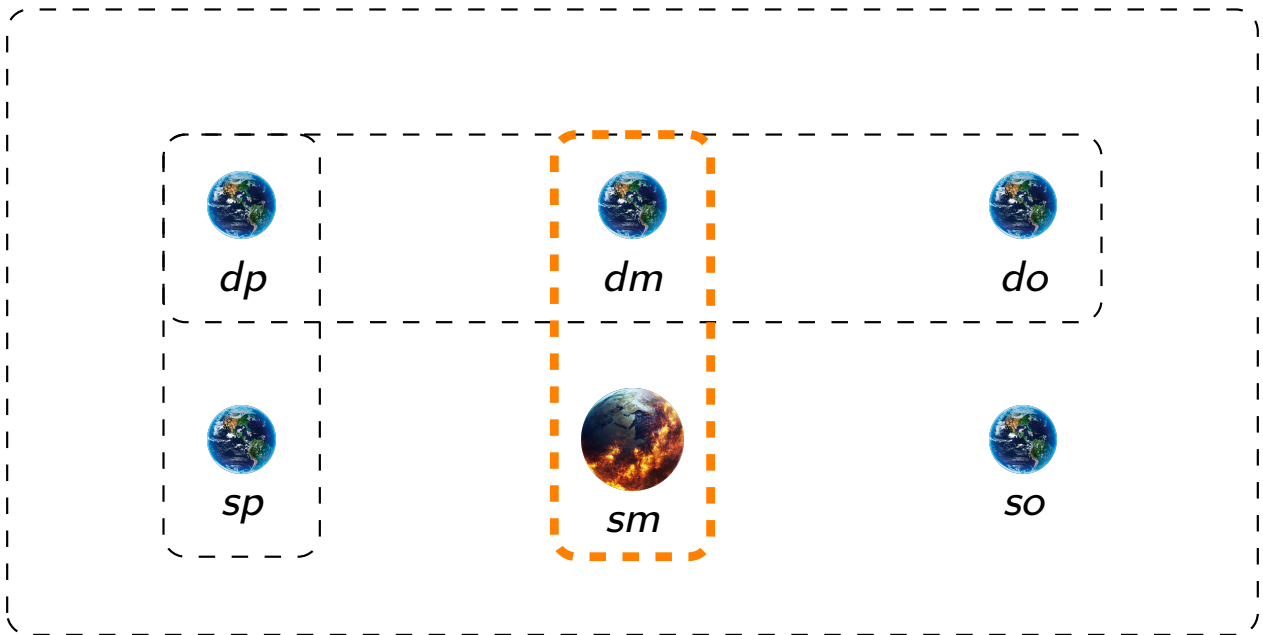

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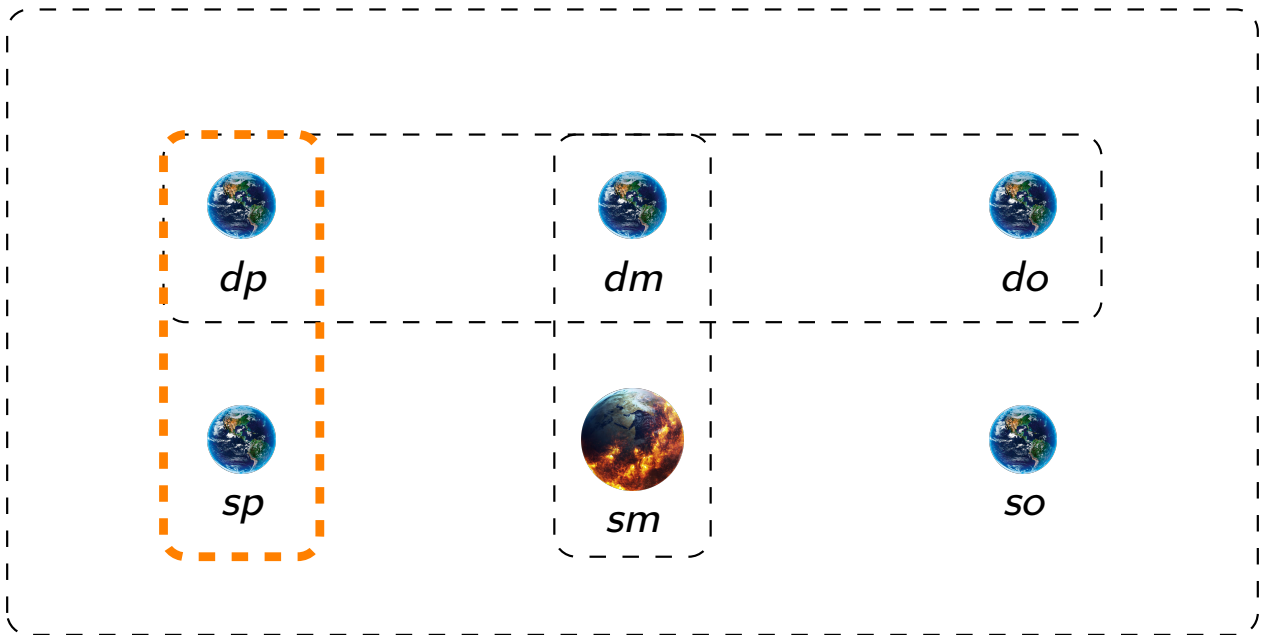
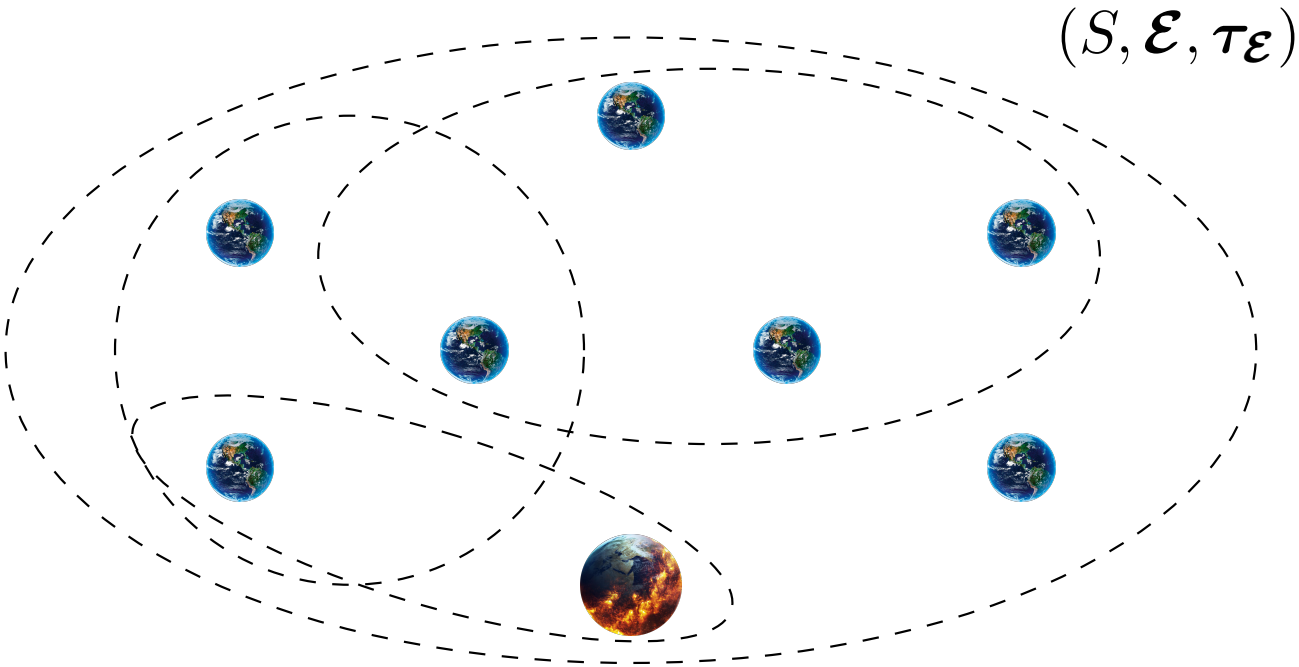


Should the car stop?

Topological Evidence Models



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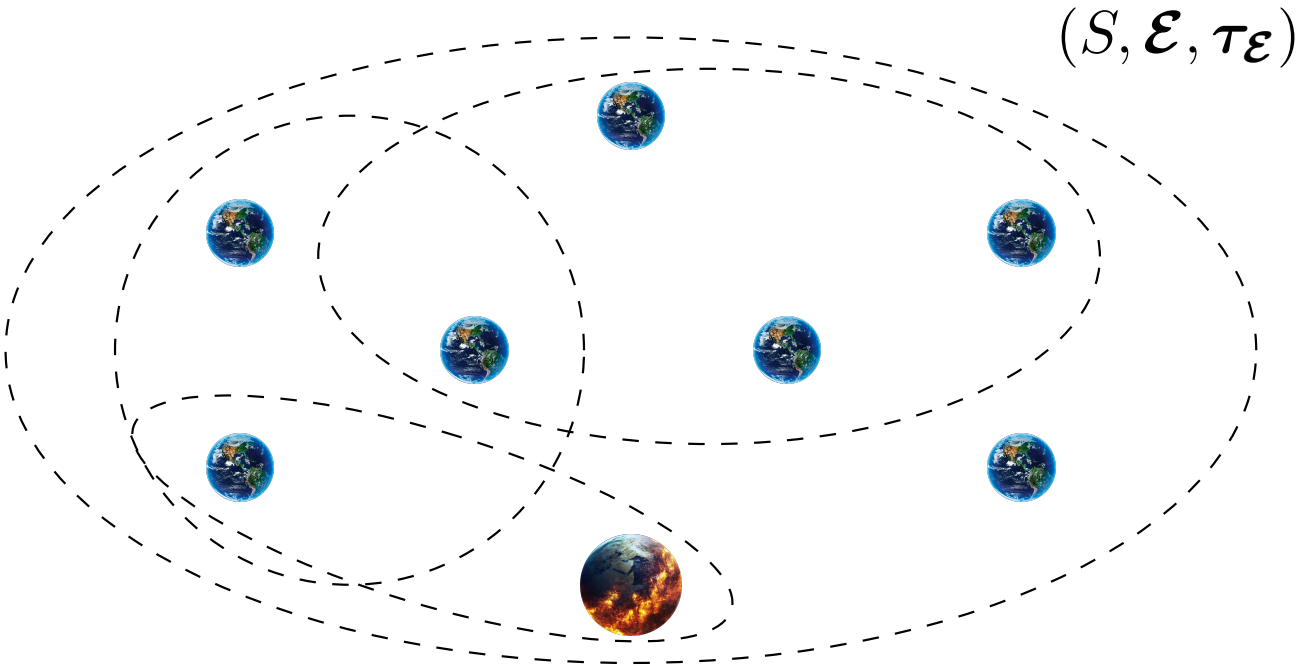

$E \in \mathcal{E}$: a piece of *direct evidence*.

