Prerequisites

for the talk on

Incompleteness of static theories & Completeness of dynamic beliefs in people and in bots

by
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@
WoLLIC 2023

Outline

1. Background: From Static to Dynamic Logic
2. States as models of theories
3. Theories as sketches
4. Models as functors
5. Universe of state spaces
6. Universality

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1. Background: From Static to Dynamic Logic
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One-slide history of logic

Aristotle

Socrates is human   Humans are mortal

Socrates is mortal

Frege ... Tarski

$\vdash a = b \vee a \neq b$

$\models a = b$ or $\models a \neq b$

morning star

evening star

Wittgenstein ... Carnap

$\xrightarrow{Q}$

One-slide history of logic

Aristotle

$$\frac{\Theta}{\alpha}$$

Frege ... Tarski

$$\begin{tikzcd}
\Theta \arrow[d, dash] & \vdash & \alpha \\
\mathcal{M} \arrow[d] & \models & \alpha \\
\mathrm{Set} & &
\end{tikzcd}$$

Wittgenstein ... Carnap

$$\begin{tikzcd}
\Theta_A \arrow[d, dash] & & \Theta_B \arrow[ll, "Q"'] \arrow[d, dash] \\
\mathcal{M}_A \arrow[dr] & A\{Q\}B & \mathcal{M}_B \arrow[dl] \\
& \mathrm{Set} &
\end{tikzcd}$$

One-slide history of logic

Aristotle

$$\frac{\Theta}{\alpha}$$

Frege ... Tarski

$$\begin{tikzcd}
\Theta \vdash \alpha \arrow[d, dash] \\
\mathcal{M} \vDash \alpha \arrow[d] \\
\mathrm{Set}
\end{tikzcd}$$

Floyd, Hoare, Pratt ...

$$\begin{tikzcd}
\Theta_A \arrow[d, dash] & b \leftarrow\!\shortmid x & \Theta_B \arrow[ll, bend right, "\ell"'] \arrow[d, dash] \\
\mathcal{M}_A \arrow[dr] & A\{x := b\}B & \mathcal{M}_B \arrow[dl] \\
& \mathrm{Set} &
\end{tikzcd}$$

Dynamics of influence

Dynamics of influence

Self-fulfilling prophecy:

The tragedy of Macbeth

Dynamics of influence

Self-fulfilling prophecy:

The tragedy of Macbeth

Witch 1: All hail Macbeth!
Hail to thee Thane of Glamis.

Witch 2: All hail Macbeth!
Hail to thee Thane of Cawdor.

Witch 3: All hail Macbeth!
Thou shalt be King
hereafter

Dynamics of influence

Self-fulfilling prophecy:

(Macbeth's reasoning:)

The tragedy of Macbeth

Witch 1: All hail Macbeth!
Hail to thee Thane of Glamis.
("How does she know me?")

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Dynamics of influence

Self-fulfilling prophecy:

The tragedy of Macbeth

(Macbeth's reasoning:)

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("It was a prophecy!")

Witch 3: All hail Macbeth!
Thou shalt be King
hereafter
("I shall be King!")

Dynamics of influence

Self-fulfilling prophecy:

Social Network Day 0

- utility $= \dfrac{\#\text{users}}{\text{population}}$

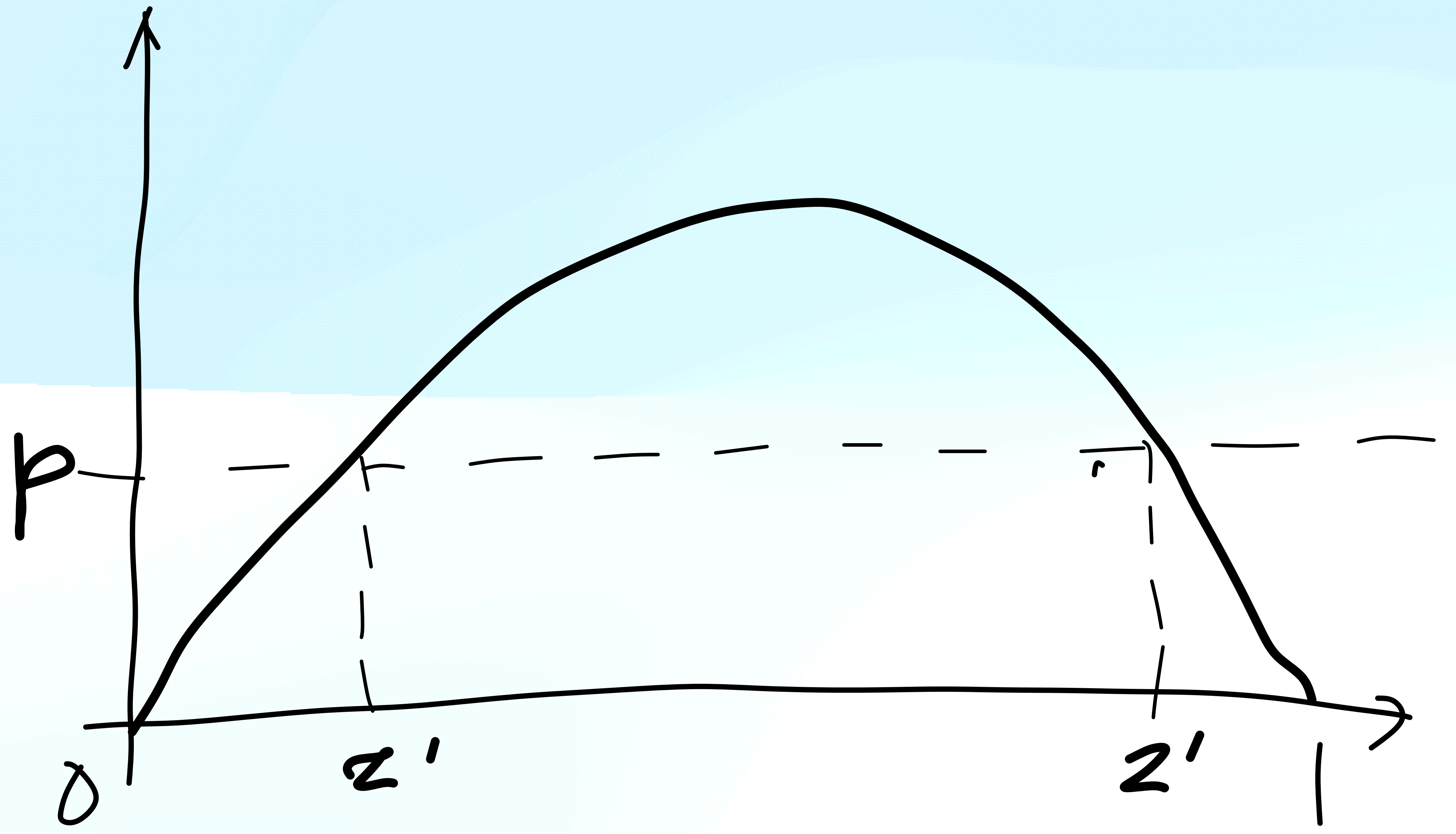
  $u(x) = x$

- demand $= \dfrac{\text{population} - \#\text{users}}{\text{population}}$

  $v(x) = 1 - x$

Dynamics of influence

Self-fulfilling prophecy:



