An Evidence Logic Perspective on Schotch-Jennings Forcing

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Definition 1. A structure $\mathfrak{F} = \langle W, \mathcal{E} \rangle$ is an evidence frame iff:

1. $W \neq \varnothing$, and
2. $\mathcal{E} : W \to \mathcal{P}(\mathcal{P}(W))$ such that for all $x \in W$
   (a) $\varnothing \notin \mathcal{E}(x)$, and
   (b) $\mathcal{E}(x) \neq \varnothing$

An evidence **model** is a structure $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ where $\mathfrak{F}$ is an evidence frame and $V : \mathbf{At} \to \mathcal{P}(W)$.

These structures provide the semantics for an operator $E\varphi$ which says there is a piece of evidence $X \in \mathcal{E}(x)$ such that $X \subseteq [\![\varphi]\!]$.

Definition 2. A cover of Γ is a *collection* of consistent sets of sentences Π such that for each $\gamma \in \Gamma$, there is $\pi \in \Pi$ such that $\pi \vdash \gamma$. (Notation: $\mathbb{C}(\Gamma) = \{ \alpha : \Gamma \vdash \alpha \}$.) Alternatively, Π is a cover of Γ when $\Gamma \subseteq \bigcup_{\pi \in \Pi} \mathbb{C}(\pi)$ and each π is consistent. Partitions of Γ into consistent sets are a special case and are referred to as 'partition covers'. The size of Π is referred to as the width of the cover.

Definition 3. The *level* of Γ, $\ell(\Gamma)$ is determined by the minimum width a set of sets must have in order to be a cover of Γ, but if there is no such minimum, its level is ∞. Thus:

$$\ell(\Gamma) = \begin{cases} 0 & \Gamma \subseteq \mathbb{C}(\varnothing) \\ \min \{ |\Pi| : \Pi \text{ is a cover of } \Gamma \} & \text{if it exists } \& \ \Gamma \nsubseteq \mathbb{C}(\varnothing) \\ \infty & \text{otherwise} \end{cases}$$

Cf Jennings et al. (2009).

Definition 4. Γ forces α, $\Gamma \Vdash \alpha$, iff for each cover Π of Γ of width $\ell(\Gamma)$, there is $\pi \in \Pi$ such that $\pi \vdash \alpha$.

Definition 5. The logic K_n:

CL All theorems of classical propositional logic.

N $\vdash_{K_n} \langle E \rangle \top$

$K_n^{\Diamond}$ $\vdash_{K_n} (\langle E \rangle p_1 \wedge \ldots \wedge \langle E \rangle p_{n+1}) \to \langle E \rangle \bigvee_{1 \leq i < j \leq n+1} (p_i \wedge p_j)$

With rules

M $\dfrac{\vdash_{K_n} p \to q}{\vdash_{K_n} \langle E \rangle p \to \langle E \rangle q}$

MP Modus Ponens, and

US Uniform Substitution.

Proposition 1. $\Gamma \Vdash \alpha$ *iff* $\langle E \rangle [\Gamma] \vdash_{K_n} \langle E \rangle \alpha$ *where* $\langle E \rangle [\Gamma] = \{ \langle E \rangle \gamma : \gamma \in \Gamma \}$ *Apostoli and Brown (1995).*

1 The Logic U

We start with the language $\mathcal{L}_{\mathbf{U}}$. It is defined by the following BNF:

$$\varphi := \bot \mid p \mid \neg\varphi \mid F\varphi \mid E\varphi \mid \Box\varphi \mid \varphi \to \varphi \mid U(\underbrace{\varphi, \ldots, \varphi}_{n-\text{times}}; \varphi) \ n \in Z^{+}$$

Where $p \in \mathbf{At}$ the set of atoms. The operators $\Diamond$, $\langle F \rangle$, and $\langle E \rangle$ are defined via their duals $\neg \blacksquare \neg\varphi$ for $\blacksquare \in \{ \Box, F, E \}$. Next we have a frame and then a model:

Definition 6. A frame $\mathfrak{F} = \langle W, \mathcal{E}, R_F \rangle$ for the logic **U** is an evidence frame $\langle W, \mathcal{E} \rangle$ along with a relation R_F on W. The frame is **augmented** when there is an equivalence relation $R_\Box \subseteq W \times W$ as well. An evidence **model** for **U** is a structure $\mathcal{M} = \langle \mathfrak{F}, R_F, V \rangle$ where $\mathfrak{F}$ is an evidence frame and $V : \mathbf{At} \to \mathcal{P}(W)$.