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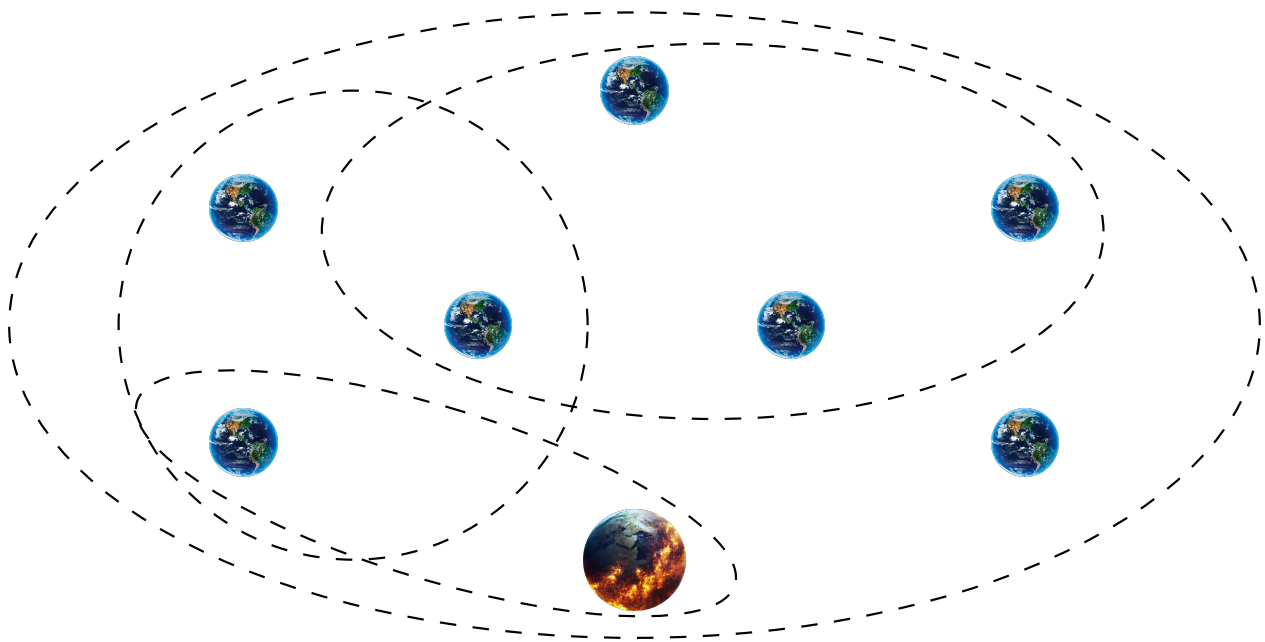
direct evidence *combined evidence* *argument/verifiable prop.*



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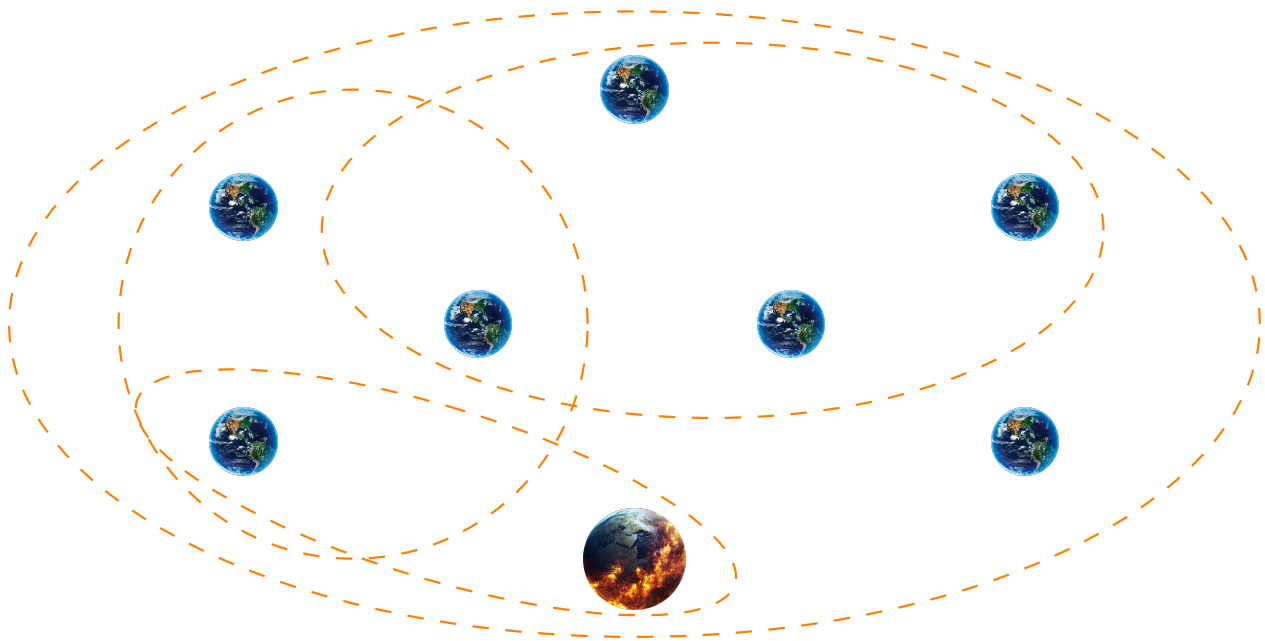

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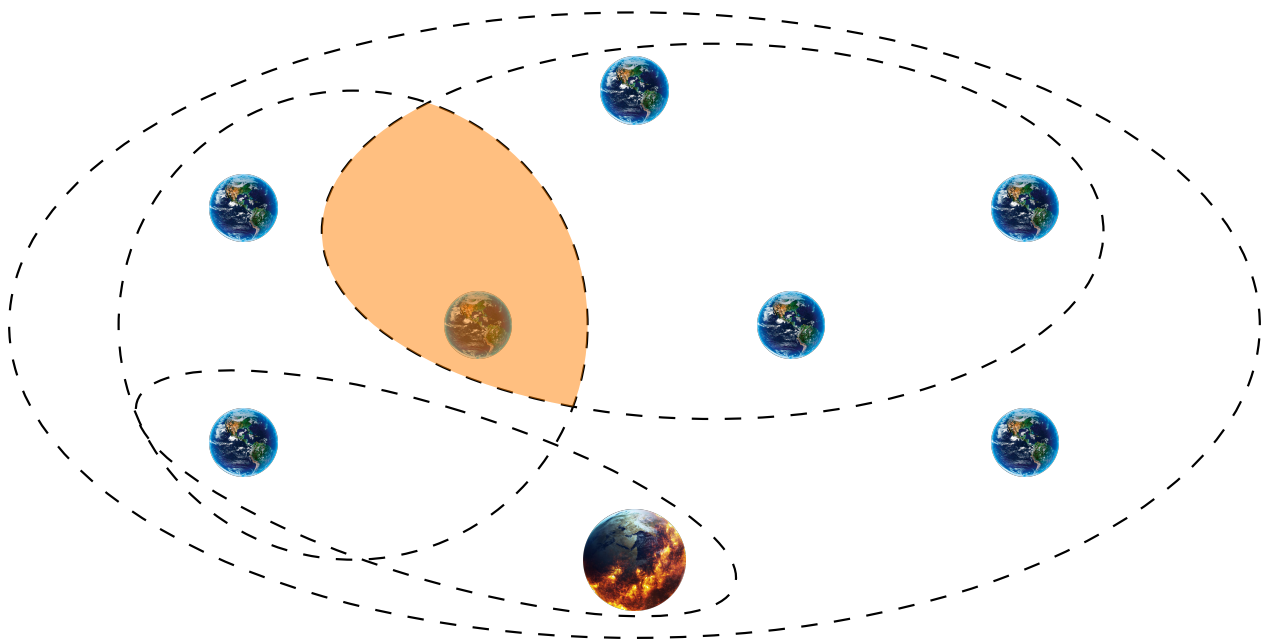