Equilibrium: $w(x) = p$

- utility $= \dfrac{\#\text{users}}{\text{population}}$

$$u(x) = x$$

- demand $= \dfrac{\text{population} - \#\text{users}}{\text{population}}$

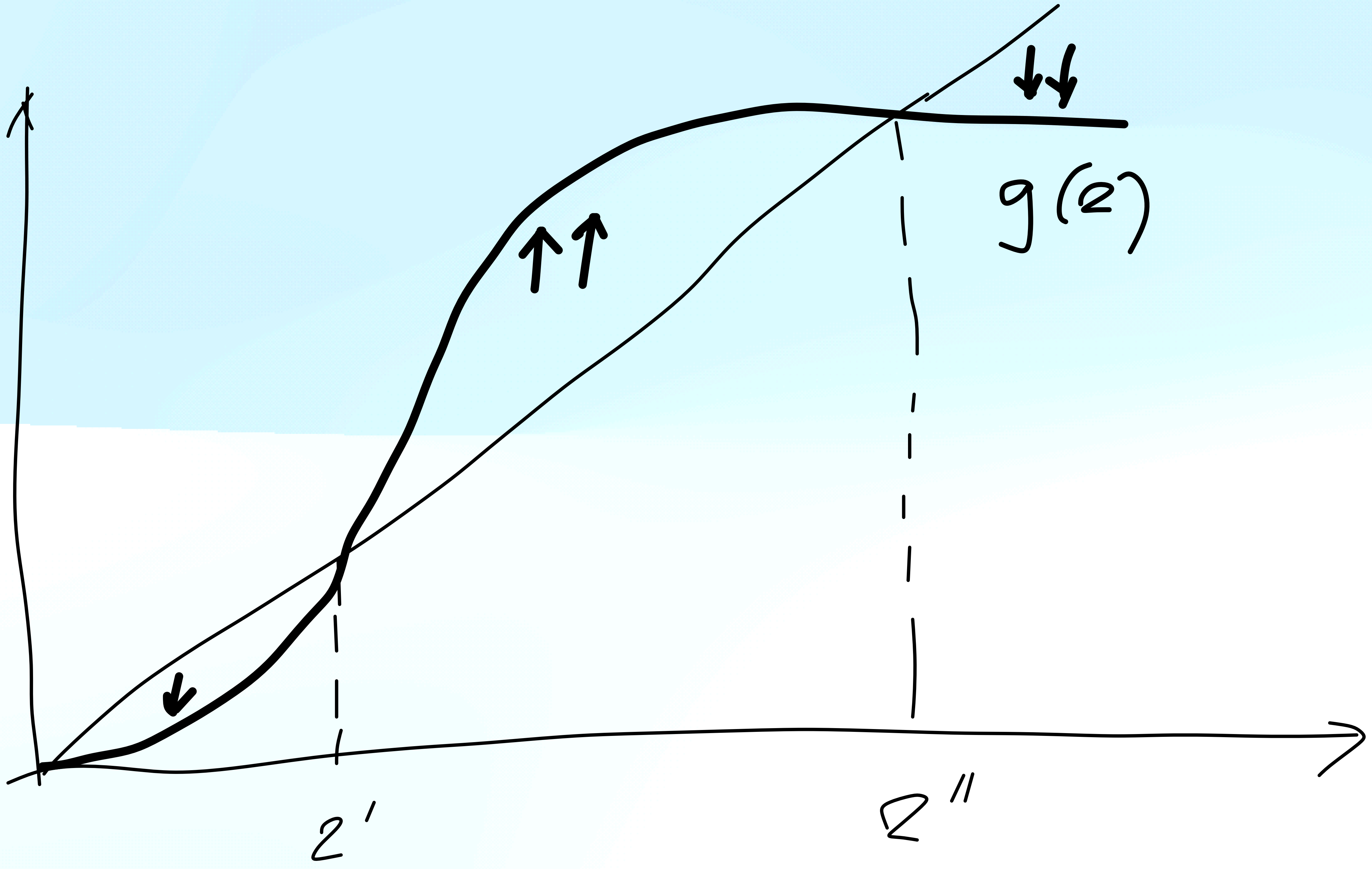
$$v(x) = 1 - x$$

- valuation = utility × demand

$$w(x) = x - x^2$$

Dynamics of influence

Self-fulfilling prophecy:



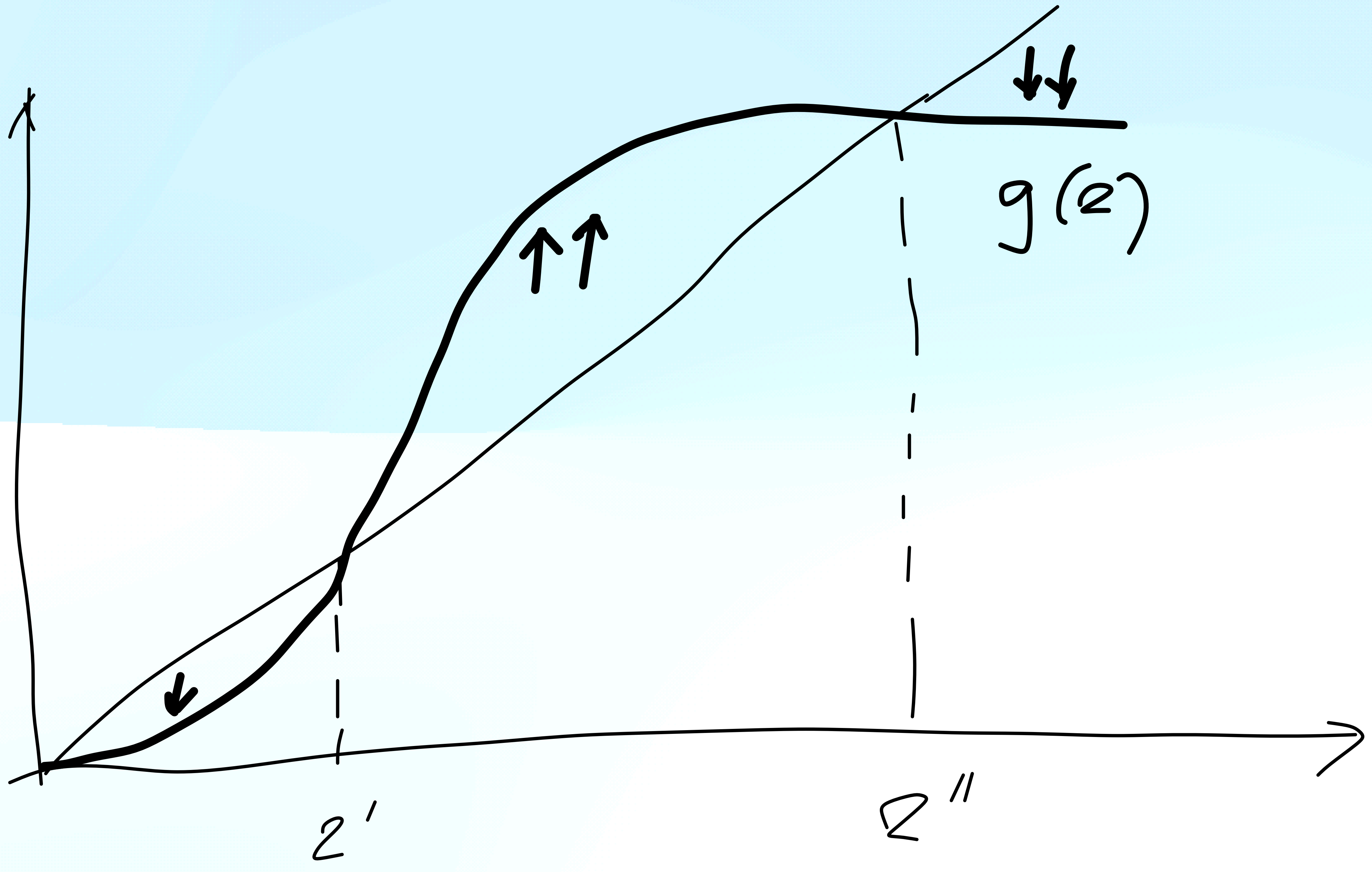
Tipping point

- adoption $\approx 1 - \frac{p}{x}$

$$g(z) = \begin{cases} 0 & \text{if } z \leq p \\ 1 - \frac{p}{z} & \text{otherwise} \end{cases}$$

Dynamics of influence

Self-fulfilling prophecy:



Tipping point

If >p people believe that you are King then you will be King.

Dynamics of influence

Q: Why do we believe?

Dynamics of influence

We believe in causes and effets.

The tragedy of Macbeth

Macbeth reasons:

- If **King** knows the prophecy, then he will kill me to prevent it.

$\Downarrow$

- I must kill **King** to defend myself.

Dynamics of influence

We believe in causes and effets.

Prophecy as a threat

Macbeth reasons:

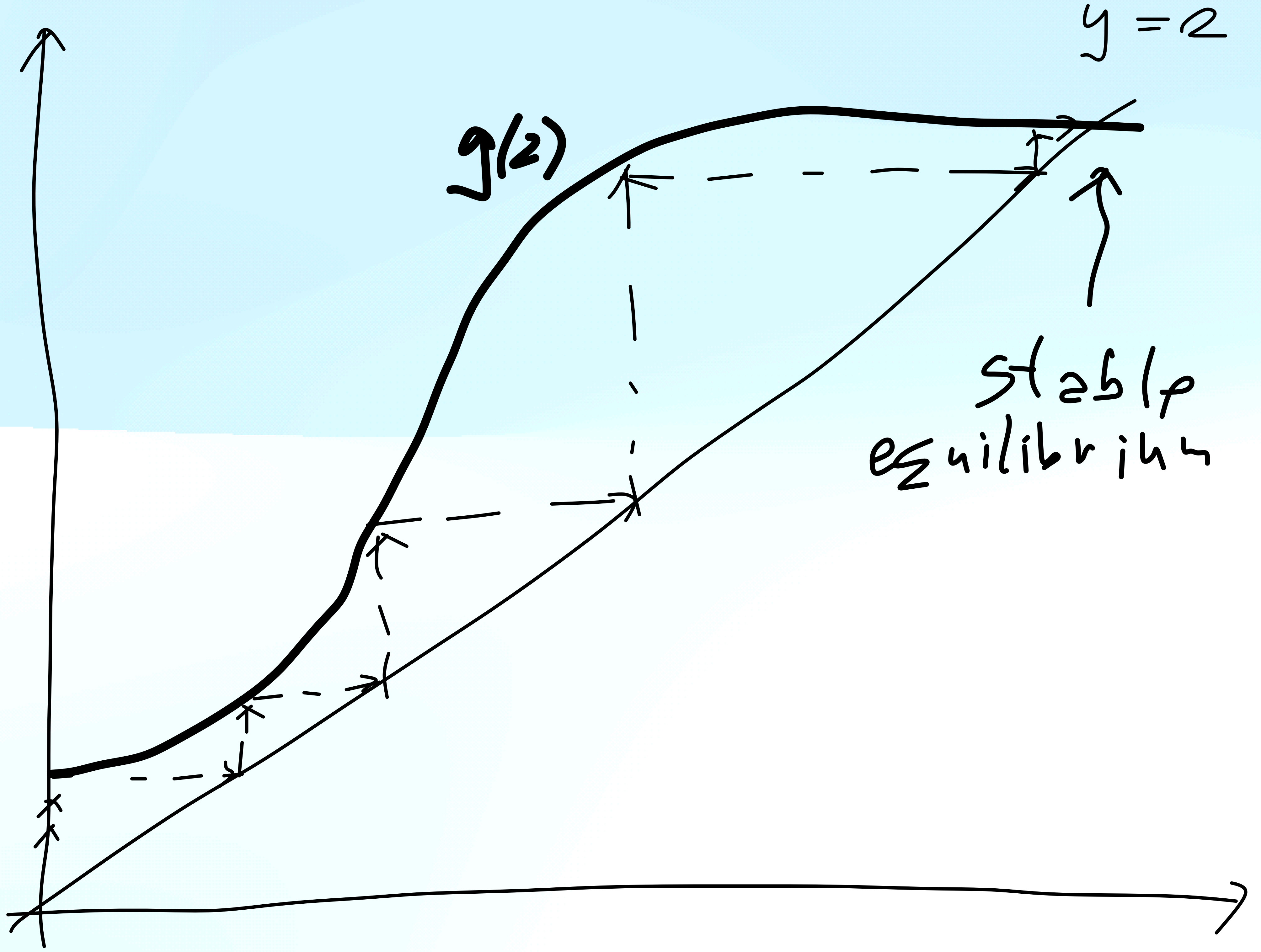
- If **King** knows the prophecy, then he will kill me to prevent it.

