An Evidence Logic Approach to Schotch-Jennings Forcing

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Dedicated to Kindly Old Professor Schotch

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- Examples can be found in van Benthem et al. (2014) and Baltag et al. (2016)

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- We allow certain inferences from inconsistent sets of premises as long as they are inferrable from certain consistent subsets of the premises
- and that they don't make the set of premises **more** inconsistent.
- This isn't inferring things from maximally consistent subsets.

What do you mean by 'more inconsistent'?

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- Can we find a way to formalize these distinctions?

Logical Cover

Definition

A tuple of sets $\mathfrak{F} = \langle \Delta_0, \ldots, \Delta_n \rangle$ *is a logical cover of the set* Γ, *indicated by* $\mathrm{COV}(\mathfrak{F}, \Gamma)$, *provided:*

- *for all* $\Delta_i \in \mathfrak{F}$ *such that* Δ_i *is consistent* ($\mathrm{CON}(\Delta_i)$) *and*
- *for all* $\alpha \in \Gamma$ *there is* $\Delta_i \in \mathfrak{F}$ *such that* $\Delta_i \vdash \alpha$

We refer to each Δ_i as a cell, and n is the 'width' of $\mathfrak{F}$ denoted by $w(\mathfrak{F})$.

Level Function

Definition

The level of the set Γ of formulas of the underlying language, indicated by $\ell(\Gamma)$ is defined:

$$\ell(\Gamma) = \begin{cases} \min\limits_{w(\mathfrak{F})} \left[\mathrm{COV}(\mathfrak{F}, \Gamma) \right] & \textit{if this limit exists} \\ \infty & \textit{otherwise} \end{cases}$$

So, the level is the width of the narrowest cover.

What are the levels?

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$\Gamma \Vdash \alpha$ *if and only if, for every logical cover* $\mathfrak{F}$ *of* Γ *which has* $\ell(\Gamma)$ *cells, i.e. of width* $\ell(\Gamma)$*, there is at least one cell* $\Delta \in \mathfrak{F}$ *such that* $\Delta \vdash \alpha$

Peter Schotch Ray Jennings

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Forcing came from the study of non-normal modal logics.

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- The generalized $[\mathbf{K_n}]$ principle:

$$[\mathbf{K_n}] \vdash \bigwedge_{0 \leq i \leq n} \Box P_i \to \Box \bigvee_{0 \leq i < j \leq n} (P_i \wedge P_j)$$

Where did forcing come from?

Figure: Barbra Partee

Partee noticed that incomplete aggregation $[\mathbf{K_2}] \vdash (\Box A \wedge \Box B \wedge \Box C) \rightarrow \Box((A \wedge B) \vee (A \wedge C) \vee (B \wedge C))$ allowed a form of non-trivial reasoning from inconsistent sets, since,

$$(\Box A \wedge \Box \neg A) \nvdash_{K_2} \Box(A \wedge \neg A)$$

The General Connection Between Forcing and K_n

If the level of a set Γ is n, then

$$\Gamma \Vdash \alpha \text{ iff } \Box[\Gamma] \vdash_{K_n} \Box\alpha$$

where $\Box[\Gamma] = \{\Box\gamma : \gamma \in \Gamma\}$.

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This was first proved in Apostoli and Brown (1995)

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- It is known as fixed level forcing
- Naturally: can we have a modal logic (based on neighbourhood semantics) where we don't have to do that?

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3. Find a translation τ from classical propositional logic into $\mathcal{L}$ such that for any $\gamma_1, \ldots, \gamma_n$ and α from propositional logic,

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4. Ideally, find a complete axiomatization for $\vDash_L$

The Intuition

- Ideally, we wish to find two translations: τ, τ' such that

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- We need a semantic notions of cover and level, and
- the semantics has to "know" the appropriate width of a cover.