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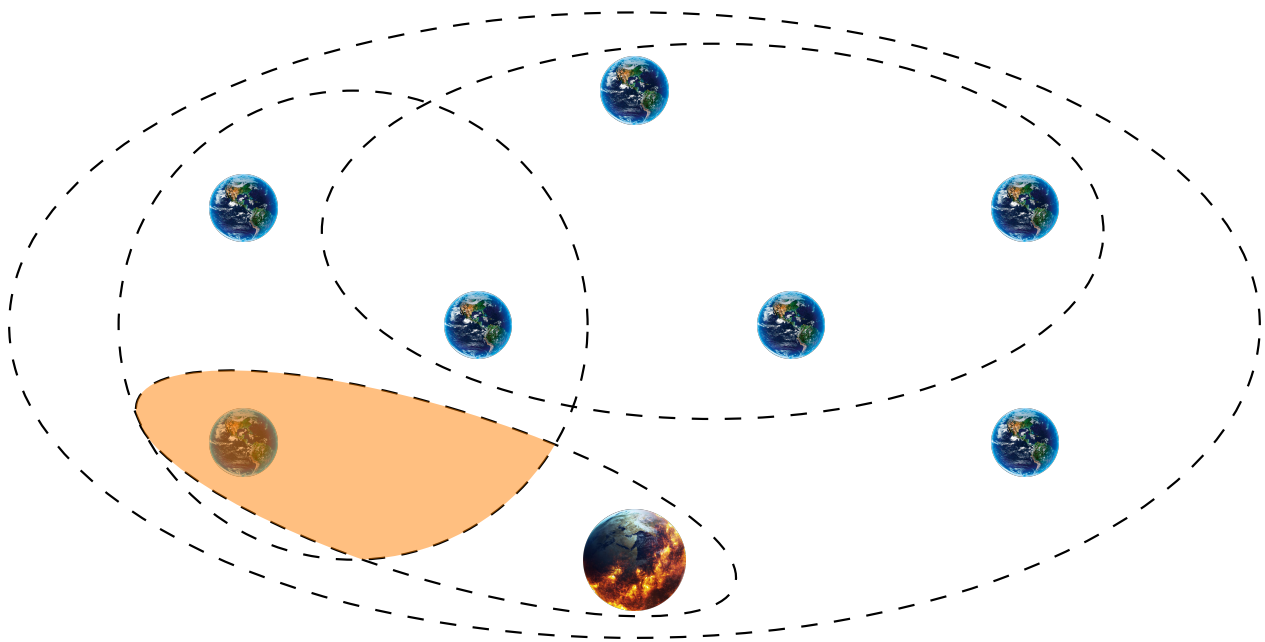

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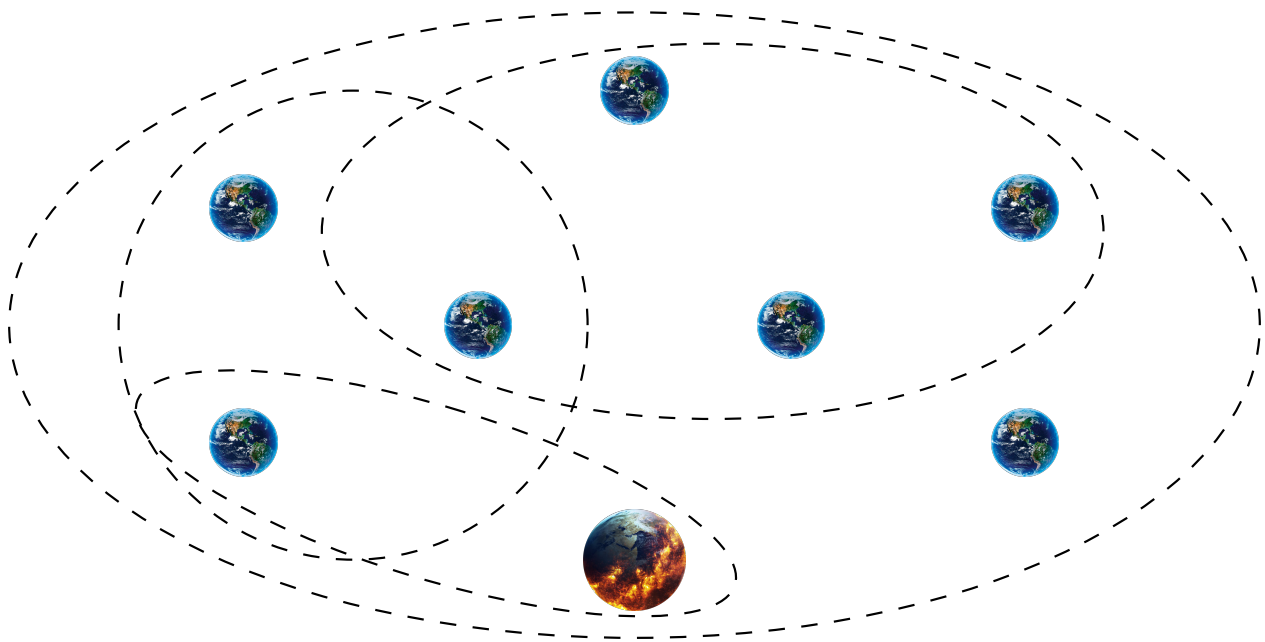

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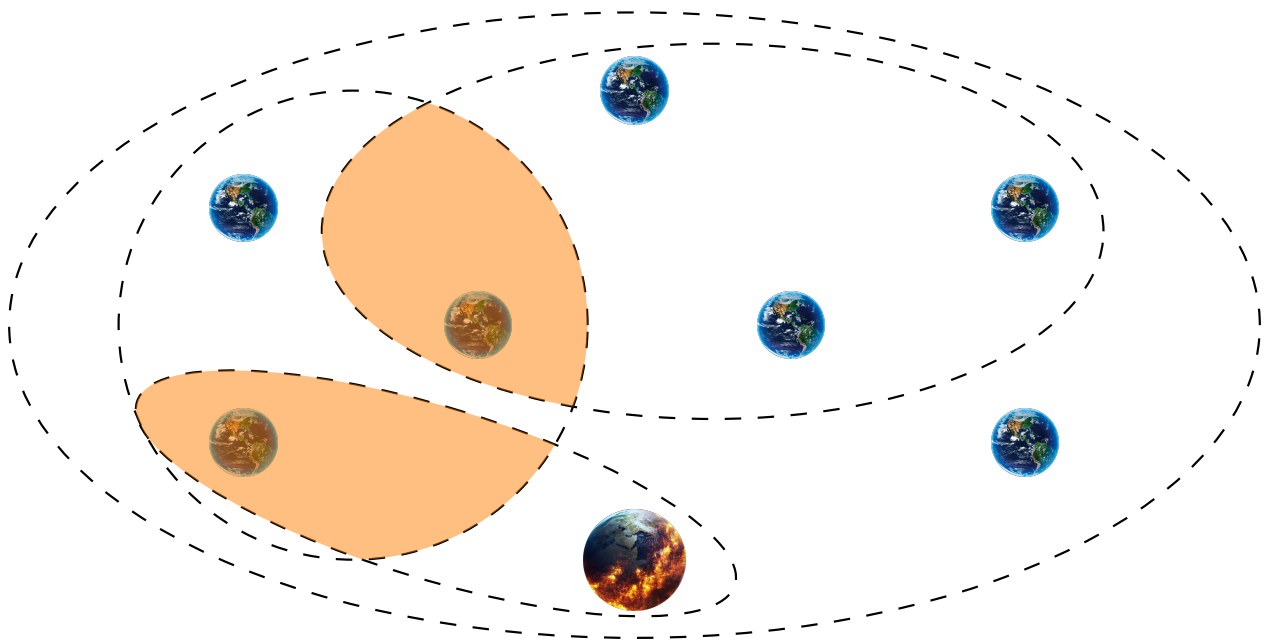

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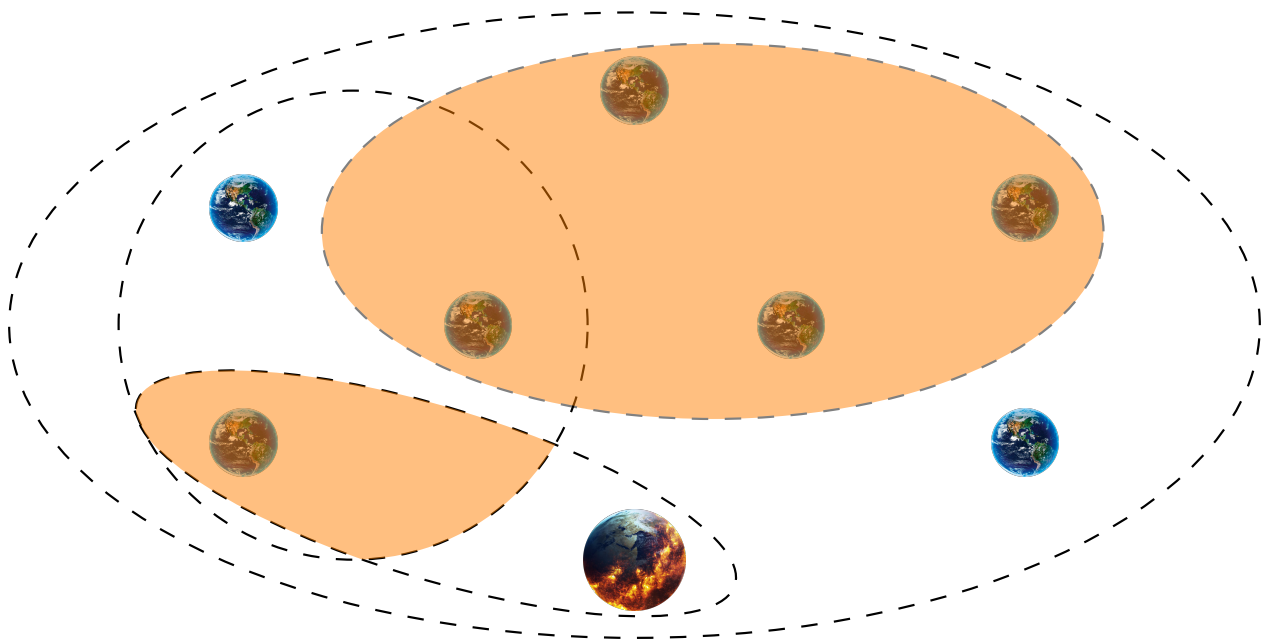