Let $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ be an evidence model for **U**. The semantics for the logic **U** is:

- $\mathcal{M}, x \models p$ iff $x \in V(p)$ for all $p \in \mathbf{At}$
- Boolean cases as usual,
- $\mathcal{M}, x \models E\varphi$ iff there is $X \in \mathcal{E}(x)$ such that $X \subseteq [\![\varphi]\!]$,
- $\mathcal{M}, x \models \Box\varphi$ iff $[\![\varphi]\!] = W$,
- $\mathcal{M}, x \models F\varphi$ iff $R_F(x) \subseteq [\![\varphi]\!]$,
- $\mathcal{M}, x \models U(\varphi_1, \ldots, \varphi_n; \psi)$ iff for all $X \in \mathcal{E}(x)$, $X \subseteq [\![\psi]\!]$ only if for some $i \leq n$, $[\![\varphi_i]\!] \subseteq X$

This semantics gives rise to a semantic consequence relation $\models_{\mathbf{U}}$, defined in the usual way. This system is complete with respect to the following axioms, which will give rise to the syntactic system $\vdash_{\mathbf{U}}$. In the following axioms $\overline{\varphi}$ refers to a tuple of formulas $\varphi_1, \ldots, \varphi_n$ as before, but in cases where it is not the only argument on the left of the ';' in a U operator it can be empty. $n!$ refers to all permutations of $\{1, 2, \ldots, n\}$ and σ will be a specific permutation in $n!$ where $\sigma(k)$ is the number that k is permuted to by the permutation σ. Let p, q, r, s, p_i be in **At**.

CL All theorems of classical propositional logic.

S5 The axioms of S5 for $\Box$.

KF $(Fp \wedge Fq) \longleftrightarrow F(p \wedge q)$

$\Box$F $\Box p \to Fp$

D $\neg E\bot$

N $E\top$

E$\Box$ $\Box(p \to q) \to (Ep \to Eq)$

MergeE $(Ep \wedge \Box q) \to E(p \wedge q)$

U$\bot$ $U(\bot; q)$

U! $U(p_1, \ldots, p_n; q) \to (\bigwedge_{\sigma \in n!} U(p_{\sigma(1)}, \ldots, p_{\sigma(n)}; q))$

UE $\neg U(\overline{p}; q) \to Eq$

U+ $U(\overline{p}; q) \to U(\overline{p}, r; q)$

U- $U(\overline{p}, r, r; q) \to U(\overline{p}, r; q)$

UV $(U(\overline{p}; q) \wedge Eq) \to \bigvee_{i=1}^{n} \Box(p_i \to q)$

U$\Box$R $\Box(q \to r) \to (U(\overline{p}; r) \to U(\overline{p}; q))$

U$\Box$L $\Box(q \to r) \to (U((\overline{p}/r)_i; s) \to U((\overline{p}/q)_i; s))$

With rules

US Uniform Substitution,

MP Modus Ponens,

Nec $\vdash \varphi$ only if $\vdash \Box\varphi$

UInf

$$\frac{\vdash \theta \to (\Box(p \to \psi) \to (\bigwedge_{j=1}^{n} \Diamond(\varphi_j \wedge \neg p) \to \neg Ep))}{\vdash \theta \to U(\varphi_1, \dots, \varphi_n; \psi)} \; p \text{ foreign to } \varphi_1, \dots, \varphi_n, \psi, \theta$$

The usual definitions for Hilbert-style proof theory are used: $\Gamma \vdash_{\mathbf{U}} \varphi$ iff there are $\gamma_1, \dots, \gamma_k \in \Gamma$ such that $\vdash_{\mathbf{U}} (\gamma_1 \wedge \dots \wedge \gamma_n) \to \varphi$. As will be shown in section 6:

Theorem 1. *The system* $\vdash_{\mathbf{U}}$ *is sound and complete with respect to* $\vDash_{\mathbf{U}}$.