⇓

- I must kill **King** to defend myself.

Dynamics of influence

We form beliefs about beliefs and their effects



Prophecy is a **promise**

Francesca reasons:

- It cannot be true that half of my friends already use FB
- But a quarter of my friends will believe that and join.
- A half of the other quarter will believe that a quarter of their friends will believe and join...
- So it will **become true...**

Dynamics of influence

- Keynes' Beauty Contest
- Soros' reflexivity

Dynamics of influence

- Keynes' Beauty Contest
- Soros' reflexivity
- Beliefs about beliefs impact reality.

Dynamics of influence

Q: Why do we believe?

A: Coordinated equilibrium:
we believe that others believe.

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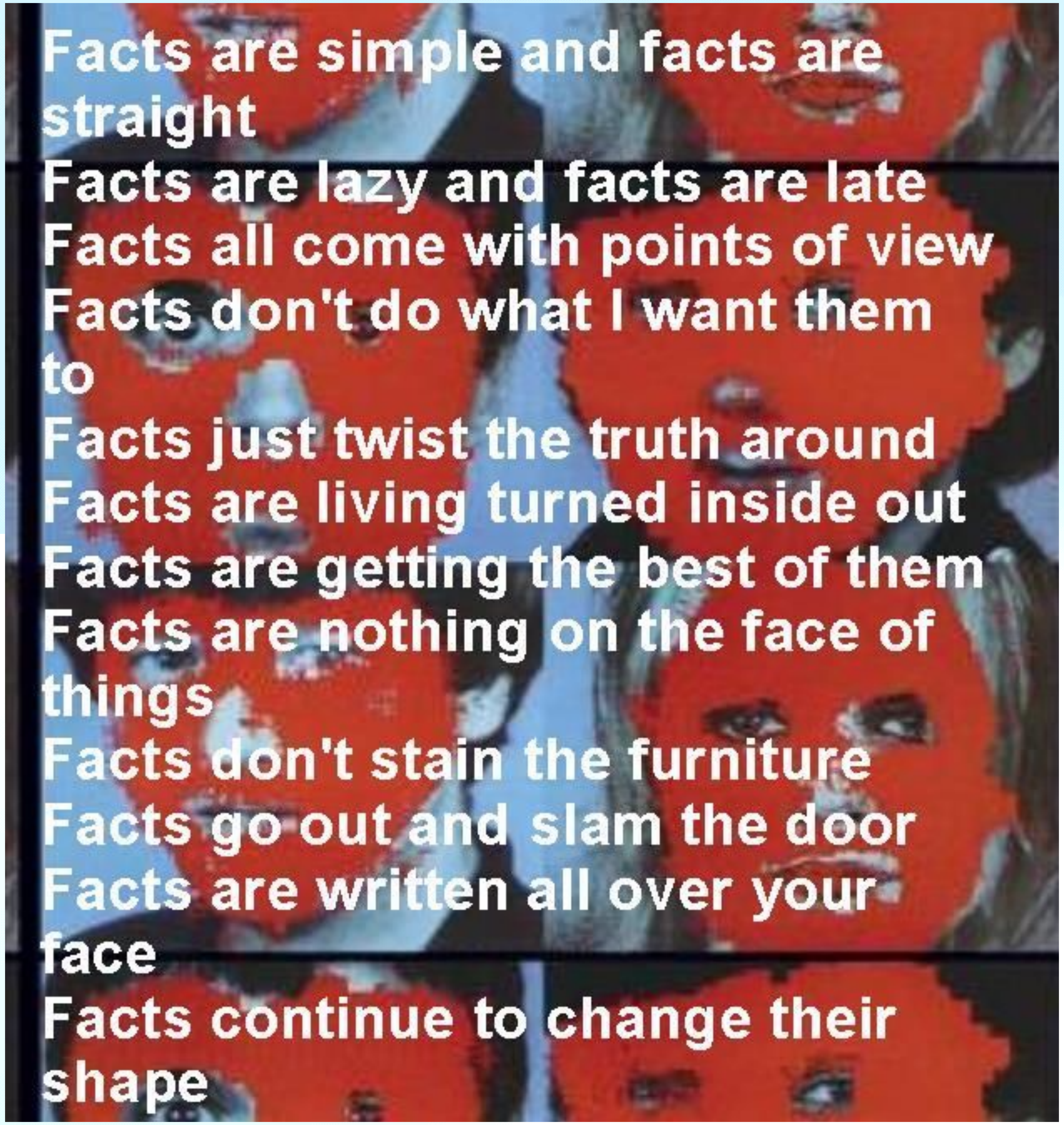
The world of facts, states, possible worlds...

Tractatus Logico-Philosophicus

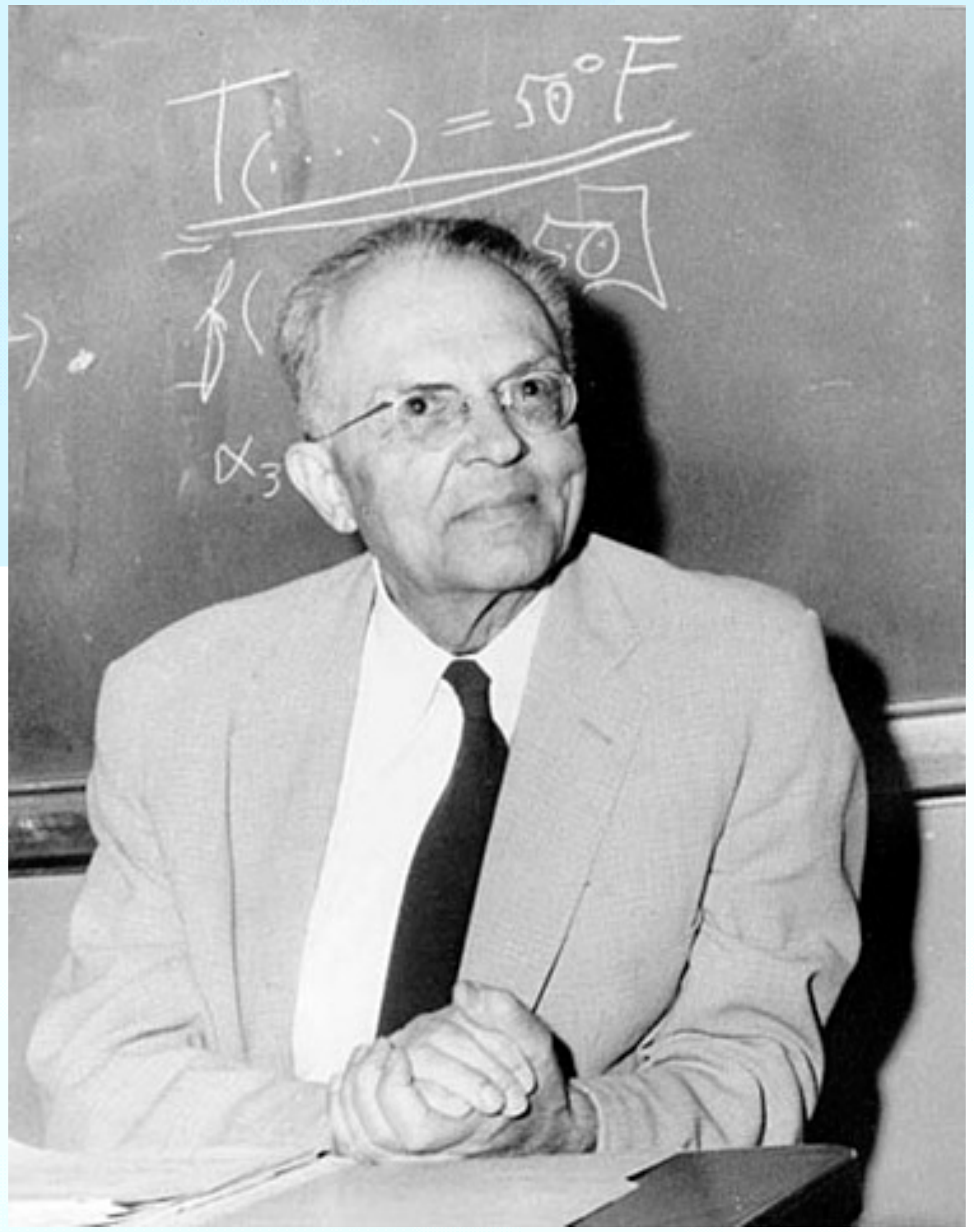
[depth level 0]