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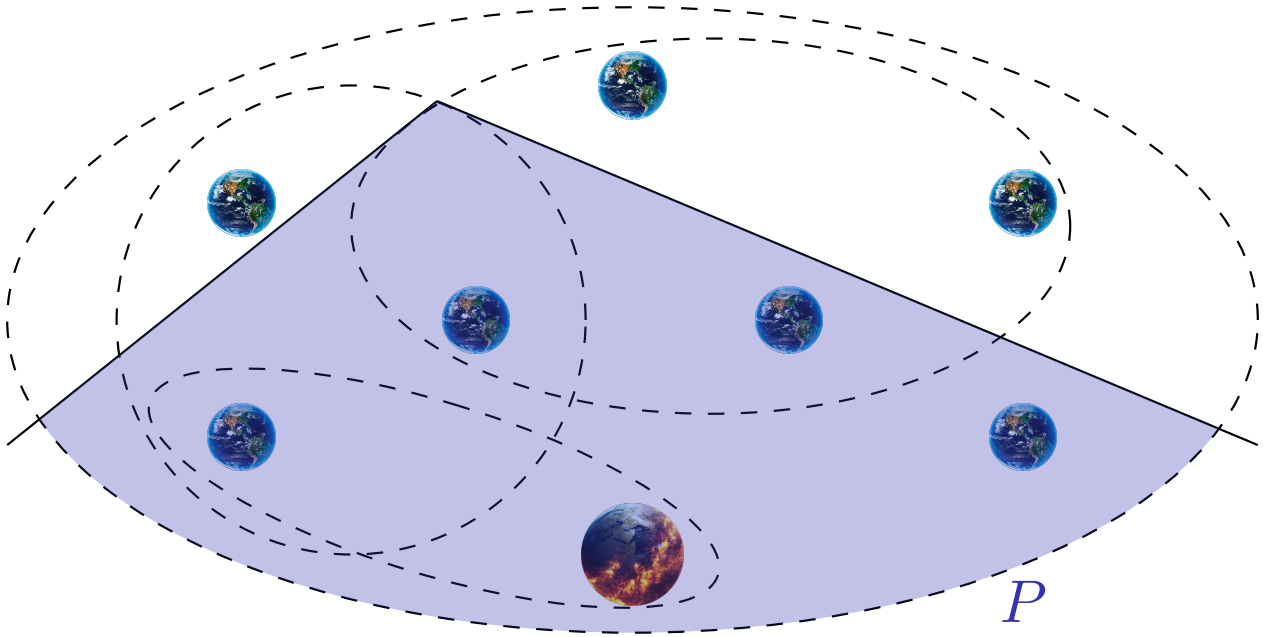

E is *basic evidence for* P iff $E \subseteq P$

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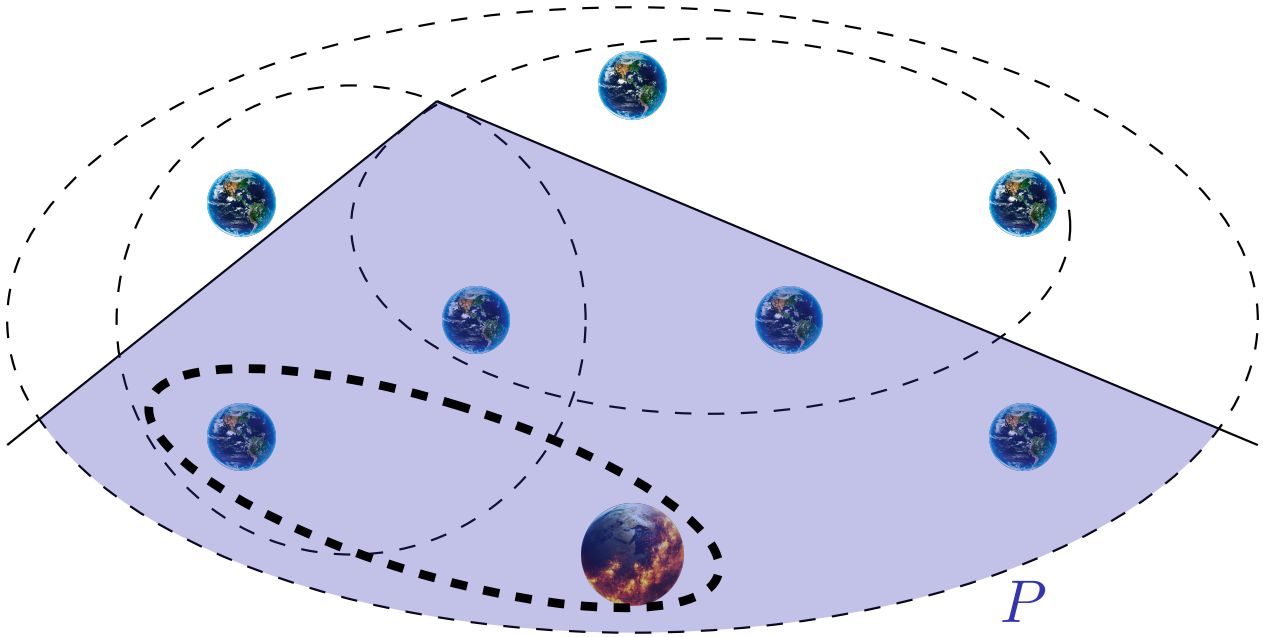

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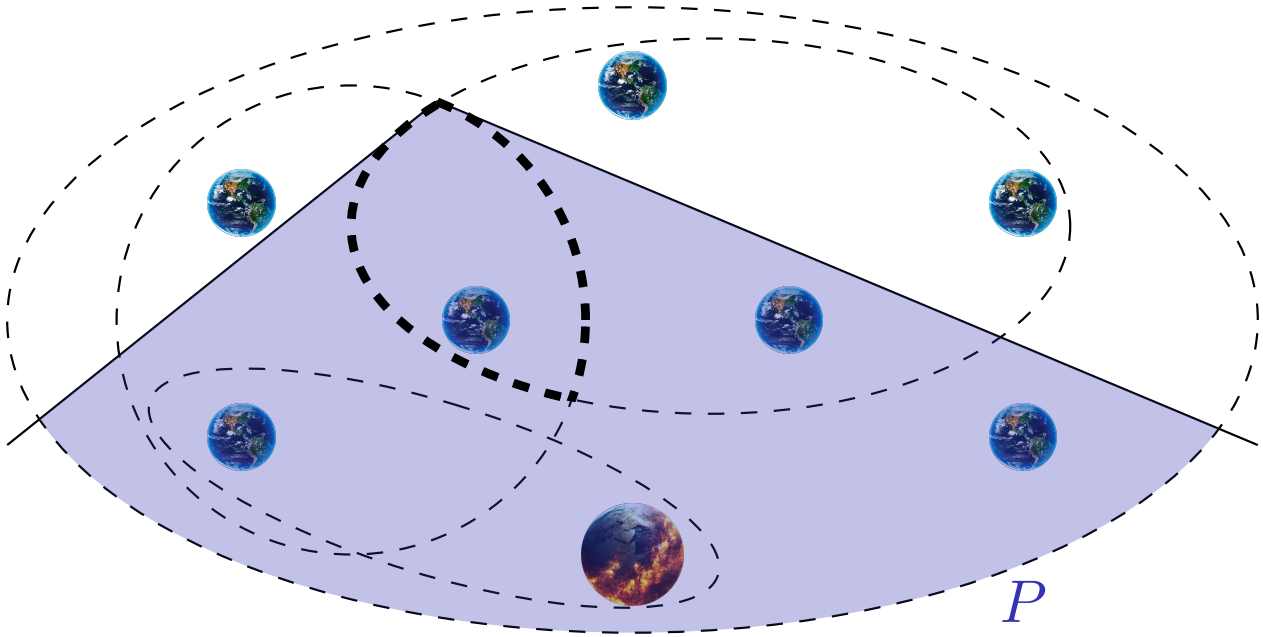
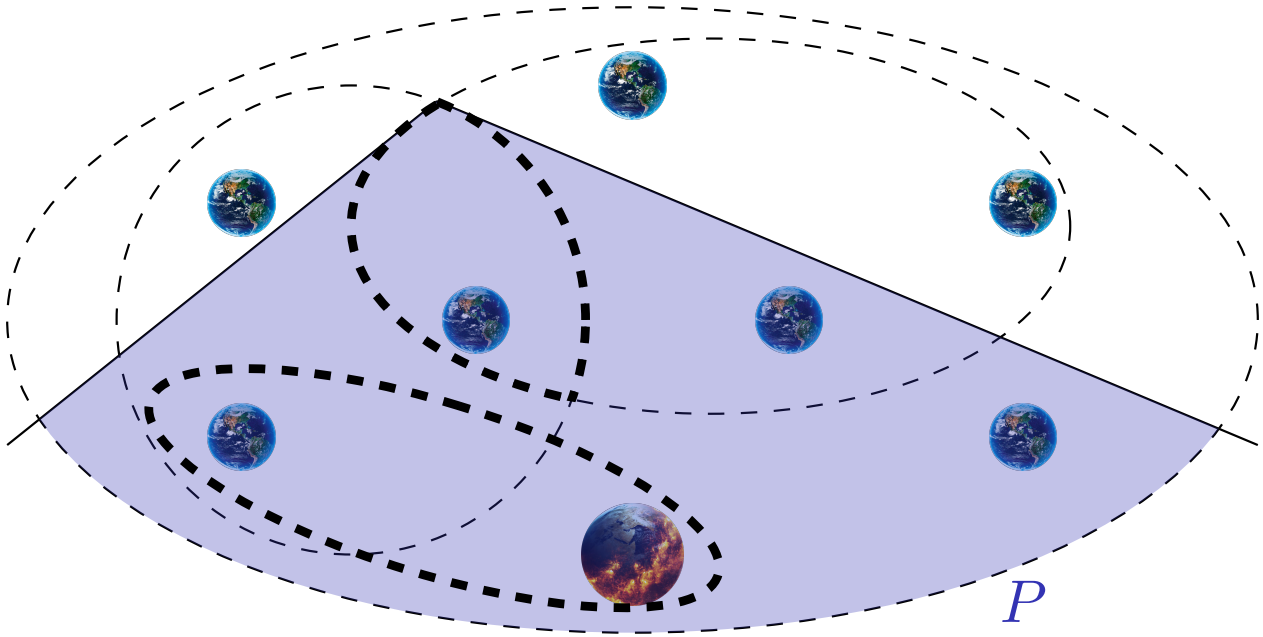