Definition 7. Let $\mathcal{X}$ be a set of sets of possible worlds W. A *cover* of $\mathcal{X}$ is a set $\mathcal{Y} \subseteq \mathcal{P}(W) \setminus \{ \varnothing \}$ such that for each $X \in \mathcal{X}$, there is $Y \in \mathcal{Y}$ and $Y \subseteq X$. Again,

$$\ell(\mathcal{X}) = \begin{cases} 0 & \text{when } \mathcal{X} = \{ W \} \\ \min \{ |\Pi| : \Pi \text{ is a cover of } \mathcal{X} \} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

Definition 8. Let's call a model $\mathcal{M}$ **consistency comprehensive** for Γ when for all $X \subseteq \mathbf{At}(\Gamma)$, there is $x \in W$ such that for all $p \in \mathbf{At}(\Gamma)$, $\mathcal{M}, x \vDash p$ iff $p \in X$, where $\mathbf{At}(\Gamma)$ is the set of atoms mentioned in Γ.

Note: $cor(\mathcal{E}(x)) = \{ X \in \mathcal{E}(x) : \nexists Y \in \mathcal{E}(x), Y \subsetneq X \}$, i.e., the set of elements of $\mathcal{E}(x)$ for which there is no proper subset also in $\mathcal{E}(x)$. Now we define a relation $\mathrm{cov}_{\mathfrak{F}} \subseteq W \times W$ as follows:

Definition 9. Let $\mathfrak{F} = \langle W, \mathcal{E} \rangle, R)F$ be an frame for $\mathbf{U}$. For all $x, y \in W$, $\mathrm{cov}_{\mathfrak{F}}(x, y)$ holds iff

1. for all $X \in \mathcal{E}(x)$ there is $Y \in \mathcal{E}(y)$ such that $Y \subseteq X$,
2. for all $Y \in cor(\mathcal{E}(y))$ there is $X \in \mathcal{E}(x)$ such that $Y \subseteq X$, and
3. $|cor(\mathcal{E}(y))| = \ell(\mathcal{E}(x))$.

2 The Logic F

The Logic $\mathbf{F}$, $\vDash_{\mathbf{F}}$ is characterized by the class of models such that when $\mathcal{E}(w)$ is of finite level and $R_F(w, y)$, then $\mathrm{cov}_{\mathfrak{F}}(w, y)$. Using the following abbreviations:

$$\mathrm{cov}(p_1, \dots, p_n) := \bigwedge_{i=1}^{n} \Diamond p_i \wedge U(p_1, \dots, p_n; \top)$$

$$\mathrm{core}(p_1, \dots, p_n) := \bigwedge_{i=1}^{n} (Ep_i \wedge U(p_i; p_i))$$

$$\mathrm{totalcore}(p_1, \dots, p_n) := \bigwedge_{i=1}^{n} (Ep_i \wedge U(p_i; p_i)) \wedge U(p_1, \dots, p_n; \top).$$

We can add the following (infinite and recursive) collection of axioms to the logic $\mathbf{U}$ and pick out the relevant collection of models:

EF $Ep \to FEp$

Cor $\mathrm{totalcore}(p_1, \dots, p_n) \to (\langle F \rangle \mathrm{core}(q) \to \bigvee_{i=1}^{n} \Box(q \to p_i))$ where $n > 0$

UpLev $\mathrm{cov}(q_1, \dots, q_k) \to (\langle F \rangle \mathrm{totalcore}(p_1, \dots, p_n) \to U(p_1, \dots, p_n; \top))$ where $n > 0$

LowLev $\mathrm{cov}(r_1, \dots, r_n) \to (\langle F \rangle \mathrm{core}(p_1, \dots, p_k) \to (U(q_1, \dots, q_m; \top) \to \bigvee_{i=1}^{m} \neg \Diamond q_i))$ where $m < k$ and $n > 0$

Theorem 2. *Suppose* $\Gamma = \{ \gamma_1, \dots, \gamma_m \}$ *and* φ *are purely Boolean.*

$$\Gamma \Vdash \varphi \iff \vDash_{\mathbf{F}} [(E\gamma_1 \wedge \dots \wedge E\gamma_m) \wedge U(\gamma_1, \dots, \gamma_m; \top) \wedge \Diamond \mathbf{At}(\Gamma)] \to FE\varphi$$

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