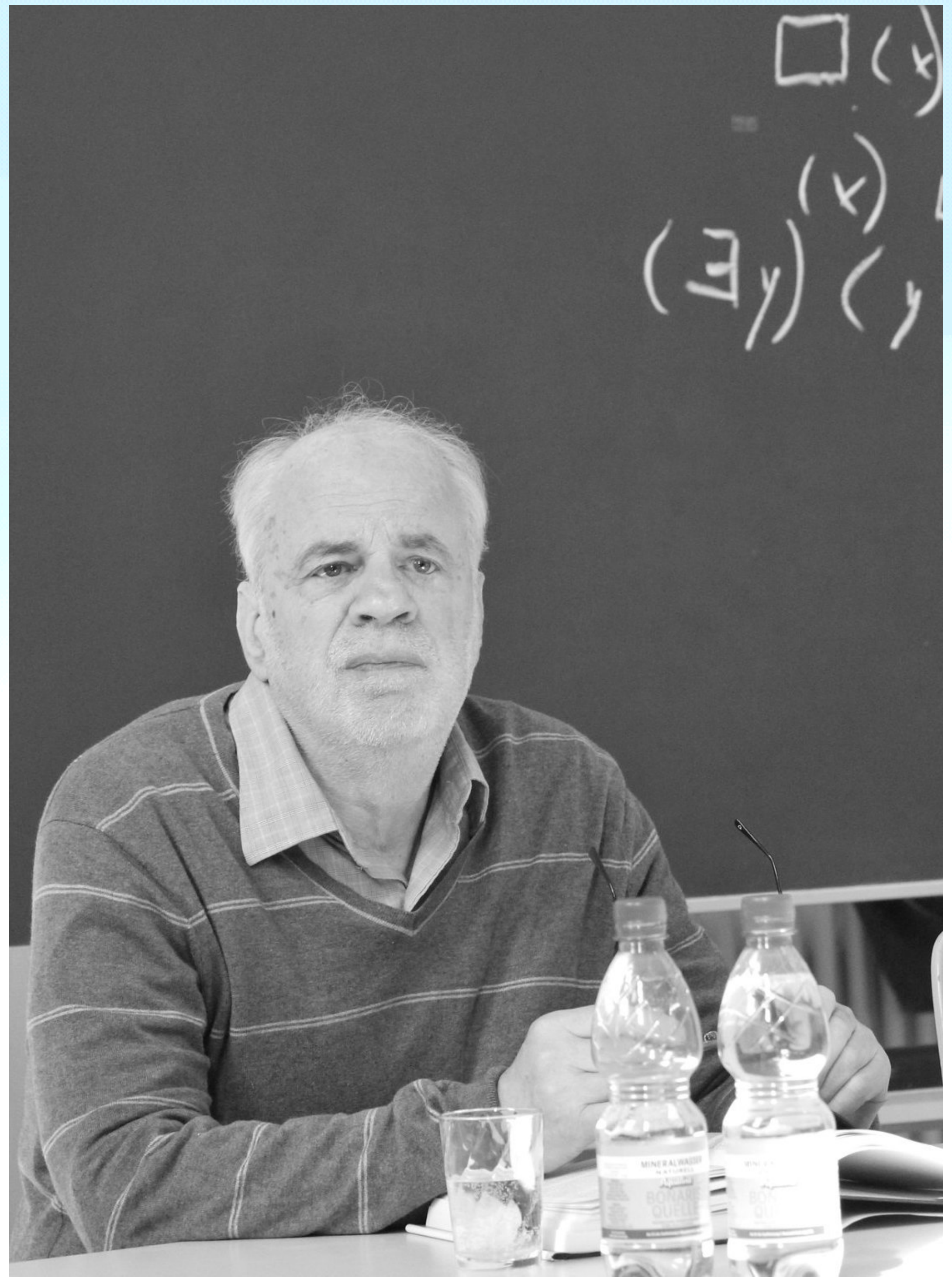
	German	Ogden	Pears/McGuinness
1*	Die Welt ist alles, was der Fall ist.	The world is everything that is the case.	The world is all that is the case.
2	Was der Fall ist, die Tatsache, ist das Bestehen von Sachverhalten.	What is the case, the fact, is the existence of atomic facts.	What is the case—a fact—is the existence of states of affairs.
3	Das logische Bild der Tatsachen ist der Gedanke.	The logical picture of the facts is the thought.	A logical picture of facts is a thought.

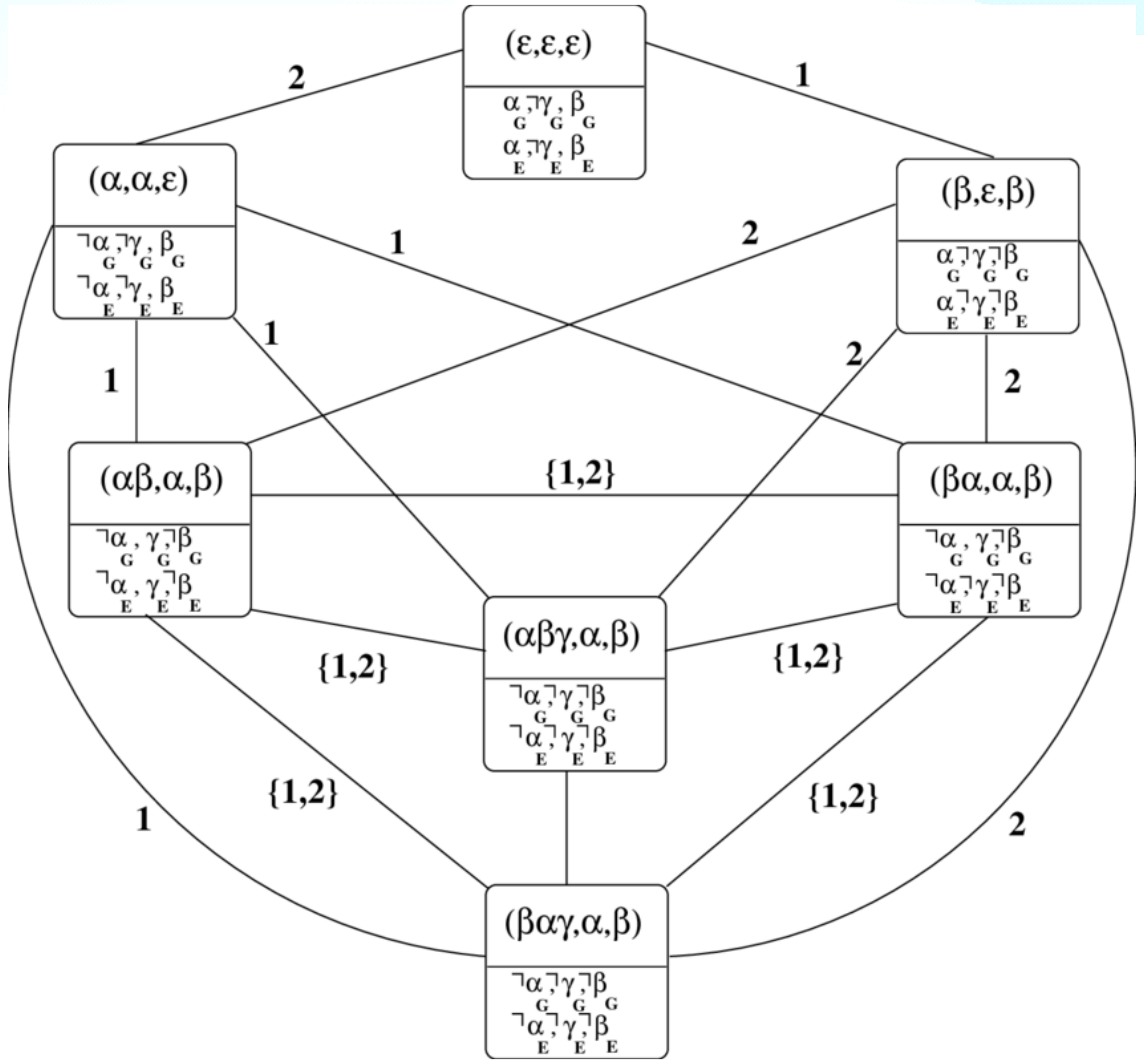
The world of facts, states, possible worlds...



The world of facts, states, possible worlds...







State
(of fact, of affairs, of the world...)

$$\mathcal{A} = \langle \Theta_A, \mathcal{M}_A \rangle$$

theory

reference model

State

(of fact, of affairs, of the world...)

$$A = \langle \Theta_A , \mathcal{M}_A \rangle$$

theory

reference model

Problem : States are static

Kripke : States change

Task : Self-reflection

Capture the change as a state

State change

$$\begin{tikzcd} A = \langle \Theta_A, & \mathcal{M}_A \rangle \arrow[rr, bend right, "f_{\mathcal{M}}"'] & B = \langle \Theta_B, \arrow[ll, bend right, "f"'] & \mathcal{M}_B \rangle \end{tikzcd}$$

$$f : A \longrightarrow B$$

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Theory

$$\Theta = \langle \underset{\text{signature}}{\Sigma}, \underset{\text{axioms}}{\Gamma} \rangle$$

Algebraic Theory

$$\Theta = \langle O, E \rangle$$

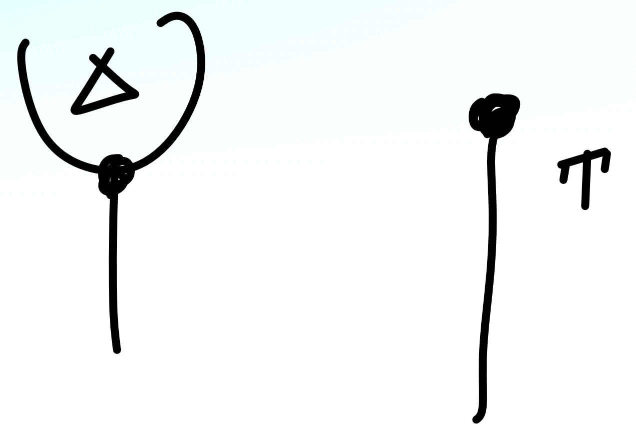
operations

equations

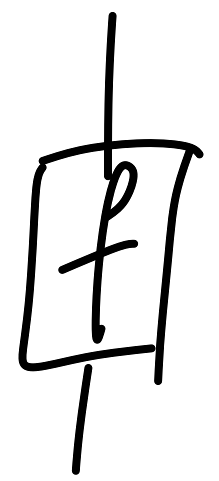
Algebraic Theory Categorically

Def (Lawvere). An algebraic theory ("clone") is a *free* category generated by

- commutative comonoid



- algebraic operations



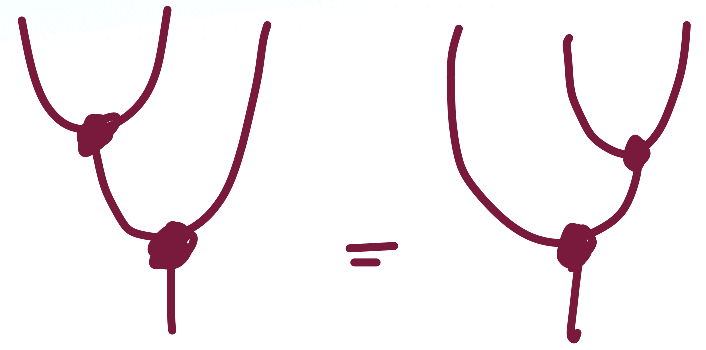
- equations

Algebraic Theory Categorically

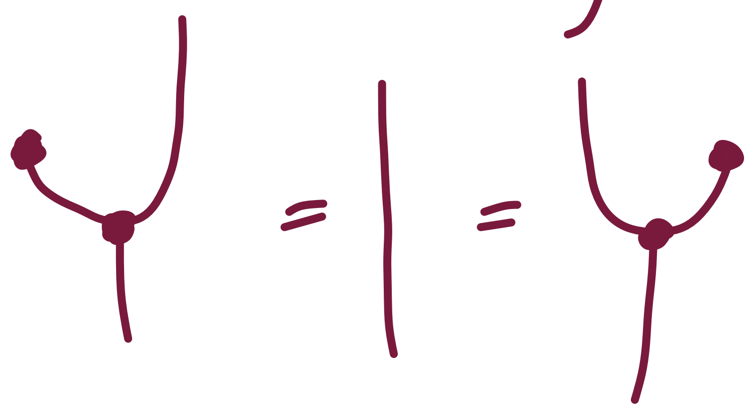
<u>**Def (Lawvere).**</u> An algebraic theory (<u>"clone"</u>) Θ is a free category generated by