E is *basic evidence for* P iff $E \subseteq P$

$E^{\cap}$ is (combined) evidence for P iff $E^{\cap} \subseteq P$

U is an *argument for* P iff $U \subseteq P$

Evidential Support



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Justification

A **justification** for P is an *argument* U *for* P *that is consistent with every available evidence.*

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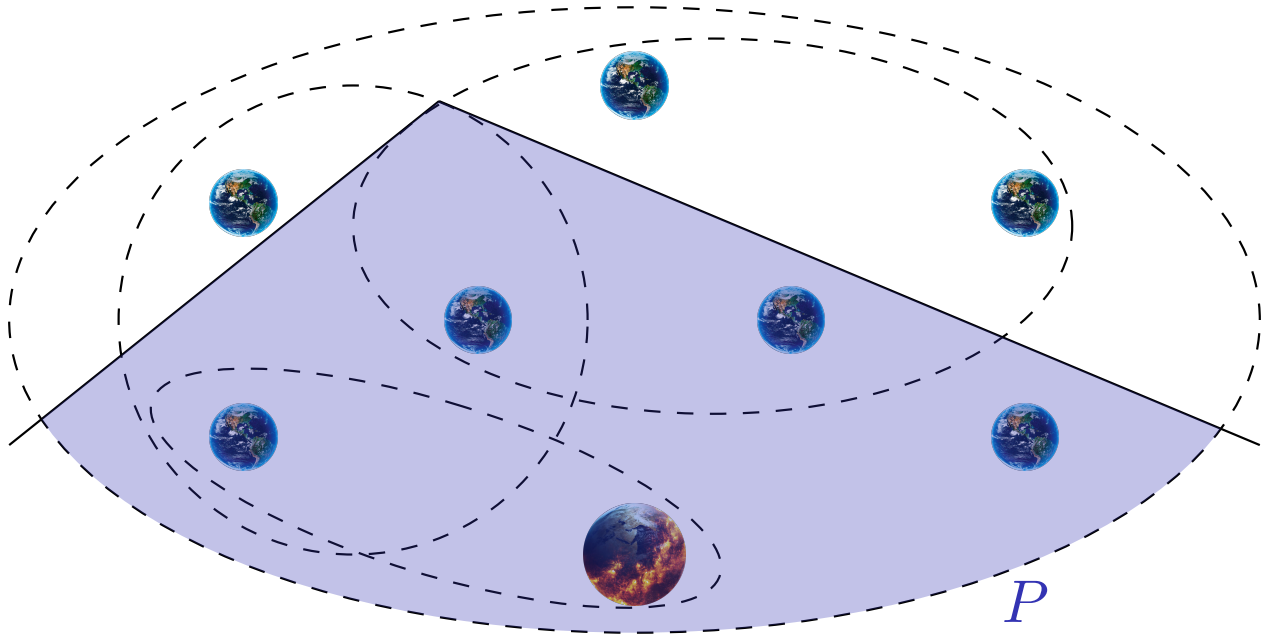

Topologically: U is a justification for P iff U is a **dense open** subset of P.

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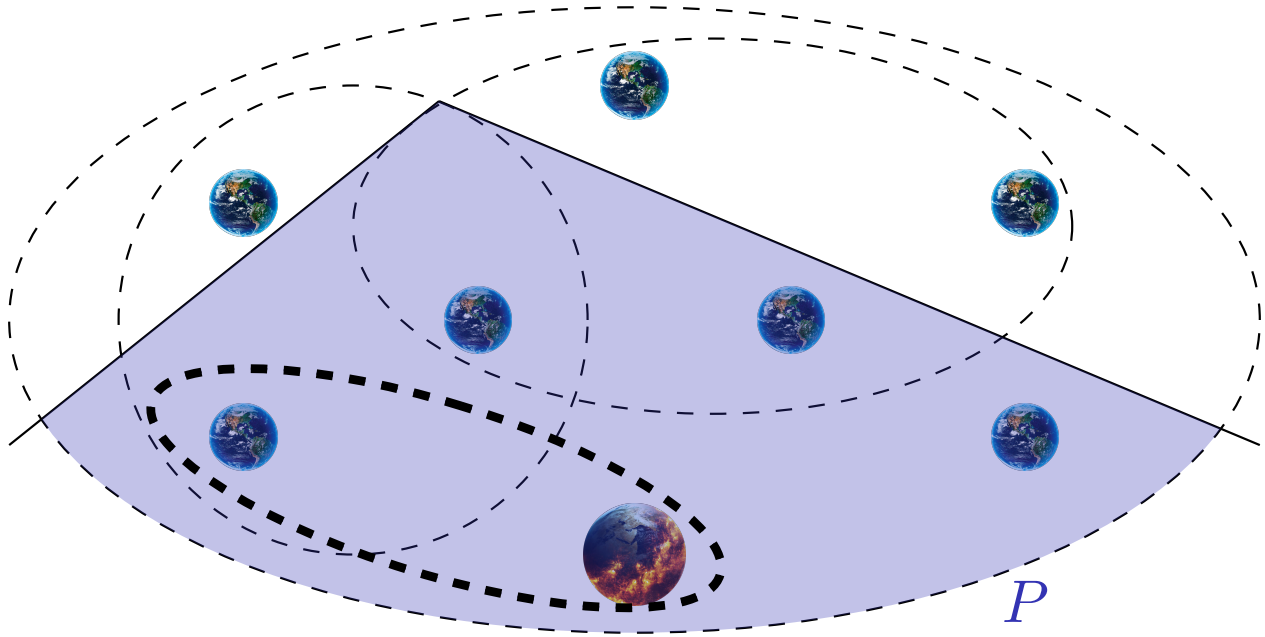

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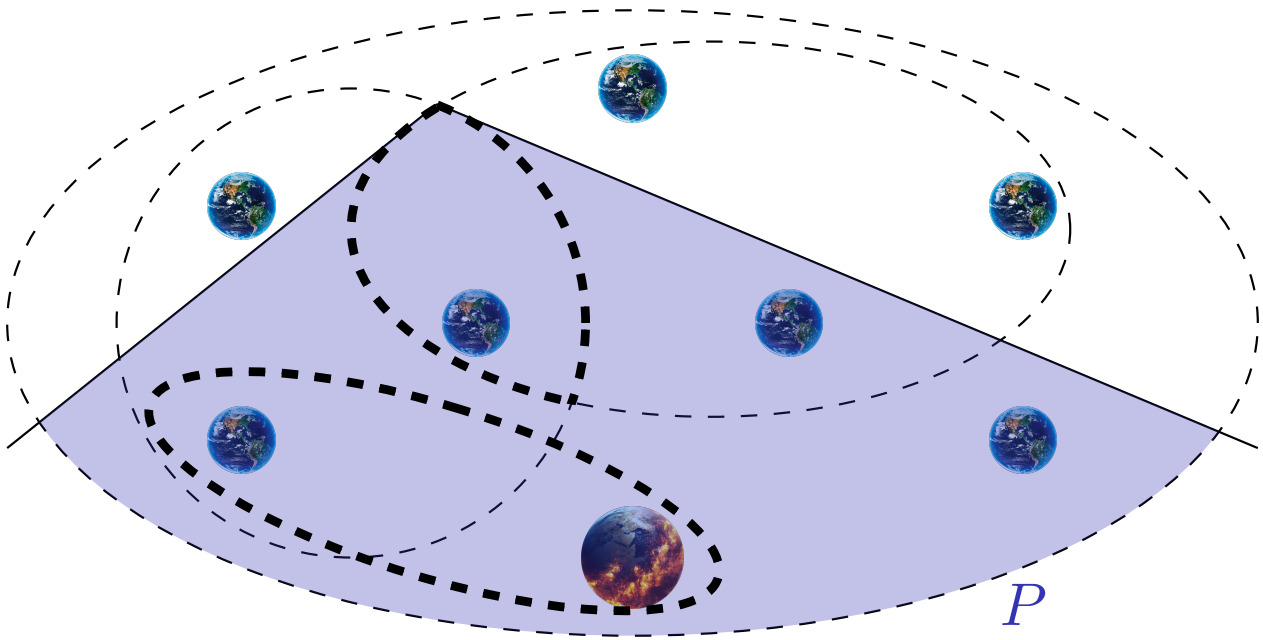

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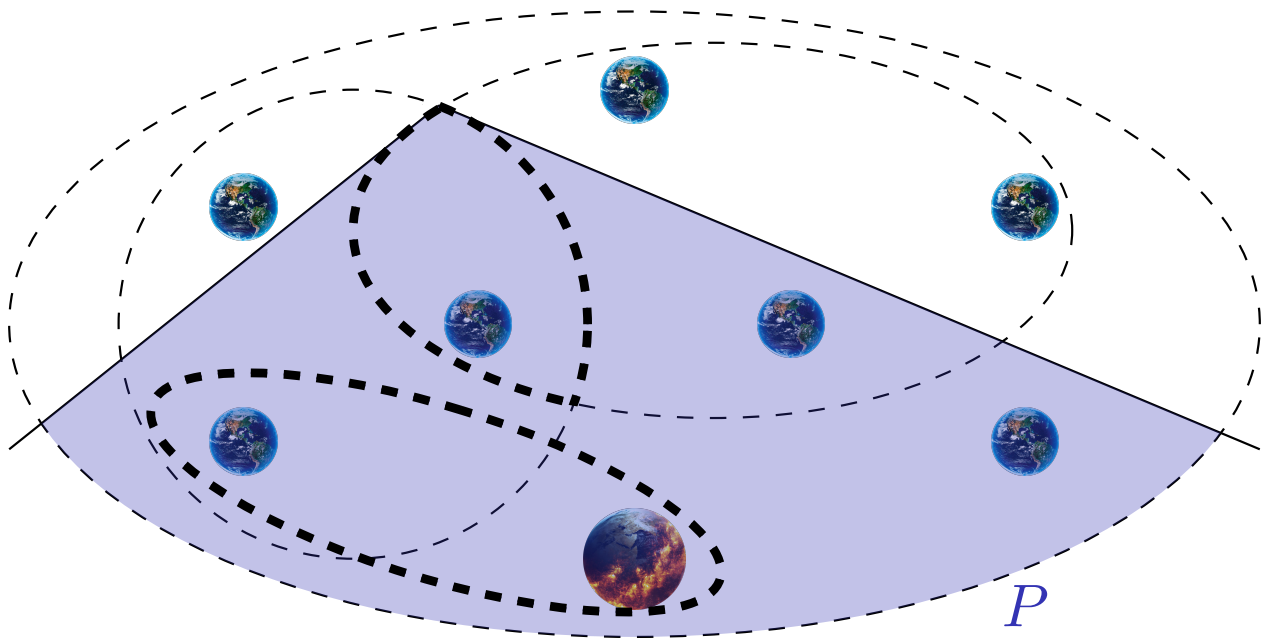
Justification & Belief

A **justification** for P is an *argument* U *for* P *that is consistent with every available evidence.*



P is **believed** iff there is a justification for P.

Topological models for evidence and belief

Pros

(+) discern various notions of evidence (*basic*, *combined*, *argument*, *justification*).

(+) produce consistent beliefs based on possibly false, possibly mutually contradictory pieces of evidence.

(+) epistemic notions receive nice topological meanings.

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How to represent uncertainty about evidence and justification?

Dempster-Schafer Theory of Belief Functions

Dempster-Shafer Theory of Belief Functions

DST is a *quantitative* theory of belief functions that assign a degree of belief to propositions based on *evidence*.

It generalizes the standard Bayesian framework of credences and has several advantages (that I will point out in passing).

Dempster-Shafer Theory of Belief Functions

Technical preliminaries

Definition (Belief function)

Given a finite set of possible states S, a *belief function* is a function $Bel : 2^S \to [0, 1]$ such that

1. $Bel(\emptyset) = 0$,
2. $Bel(S) = 1$, and
3. $Bel\big(\bigcup_1^n A_i\big) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} Bel\big(\bigcap_{i \in I} A_i\big)$.

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When $|I| = 2$ and $A_1 \cap A_2 = \emptyset$, item (3) is

$$Bel(A_1 \cup A_2) \geq Bel(A_1) + Bel(A_2).$$

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Belief functions are superadditive (not necessarily additive)!

Definition (Basic probability assignment/mass functions)

Given a finite set of possible states S, a *basic probability assignment* over the set S is a function $m : 2^S \to [0, 1]$ such that $m(\emptyset) = 0$ and $\sum_{A \subseteq S} m(A) = 1$.

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$m(A)$: degree of certainty in A (evidential support supplied by the evidence source to the proposition A).

When $m(A) \neq 0$, we call A a *focal element*.

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There is limited division of beliefs!

Given a basic probability assignment m, we can define a belief function:

$$Bel(P) = \sum_{A \subseteq P} m(A)$$

Definition (Dempster's rule of combination (DRC))

Let m_1 and m_2 be basic probability assignments over the same finite set S of possible states and $A_1, \ldots, A_k, B_1, \ldots, B_\ell \subseteq S$ such that $m_1(A_i) \neq 0$ and $m_2(B_i) \neq 0$, respectively. Moreover, suppose that $\sum_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j) < 1$. Then:

$$m_{12}(C) = \frac{\sum_{A_i \cap B_j = C} m_1(A_i) m_2(B_j)}{\sum_{A_i \cap B_j \neq \emptyset} m_1(A_i) m_2(B_j)}$$

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(+) generalizes the Bayesian framework for credences via *limited division of beliefs*.

(+) not every subset of the space is a piece of evidence.

(+) represents degrees of beliefs based on uncertain and incomplete evidence.

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(–) ignores conflict and, in turn, leads to unintuitive results.

(–) does not discern various notions of evidence, does not represent *evidential demands* of an agent.

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How to better model conflict by taking into account the evidential demands of an agent?

Beliefs based on Conflicting and Uncertain evidence
-
Connecting DST and TME

Joint work with Daira Pinto Prieto and Ronald de Haan.

Multi-Layer Belief Model

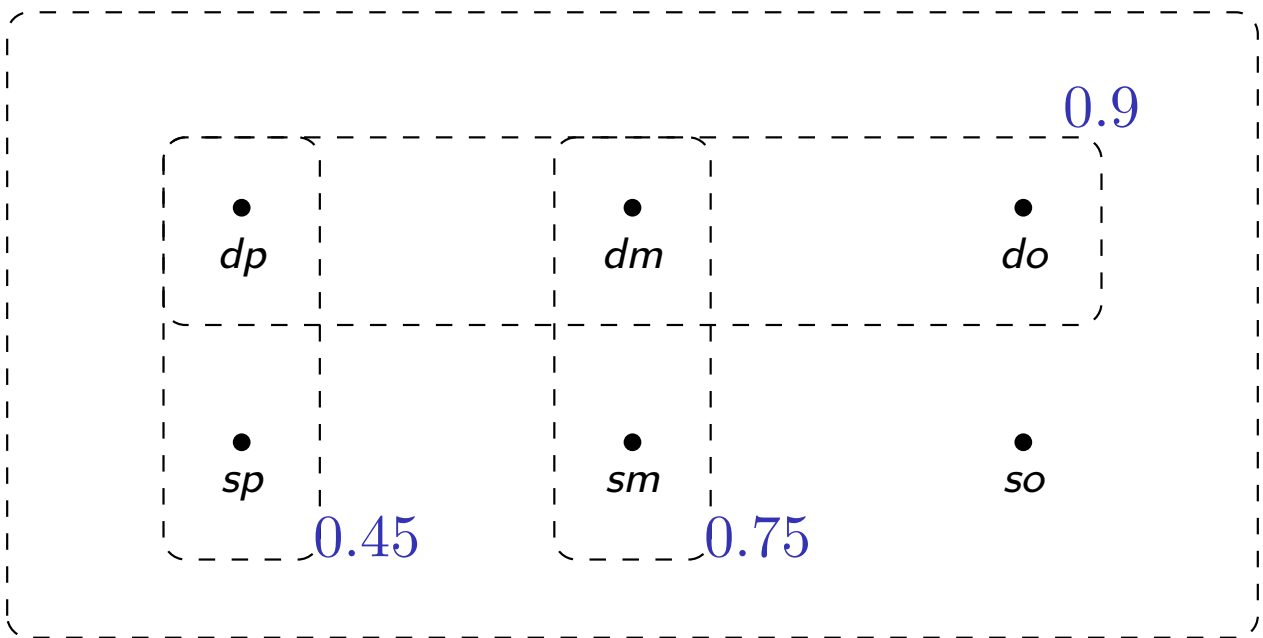
1. Qualitative Layer
   - represents the propositional content of the basic pieces of evidence
   - identifies a set of justifications that represents agents' evidential demands
2. Quantitative Layer
   - represents degrees of support supplied by uncertain evidence
   - transforms the degrees of uncertainties of the basic pieces of evidence into a mass function defined over the set of basic pieces of evidence $\mathcal{E}$.
3. Bridging Layer
   - connects the values of the mass function obtained in the second layer to the justifications of the first one.

Multi-Layer Belief Model

Example

Let Red be a fully autonomous car and Blue a driver-assist car. They both detect an object at the road crossing in front of them. They receive the following pieces of evidence:

$(E_1, 0.9)$:= 'It is dynamic' with 90% certainty.
$(E_2, 0.75)$:= 'It is a motorbike' with 75% certainty.
$(E_3, 0.45)$:= 'It is a pedestrian' with 45% certainty.