- commutative comonoid
- algebraic operations
- equations

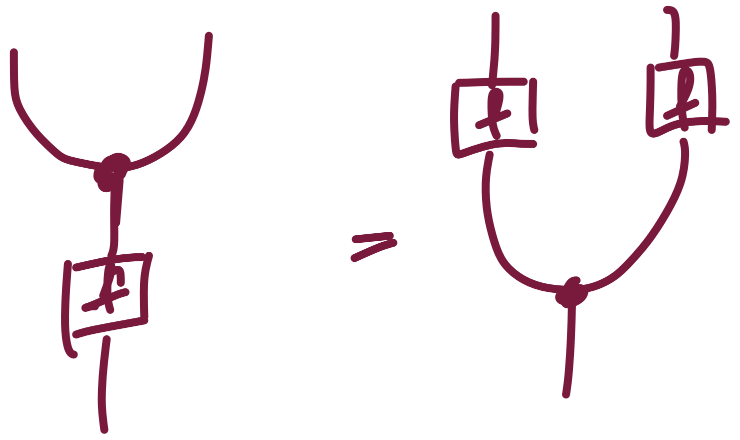
associative



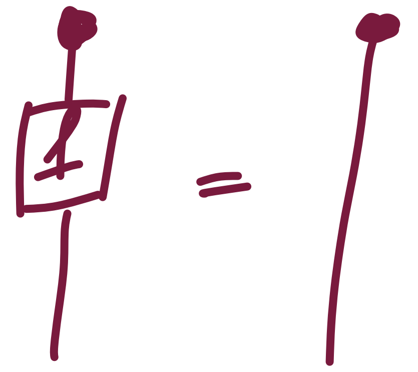
unitary



single-valued



total



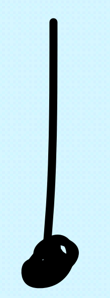
Upshot: Functorial semantics

Thm.

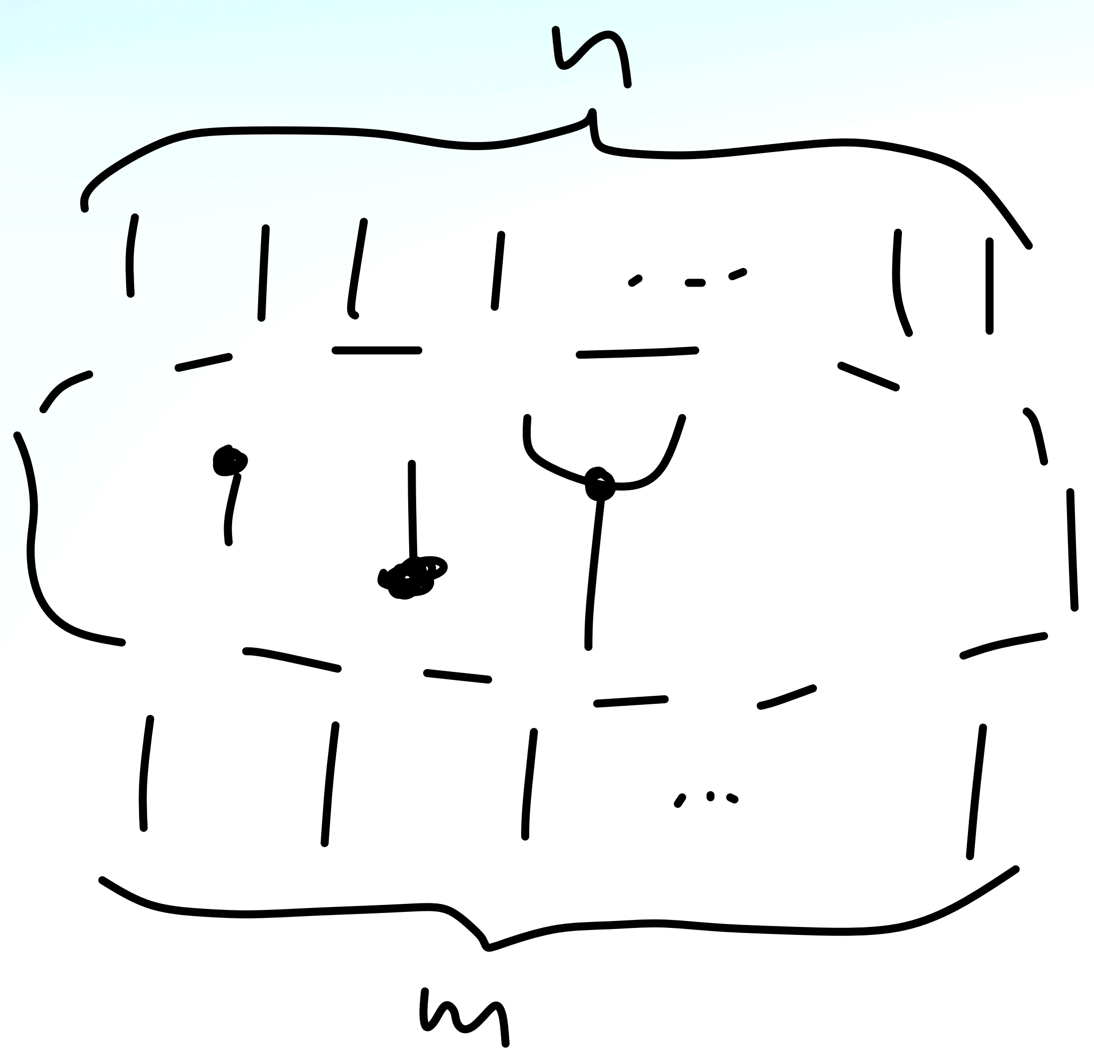
- A Θ-algebra is a $\times$-preserving functor $\Theta \xrightarrow{A} \mathrm{Set}$
- A Θ-algebra homomorphism $h : A \to B$ is a natural transformation

$$\begin{tikzcd} \Theta \arrow[d, bend right=60, "A"'] \arrow[d, bend left=60, "B"] \arrow[d, phantom, "\xrightarrow{h}"] \\ \mathrm{Set} \end{tikzcd}$$

Example 1

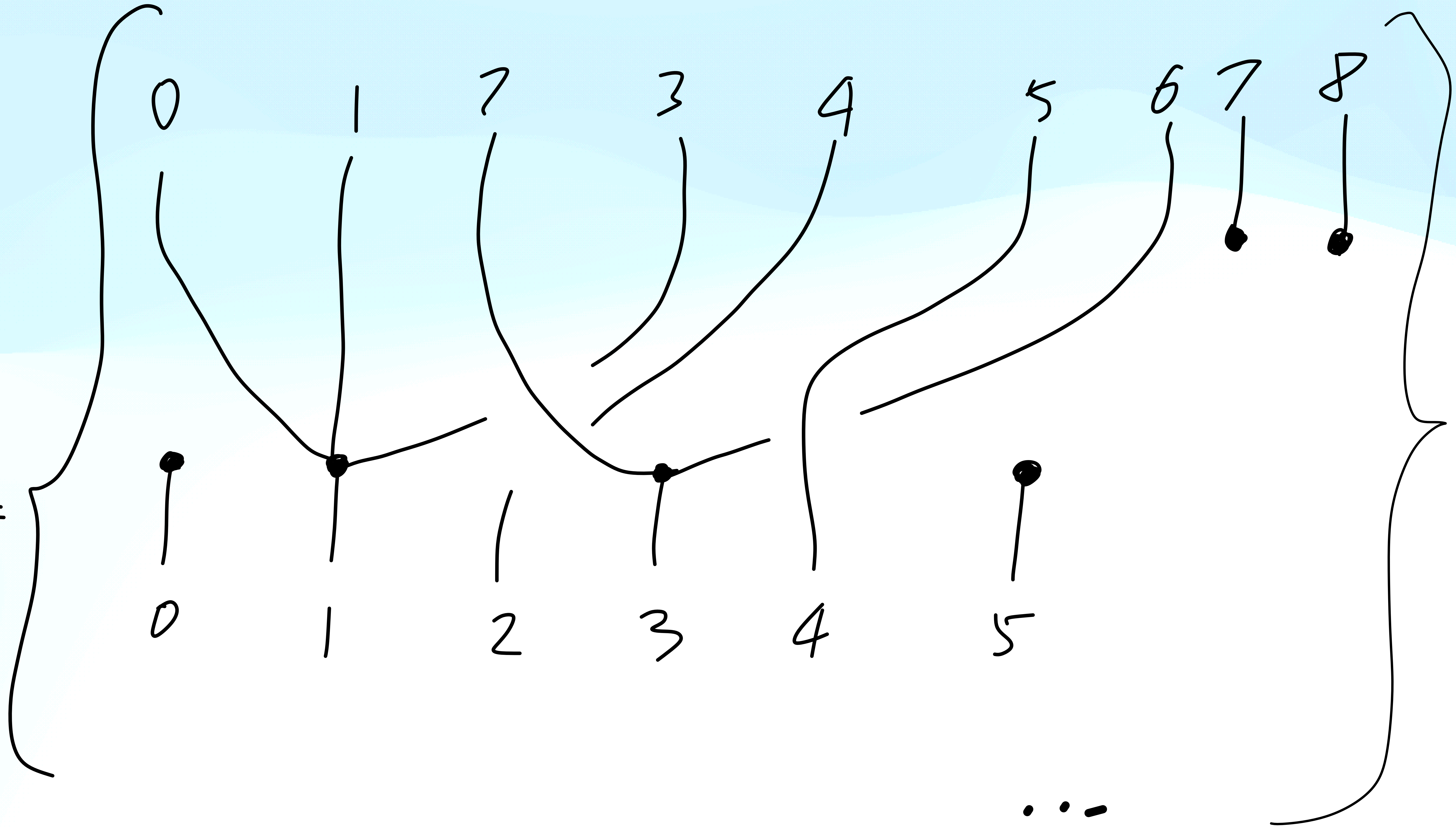
$\mathcal{O} = \{$  $\}$

$|\Theta| = \{0, 1, 2, \ldots\}$

$\Theta(m, n) = \{$  $\}$

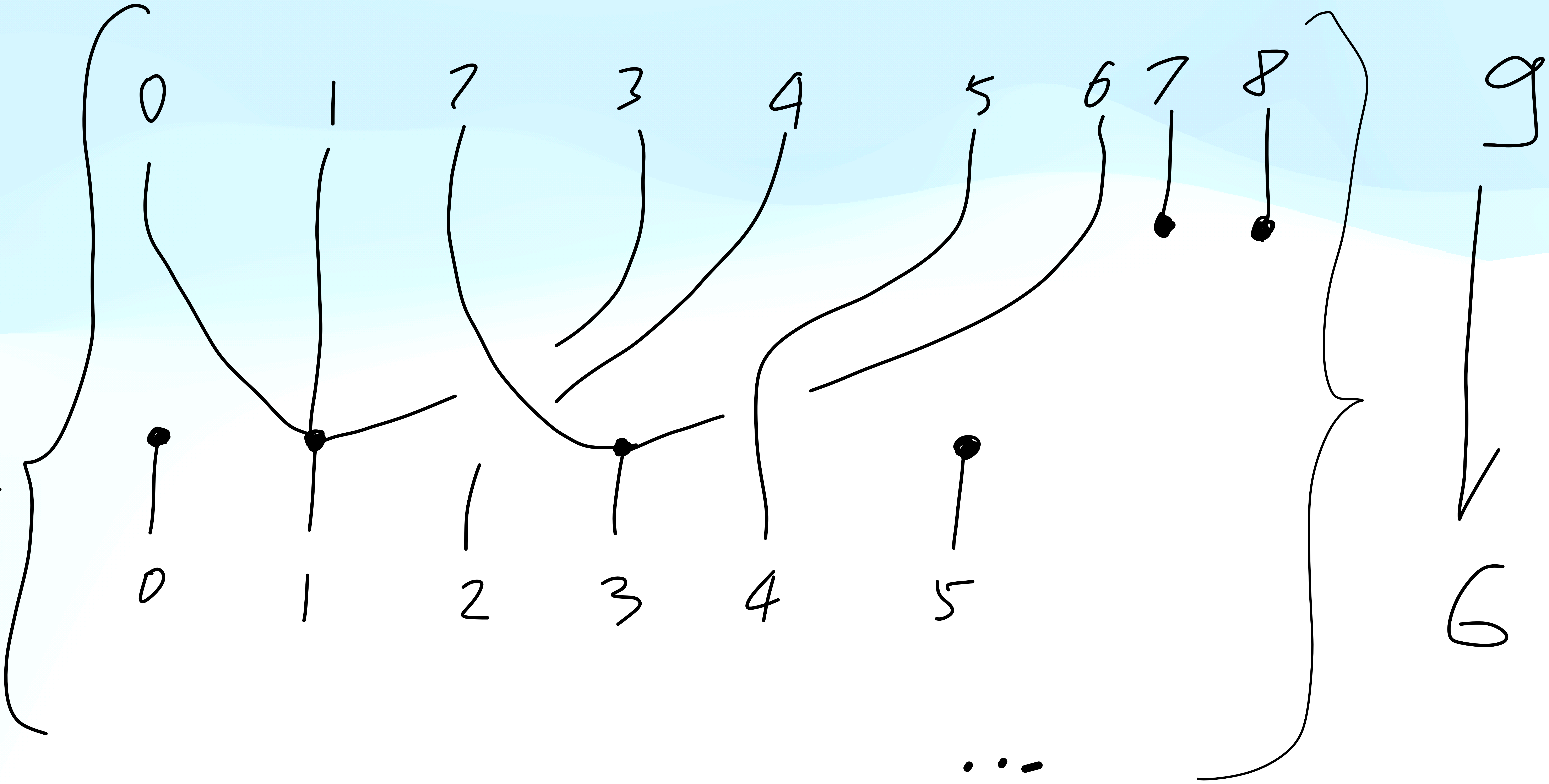
Example 1

$\Theta(6,9) =$

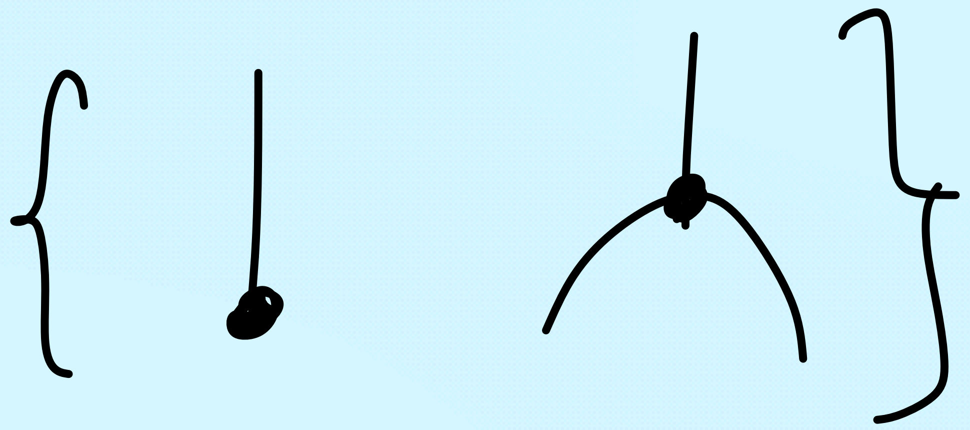


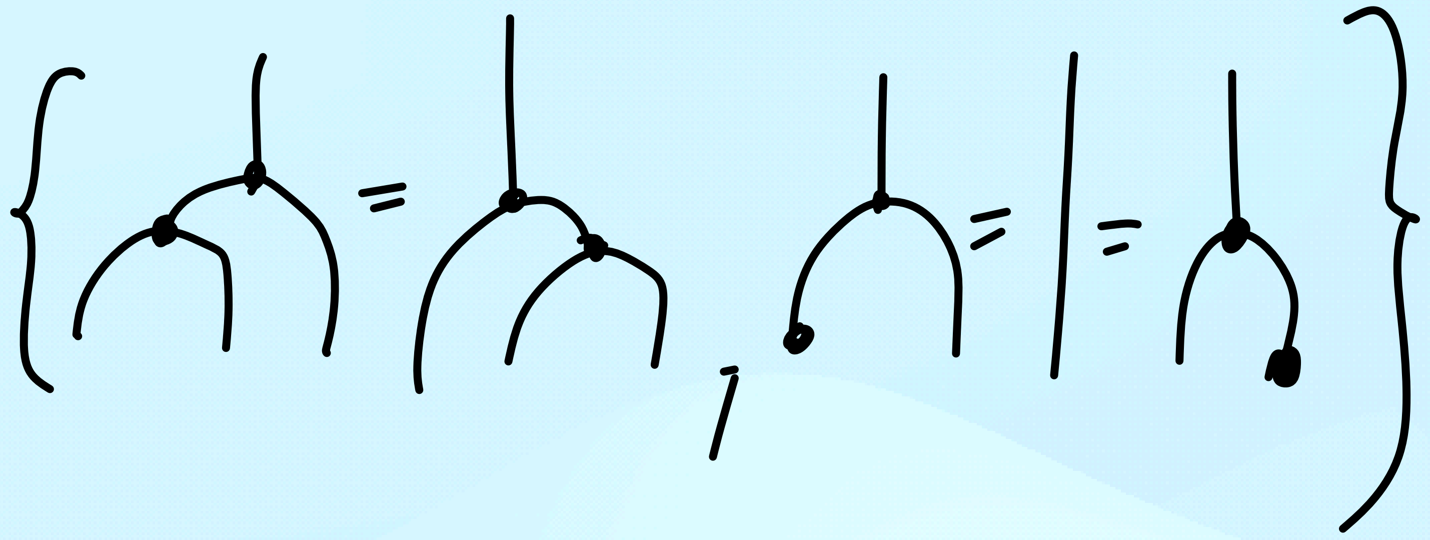
Example 1

$\Theta(6,9) =$

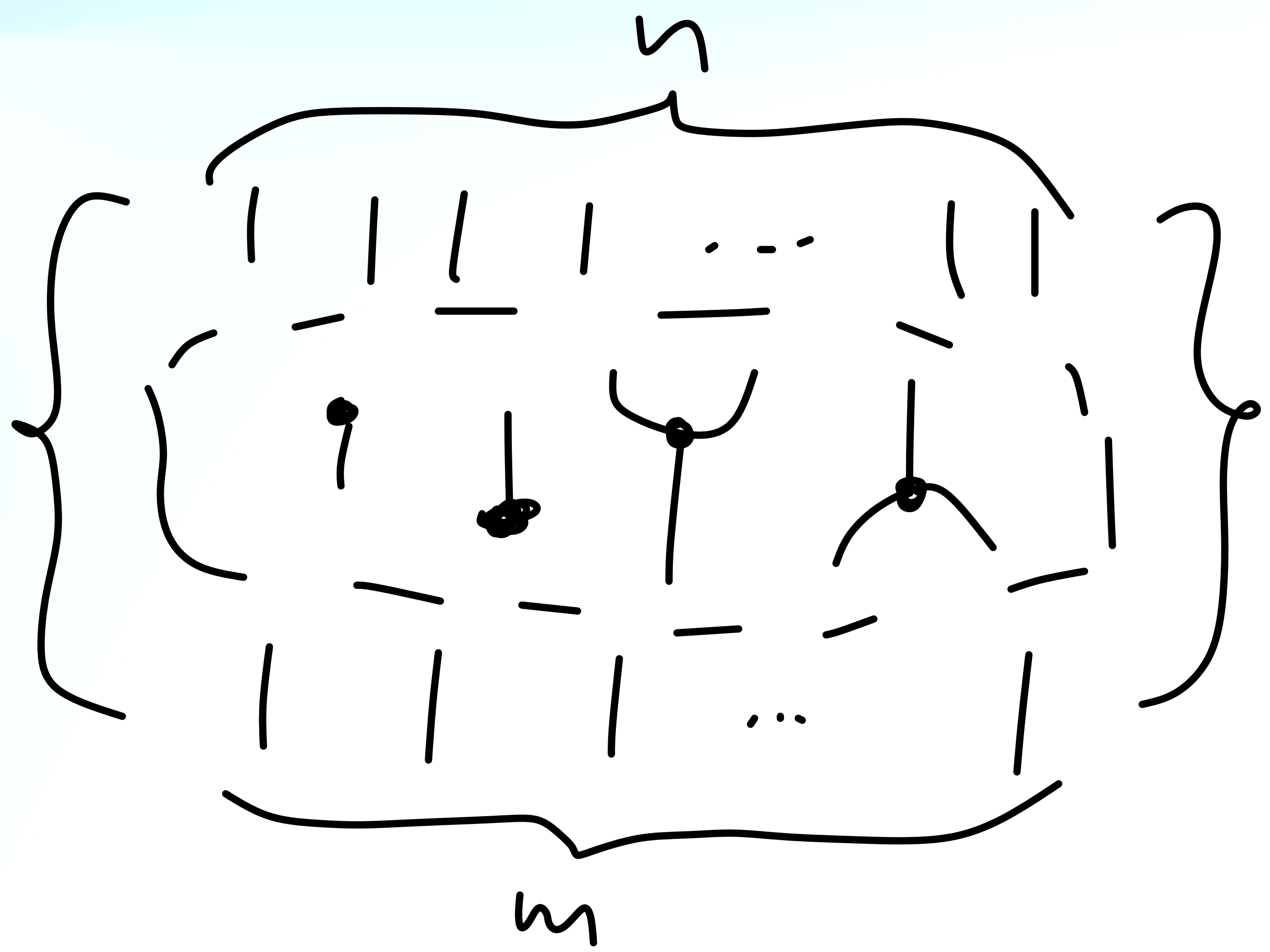


Example 2

$O =$ 

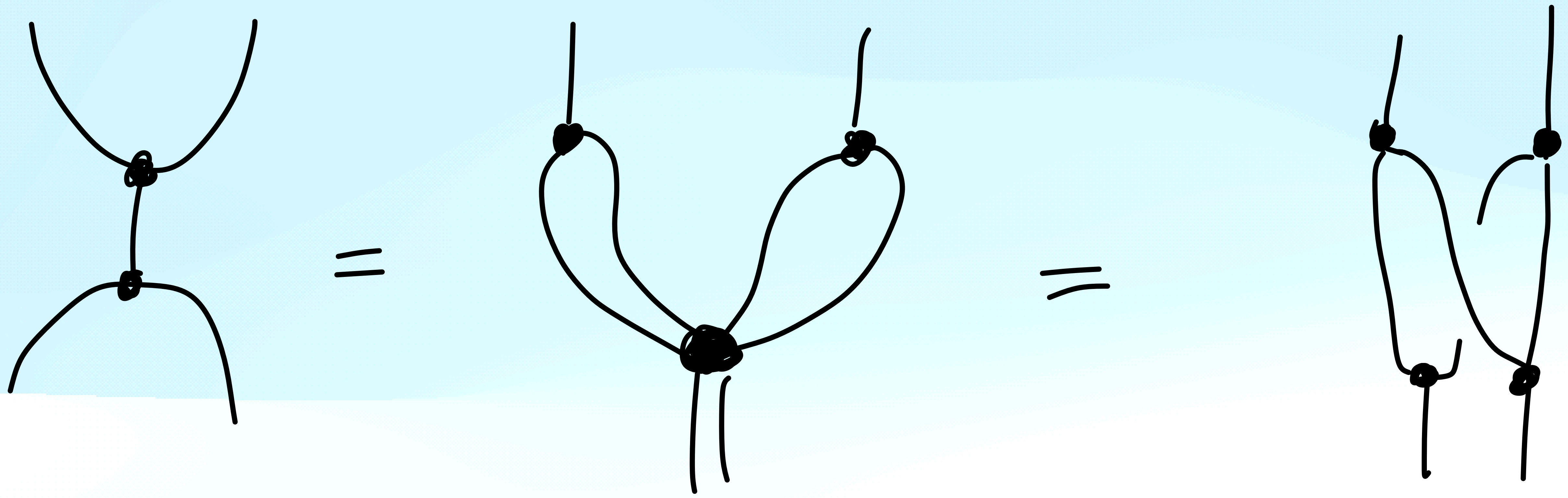
$E =$ 

$|\Theta| = \{0, 1, 2, \ldots\}$

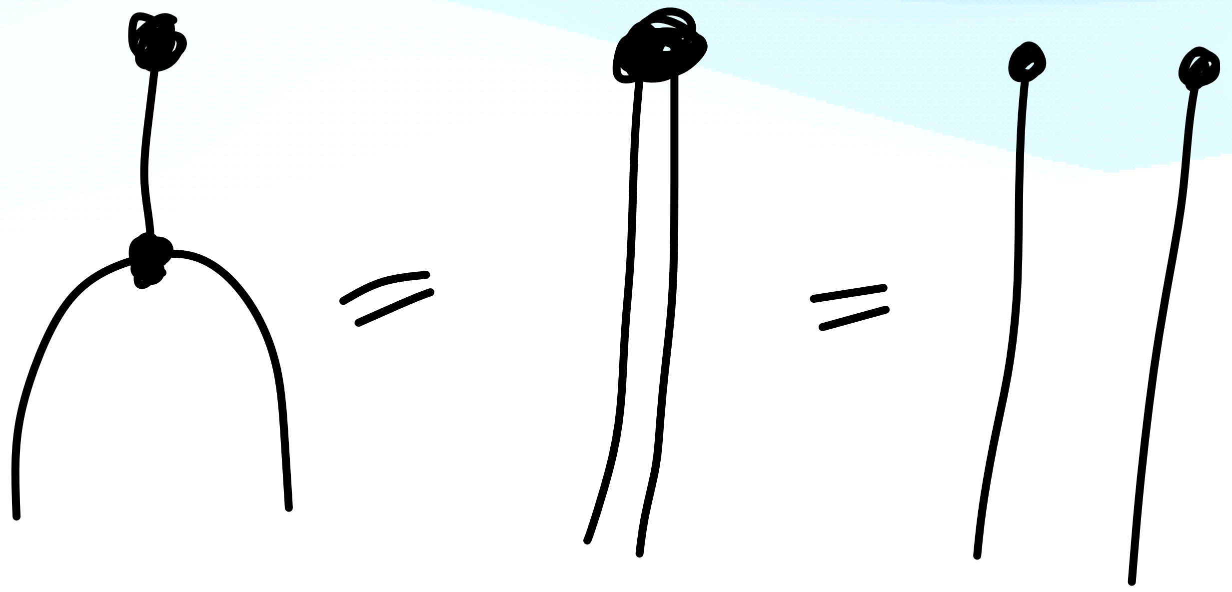
$\Theta(m, n) =$ 

Example 2

single-valued

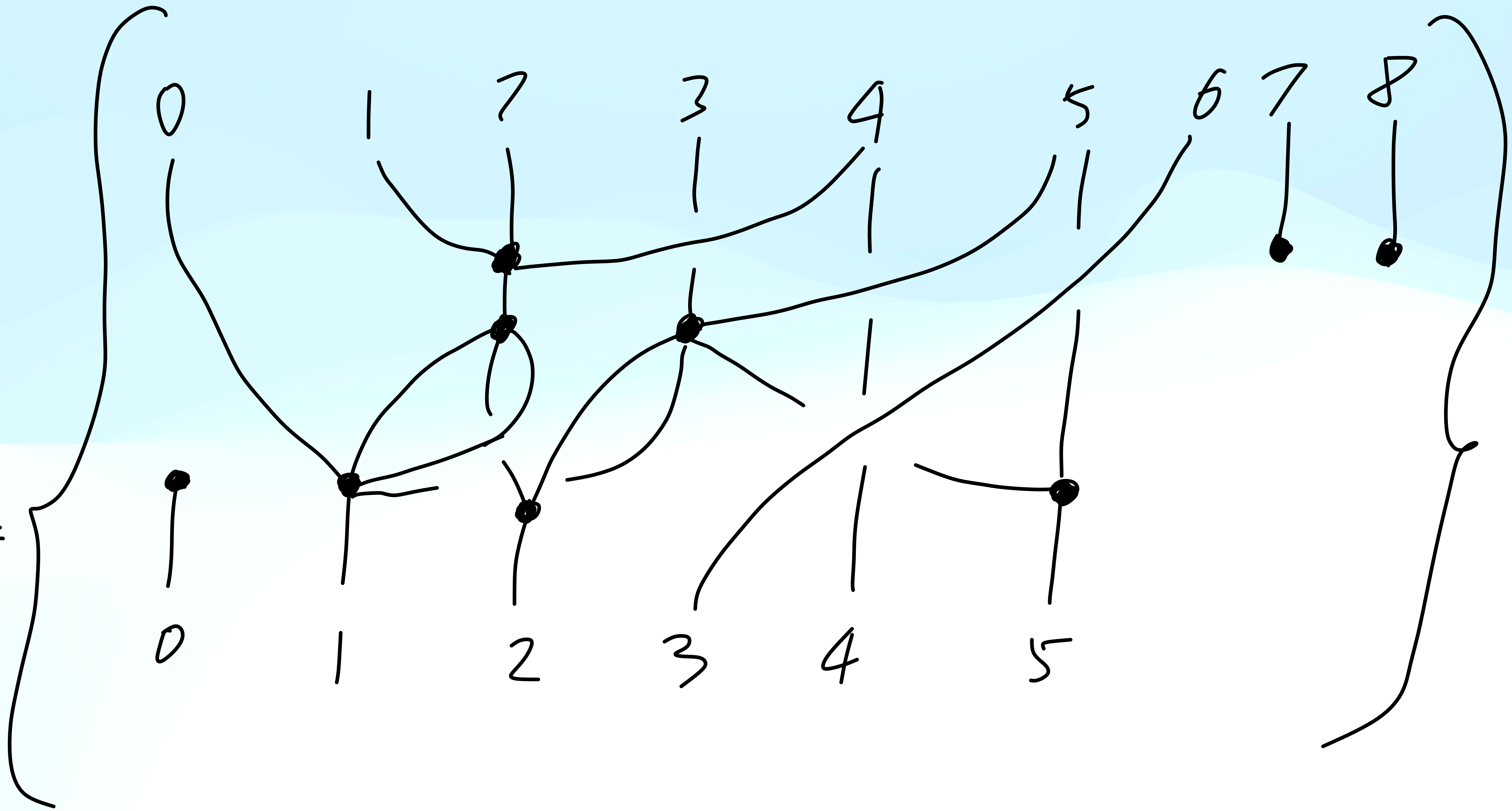


total



Example 2

$$\Theta(6,9) =$$



$$9 \longrightarrow 6^*$$

...

Theory

$$\Theta = \langle \underbrace{\Sigma}_{\text{signature}} , \underbrace{\Gamma}_{\text{axioms}} \rangle$$

First-Order Theory

$$\Theta = \langle \Sigma, \Gamma \rangle$$

signature

axioms

First-Order Theory Categorically

Def (Ehresmann). A first-order theory ("sketch") Θ is comprised of

- category Σ — usually cartesian, for arities
- set of diagrams Γ — required to be limit cones or colimit cocones.

First-Order Theory Categorically

Idea.

- Assume that the theory is given in $L_{\infty\infty}$-logic $\rightsquigarrow$ with infinite

  $\bigwedge$ presenting $\forall x.\, P(x)$ by $\bigwedge_{x\in A} P(x)$

  $\bigvee$ presenting $\exists x.\, P(x)$ by $\bigvee_{x\in A} P(x)$

- ...

First-Order Theory Categorically

Idea. ...

- $|\Sigma| = \{\varphi(\vec{x}) \mid \vec{x} : \mathrm{Sorts}^*\}/\alpha\text{-conv}$

  $\Sigma(\varphi(x), \psi(y)) = \left\{ \theta(x,y) \vdash \varphi(x) \wedge \psi(y) \;\middle|\; \begin{array}{l} \varphi(x) \vdash \exists y.\, \theta(x,y) \\ \theta(x,y) \wedge \theta(x,y') \vdash y = y' \end{array} \right\}$

- Γ expresses the axioms using $\varprojlim$ and $\varinjlim$

First-Order Theory Categorically

Example. Sketch for fields $\Theta = (\Sigma, \Gamma)$

- Σ = clone over sorts $r, i, 2$ with

  r = operations & equations for ring
  i = operations & equations for group