Multi-Layer Belief Model

Underlying structure

A *quantitative evidence frame* is a tuple $(S, \boldsymbol{\mathcal{E}}^Q)$, where

- $S \neq \emptyset$ is the set of *possible worlds/states*;
- $\boldsymbol{\mathcal{E}} \subseteq \mathcal{P}(S)$ is the set of *basic evidence sets* s.t $\emptyset, S \notin \boldsymbol{\mathcal{E}}$.
- $\boldsymbol{\mathcal{E}}^Q \subseteq \boldsymbol{\mathcal{E}} \times (0, 1)$.

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- $\boldsymbol{\mathcal{E}}^Q \subseteq \boldsymbol{\mathcal{E}} \times (0, 1)$.

$(E, p) \in \boldsymbol{\mathcal{E}} \times (0, 1)$: E is the *propositional content of the evidence* and p is its *degree of certainty*.

$1 - p$: represents the *uncertainty of the given piece of evidence* (and **not the certainty of** $S \setminus E$).

$\boldsymbol{\tau_{\mathcal{E}}} \setminus \{\emptyset\}$: the set of *arguments* available to the agent.

Multi-Layer Belief Model

Qualitative Layer - Frames of Justification

$\mathcal{J} \subseteq \boldsymbol{\tau_{\mathcal{E}}} \backslash \{\emptyset\}$: *frame of justification*

Given $(S, \boldsymbol{\mathcal{E}})$ and a proposition P:

T is justification for P (w.r.t. $\mathcal{J}$) iff $T \subseteq P$ and $T \in \mathcal{J}$.

A frame of justification represents an agent's evidential demands.

Multi-Layer Belief Model

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A frame of justification represents an agent's evidential demands.

Depending on the situation modelled, one can think of natural constraints on frames of justifications.

In this talk, I focus on two kinds of frame of justifications:

1. The Dempster-Shafer frame of justification
2. The strong denseness frame of justification

Multi-Layer Belief Model

Qualitative Layer - Frames of Justification

1. The *Dempster-Shafer frame of justification*:

$$\mathcal{J}^{DS} = \tau_{\mathcal{E}} \setminus \{\emptyset\}$$

Low evidential demands: having an argument for P is sufficient to justify P, *regardless whether the argument contradicts with the other available arguments*.

2. The *strong denseness frame of justification*:

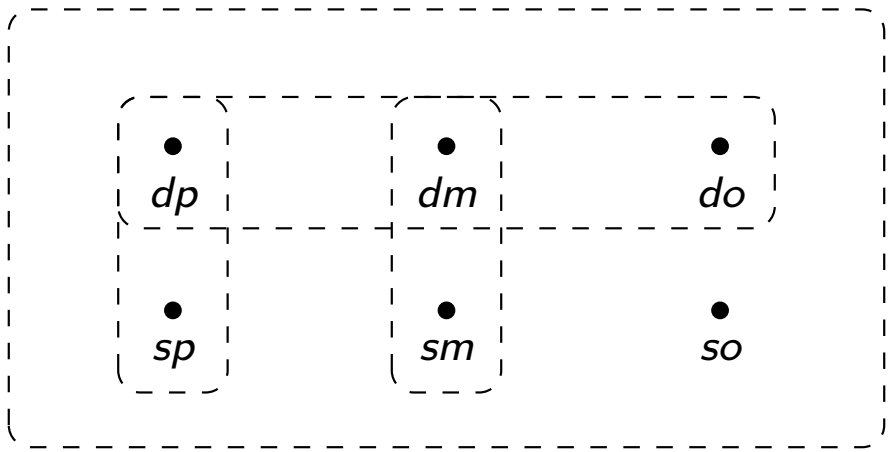
$$\mathcal{J}^{SD} := \{U \in \tau_{\mathcal{E}} : U \text{ is dense } \}$$

High evidential demands: beliefs are formed only based on arguments which do not contradict with any other argument.

Consequently, (nonzero) degrees of beliefs are assigned to those propositions that do not contradict with any available argument.

Multi-Layer Belief Model

Qualitative Layer - Frames of Justification - Example

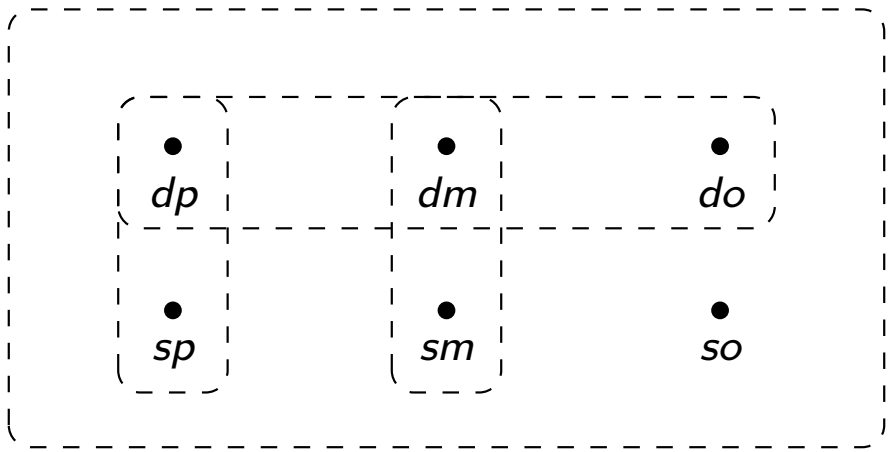


$$\mathcal{J}^{DS} = \tau_{\mathcal{E}} \backslash \{\emptyset\}$$

$$\mathcal{J}^{SD} = \{E_1, \{dp, dm\}, \{sp, dp, do, dm\}, \{dp, do, dm, sm\}, \{sp, dp, dm, sm\}, S\}$$

Multi-Layer Belief Model

Qualitative Layer - Frames of Justification - Example



$$\mathcal{J}^{DS} = \tau_{\mathcal{E}} \backslash \{\emptyset\}$$

$$\mathcal{J}^{SD} = \{E_1, \{dp, dm\}, \{sp, dp, do, dm\}, \{dp, do, dm, sm\},$$
$$\{sp, dp, dm, sm\}, S\} = \mathcal{J}^{DS} \setminus \{E_2, E_3, \{dp\}, \{dm\}\}$$

Multi-Layer Belief Model

Quantitative Layer

We have the degrees of certainty over the basic evidence pieces.
What about the degrees of certainty of justifications?

Multi-Layer Belief Model

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What about the degrees of certainty of justifications?

We measure this in three steps:

1. define a mass function over $2^{\mathcal{E}}$
2. allocate the masses over $2^{\mathcal{E}}$ to the elements of $\tau_{\mathcal{E}}$
3. define basic probability assignments over $\mathcal{J}$

Multi-Layer Belief Model

Quantitative Layer

We have the degrees of certainty over the basic evidence pieces.
What about the degrees of certainty of justifications?

We measure this in three steps:

1. define a mass function over $2^{\mathcal{E}}$ *(Quantitative layer)*
2. allocate the masses over $2^{\mathcal{E}}$ to the elements of $\tau_{\mathcal{E}}$ *(Bridging)*
3. define basic probability assignments over $\mathcal{J}$ *(Bridging)*

Multi-Layer Belief Model

Quantitative Layer

Given a quantitative evidence frame $(S, \boldsymbol{\mathcal{E}}^Q)$ such that $\boldsymbol{\mathcal{E}}^Q = \{(E_1, p_1), \ldots, (E_m, p_m)\}$, we merge these different certainty values by the following function $\delta : 2^{\boldsymbol{\mathcal{E}}} \to [0, 1]$:

$$\delta(\boldsymbol{E}) = \prod_{E_i \in \boldsymbol{E}} p_i \prod_{E_i \notin \boldsymbol{E}} 1 - p_i$$

Intuitively, the function δ distributes the degree of certainty of a piece of evidence $E \in \boldsymbol{\mathcal{E}}$ among all the possible occurrences of E in presence of other pieces of evidence.

$\delta(\{E_1, E_2\}) = p_1 p_2 (1 - p_3)$: the induced certainty of the event 'at least E_1 and E_2 are true' in the context of having exactly three pieces of evidence.

Multi-Layer Belief Model

Quantitative Layer

Given a quantitative evidence frame $(S, \boldsymbol{\mathcal{E}}^Q)$ such that $\boldsymbol{\mathcal{E}}^Q = \{(E_1, p_1), \ldots, (E_m, p_m)\}$, we merge these different certainty values by the following function $\delta : 2^{\boldsymbol{\mathcal{E}}} \to [0, 1]$:

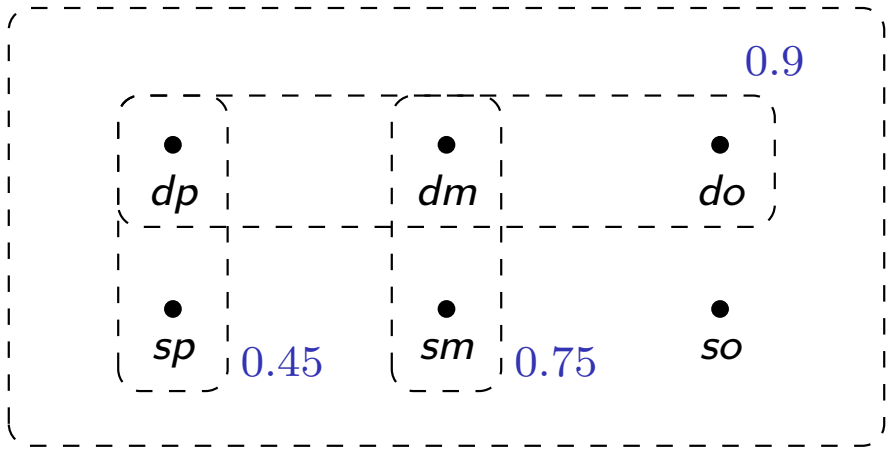
$$\delta(\boldsymbol{E}) = \prod_{E_i \in \boldsymbol{E}} p_i \prod_{E_i \notin \boldsymbol{E}} 1 - p_i$$

Nevertheless, it is easier to read δ as a system of certainty assignments since it assigns values to every combination of pieces of evidence.

It is merely a method to distribute the certainty of a single piece of evidence to all the ways it can be observed in combination with the other evidence pieces.

Multi-Layer Belief Model

Quantitative Layer - Example



$$
\begin{aligned}
\delta(\{\emptyset\}) &= 0.01 & \delta(\{E_1, E_2\}) &= 0.37 \\
\delta(\{E_1\}) &= 0.12 & \delta(\{E_1, E_3\}) &= 0.10 \\
\delta(\{E_2\}) &= 0.04 & \delta(\{E_2, E_3\}) &= 0.03 \\
\delta(\{E_3\}) &= 0.01 & \delta(\{E_1, E_2, E_3\}) &= 0.30
\end{aligned}
$$

Multi-Layer Belief Model

Bridging Layer

What about the degree of certainty of justifications?

Where to map $\delta(\boldsymbol{E})$*?* $\bigcup \boldsymbol{E}$*?* $\bigcap \boldsymbol{E}$*? Something else?*

Multi-Layer Belief Model

Bridging Layer

What about the degree of certainty of justifications?

Where to map $\delta(\boldsymbol{E})$? $\bigcup \boldsymbol{E}$? $\bigcap \boldsymbol{E}$? Something else?

Definition

Let $(S, \boldsymbol{\mathcal{E}})$ be a qualitative evidence frame. A *set of evidence allocation functions* F on $(S, \boldsymbol{\mathcal{E}})$ is a set of of functions from $2^{\boldsymbol{\mathcal{E}}}$ to $\boldsymbol{\tau_{\mathcal{E}}}$ (the topology generated by $\boldsymbol{\mathcal{E}}$) such that for all $f, g \in F$:

1. $f(\emptyset) = S$,
2. for all nonempty $\boldsymbol{E} \subseteq \boldsymbol{\mathcal{E}}$, $f(\boldsymbol{E}) \in \boldsymbol{\tau_E}$ (the topology generated by $\boldsymbol{E}$) and it is dense in $\bigcup \boldsymbol{E}$ w.r.t. $\boldsymbol{\tau_E}$; or $f(\boldsymbol{E}) = \emptyset$.
3. for all $\boldsymbol{E} \subseteq \boldsymbol{\mathcal{E}}$ and every f, g in F, $f(\boldsymbol{E}) \subseteq g(\boldsymbol{E})$ or $g(\boldsymbol{E}) \subseteq f(\boldsymbol{E})$.

Multi-Layer Belief Model

Bridging Layer

Examples of evidence allocation functions:

- $i(\boldsymbol{E}) = \begin{cases} \bigcap \boldsymbol{E} & \text{if } \boldsymbol{E} \neq \emptyset, \\ S & \text{otherwise.} \end{cases}$
- $u(\boldsymbol{E}) = \begin{cases} \bigcup \boldsymbol{E} & \text{if } \boldsymbol{E} \neq \emptyset, \\ S & \text{otherwise.} \end{cases}$
- $d(\boldsymbol{E}) = \begin{cases} \min\big((\text{dense}(\boldsymbol{E}), \subseteq\big)\big) & \text{if } \boldsymbol{E} \neq \emptyset, \\ S & \text{otherwise.} \end{cases}$

Multi-Layer Belief Model

Bridging Layer

$\boldsymbol{E}$	$\boldsymbol{i(E)}$	$\boldsymbol{u(E)}$	$\boldsymbol{d(E)}$	$\boldsymbol{\delta(E)}$
$\emptyset$	S	S	S	0.01
E_1	E_1	E_1	E_1	0.12
E_2	E_2	E_2	E_2	0.04
E_3	E_3	E_3	E_3	0.01
E_1, E_2	$\{dm\}$	$\{dp, do, dm, sm\}$	$\{dm\}$	0.37
E_1, E_3	$\{dp\}$	$\{sp, dp, do, dm\}$	$\{dp\}$	0.10
E_2, E_3	$\emptyset$	$\{sp, dp, dm, sm\}$	$\{sp, dp, dm, sm\}$	0.03
E_1, E_2, E_3	$\emptyset$	S	$\{dp, dm\}$	0.30

Multi-Layer Belief Model

Bridging Layer

Masses over $\boldsymbol{\tau_{\mathcal{E}}}$, $\delta_\tau : F \times 2^S \to [0,1]$:

$$\delta_{\boldsymbol{\tau}}(f,T) = \begin{cases} \sum_{\boldsymbol{E}: f(\boldsymbol{E})=T} \delta(\boldsymbol{E}) & \text{if } T \in \boldsymbol{\tau_{\mathcal{E}}}, \\ 0 & \text{otherwise.} \end{cases}$$

Basic probability assignment over $\boldsymbol{\mathcal{J}}$, $\delta_{\mathcal{J}} : F \times 2^S \to [0,1]$:

$$\delta_{\boldsymbol{\mathcal{J}}}(f,A) = \begin{cases} \frac{\delta_{\boldsymbol{\tau}}(f,A)}{\sum_{T \in \boldsymbol{\mathcal{J}}} \delta_{\boldsymbol{\tau}}(f,T)} & \text{if } A \in \boldsymbol{\mathcal{J}}, \\ 0 & \text{otherwise.} \end{cases}$$

Belief function $\mathrm{Bel}_{\boldsymbol{\mathcal{J}}} : F \times 2^S \to [0,1]$:

$$\mathrm{Bel}_{\boldsymbol{\mathcal{J}}}(f,P) = \sum_{A \subseteq P} \delta_{\boldsymbol{\mathcal{J}}}(f,A)$$

Multi-Layer Belief Model

Bridging Layer - Example: degrees of beliefs

Red car, with $\mathcal{J}^{DS}$:

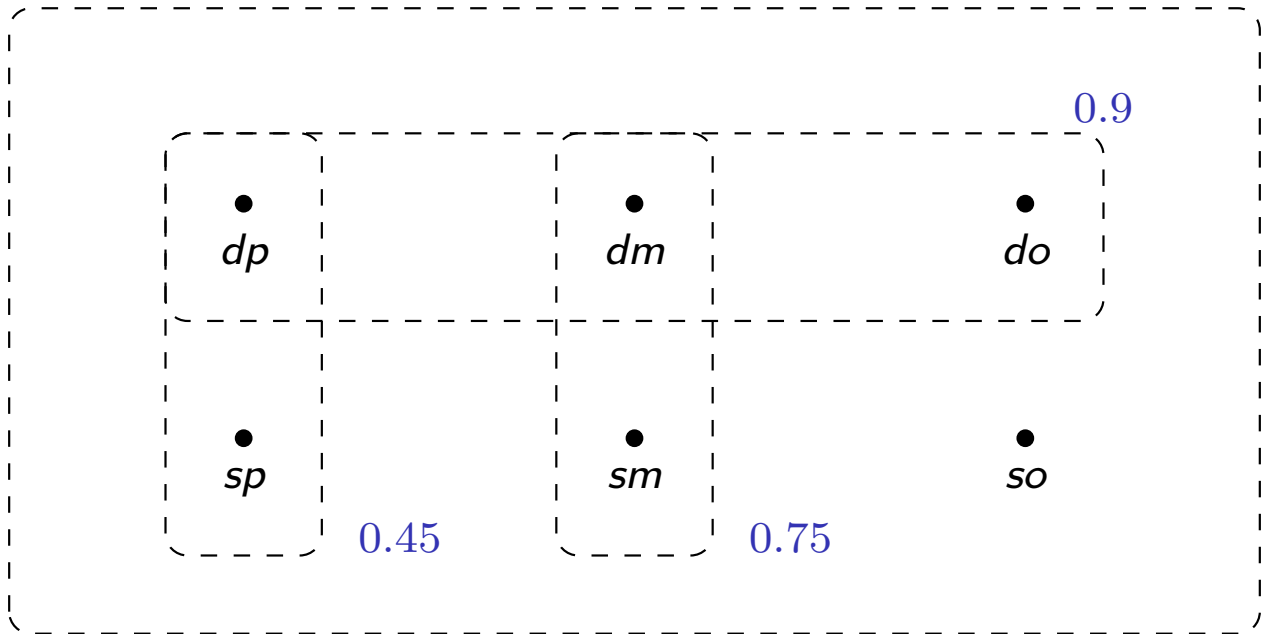
Proposition	P	i	u	d
(1)	$\{dp, do, dm\}$	0.88	0.59	0.89
(2)	$\{sp, dp\}$	0.16	0.11	0.11

Blue car, with $\mathcal{J}^{SD}$:

Proposition	P	i	u	d
(1)	$\{dp, do, dm\}$	0.92	0.13	0.91
(2)	$\{sp, dp\}$	0	0	0

Multi-Layer Belief Model

Qualitative Layer - Frames of Justification - Example



Summary

- A new model for measuring degrees of beliefs based on possibly inconsistent, incomplete, and uncertain evidence.
- This model generalizes DST of belief functions and topological notion of belief.
- Flexible new components: *evidence allocation functions* and *frames of justification*.
- For preliminary complexity results, please see the paper!

Future Work

- Formal comparison to other methods of evidence combinations
- Other frames of justification
- Logics of evidence and belief functions

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Thank you!