  2 = singleton : a constant and all terms equal

- $\Gamma = \{ i \longrightarrow r \longleftarrow 2 \}$

  colimit cocone
  preserving $\times$
  and 0

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Model of theory $\Theta = \langle \Sigma, \Gamma \rangle$

$$\begin{tikzcd} \Sigma \arrow[dr, "\Gamma\text{-preserving}\ \mathcal{M}"', "\text{functor}"] & \\ & \mathbf{Set} \end{tikzcd}$$

Model of theory $\Theta = \langle \Sigma, \Pi \rangle$

$$\begin{tikzcd} \Sigma \arrow[d, shift right=2, "\mathcal{M}"'] \arrow[d, shift left=2, "\mathcal{N}"] \\ \mathbf{Set} \end{tikzcd}$$

Π-preserving $\mathcal{M}$ $\mathcal{N}$ functor

Homomorphism $\mathcal{M} \xrightarrow{h} \mathcal{N}$

$$
\begin{tikzcd}
\Sigma \arrow[dr] \arrow[drr] & & \\
& \mathcal{M} \arrow[r, "h"] \arrow[dr] & \mathcal{N} \arrow[d] \\
& & \mathrm{Set}
\end{tikzcd}
$$

natural

Homomorphism $\mathcal{M} \xrightarrow{h} \mathcal{N}$

$$\begin{tikzcd}
\Sigma \arrow[d, dash] \arrow[dr, dash] & & \\
\mathcal{M} \arrow[r, "h"] \arrow[dr] & \mathcal{N} \arrow[d] & \\
& \mathbf{Set} &
\end{tikzcd}$$

natural

Model-categories for first-order theories

=

Accessible Categories

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State spaces

$$\begin{tikzcd}
 & \mathcal{A} = \langle \Sigma_A, \Gamma_A, \mathcal{M}_A \rangle & & & & \mathcal{B} = \langle \Sigma_B, \Gamma_B, \mathcal{M}_B \rangle & \\
 & \Theta_A \arrow[dr, dash] & & & & \Theta_B \arrow[dl, dash] & \\
\Gamma_A\text{-pres.} & & \mathcal{M}_A \arrow[dr] & & \mathcal{M}_B \arrow[dl] & & \Gamma_B\text{-pres} \\
 & & & \mathbf{Set} & & &
\end{tikzcd}$$

State spaces

$$A = \langle \Sigma_A, \Gamma_A, \mathcal{M}_A \rangle \qquad \qquad B = \langle \Sigma_B, \Gamma_B, \mathcal{M}_B \rangle$$

$$\begin{tikzcd}
 & \Theta_A \arrow[d, dash] & \Gamma \text{ completions} \arrow[l] \arrow[r] & \Theta_B \arrow[d, dash] & \\
\Gamma_A\text{-pres.} & \mathcal{M}_A \arrow[dr] & & \mathcal{M}_B \arrow[dl] & \Gamma_B\text{-pres} \\
 & & \mathbf{Set} & &
\end{tikzcd}$$

State transition

$$\begin{tikzcd}
A \arrow[rr, "F"] & & B \\
\Theta_A \arrow[d, dash] & & \Theta_B \arrow[ll, "F_\Theta"'] \arrow[d, dash] \\
\mathcal{M}_A \arrow[rr, Rightarrow, "F_{\mathcal{M}}"] \arrow[dr] & & \mathcal{M}_B \arrow[dl] \\
& \mathrm{Set} &
\end{tikzcd}$$

Γ_A-pres.

Γ_B-pres

State transition

$$\frac{F_\Gamma : \Gamma_B \to \Gamma_A}{F_\Theta \lim_\gamma D = \lim_{F_\Gamma(\gamma)} F_\Theta D}$$

$$\begin{tikzcd}
& A \arrow[rr, "F"] & & B & \\
& \Theta_A \arrow[d, dash] & & \Theta_B \arrow[ll, "F_\Theta"'] \arrow[d, dash] & \\
\Gamma_A\text{-pres.} & \mathcal{M}_A \arrow[rr, Rightarrow, "F_\mathcal{M}"] \arrow[dr] & & \mathcal{M}_B \arrow[dl] & \Gamma_B\text{-pres} \\
& & \mathrm{Set} & &
\end{tikzcd}$$

Universe $\mathcal{U}$ of states & transitions

- state $A = \langle \Sigma_A, \Gamma_A, \mathcal{M}_A \rangle$ $\begin{cases} \bullet\ \Theta_A = \langle \Sigma_A, \Gamma_A \rangle & \leftsquigarrow \text{belief} \\ \bullet\ \mathcal{M}_A & \leftsquigarrow \text{reference} \end{cases}$

- state transition $\leftsquigarrow$ belief update

UPSHOT

- state space of state spaces:

$$\mathfrak{S} = \left\langle \odot_{\mathfrak{s}} , \mathcal{M}_{\mathfrak{S}} \right\rangle$$

sketch of sketches

1

UPSHOT

$$\begin{array}{c} A \\ \rotatebox[origin=c]{-90}{$\in$} \\ \mathcal{U} \end{array} \longleftrightarrow \ulcorner A \urcorner \in \mathcal{U}(I, S)$$

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UNIVERSAL TRANSITIONS

- state space of state transitions:

$$\mathbb{L} = \langle \Theta_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}} \rangle$$

sketch of sketch morphisms

1

UNIVERSAL TRANSITIONS

$$f \in \mathcal{U}(A,B) \longleftarrow \mathcal{U}^\bullet(I,\mathbb{L}) \ni \ulcorner f \urcorner$$

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "{\langle \ulcorner f \urcorner, id \rangle}"'] & & B \\
& \mathbb{L} \times A \arrow[ur, dashed, "\{\}"'] &
\end{tikzcd}$$

$$f = \{\ulcorner f \urcorner\}$$