Semantic Notions of Cover and Level

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A cover of $\mathcal{X} \subseteq \mathcal{P}(W)$ *is a set* $\mathcal{Y} \subseteq \mathcal{P}(W) \setminus \{\varnothing\}$ *such that for each* $X \in \mathcal{X}$*, there is* $Y \in \mathcal{Y}$ *for which* $Y \subseteq X$*.*

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Definition

$$\ell(\mathcal{X}) = \begin{cases} 0 & \textit{when } \mathcal{X} = \{W\} \\ \min\{\,|\Pi| : \Pi \textit{ is a cover of } \mathcal{X}\,\} & \textit{if it exists} \\ \infty & \textit{otherwise} \end{cases}$$

The language $\mathcal{L}_{\mathbf{U}}$

The language $\mathcal{L}_{\mathbf{U}}$ is given by the follow BNF:

$$\varphi := \bot \mid p \mid \neg\varphi \mid F\varphi \mid E\varphi \mid \Box\varphi \mid \varphi \to \varphi \mid U(\underbrace{\varphi, \ldots, \varphi}_{n\text{–times}}; \varphi) \;\; n \in Z^{+}$$

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- U is the odd one out.

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- The φ_is are a **cover** of the evidence

Models for **U**

Definition

A structure $\mathfrak{F} = \langle W, \mathcal{E}, R_F \rangle$ *is an* **evidence frame** *iff:*

1. $W \neq \varnothing$, *and*
2. $\mathcal{E} : W \to \mathcal{P}(\mathcal{P}(W))$ *such that for all* $x \in W$
    1. $\varnothing \notin \mathcal{E}(x)$, *and*
    2. $\mathcal{E}(x) \neq \varnothing$
3. R_F *is a relation on* W
4. *The frame is* **augmented** *when there is an equivalence relation* $R_\Box \subseteq W \times W$ *added to the frame.*

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It is an evidence **model** *when we add a truth assignment* $V : \mathbf{At} \to \mathcal{P}(W)$.

Semantics for **U**

Let $\mathcal{M} = \langle W, \mathcal{E}, R_F, V \rangle$ be an evidence model and $x \in W$.

- $\mathcal{M}, x \vDash p$ iff $x \in V(p)$ for all $p \in \mathbf{At}$
- Boolean cases as usual,
- $\mathcal{M}, x \vDash E\varphi$ iff there is $X \in \mathcal{E}(x)$ such that $X \subseteq [\![\varphi]\!]$,
- $\mathcal{M}, x \vDash \Box\varphi$ iff $[\![\varphi]\!] = W$,
- $\mathcal{M}, x \vDash F\varphi$ iff $R_F(x) \subseteq [\![\varphi]\!]$,
- $\mathcal{M}, x \vDash U(\varphi_1, \dots, \varphi_n; \psi)$ iff for all $X \in \mathcal{E}(x)$, $X \subseteq [\![\psi]\!]$ only if for some $i \leq n$, $[\![\varphi_i]\!] \subseteq X$
- The Logic **U**, $\vDash_{\mathbf{U}}$, is that determined by this semantics for the class of evidence models
- The logic to achieve our goal requires that we impose more structure on the relation R_F

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- Note: $cor(\mathcal{E}(x)) = \{ X \in \mathcal{E}(x) : \nexists\ Y \in \mathcal{E}(x), Y \subsetneq X \}$, i.e., the set of elements of $\mathcal{E}(x)$ for which there is no proper subset also in $\mathcal{E}(x)$.

Definition

Let $\mathfrak{F} = \langle W, \mathcal{E}, R_F \rangle$ be an evidence frame. For all $x, y \in W$, $\mathrm{COV}_{\mathfrak{F}}(x, y)$ holds iff

1. *for all $X \in \mathcal{E}(x)$ there is $Y \in \mathcal{E}(y)$ such that $Y \subseteq X$,*
2. *for all $Y \in cor(\mathcal{E}(y))$ there is $X \in \mathcal{E}(x)$ such that $Y \subseteq X$, and*
3. $|cor(\mathcal{E}(y))| = \ell(\mathcal{E}(x))$.

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Theorem

Suppose $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ *and* α *are purely Boolean.*

$$\Gamma \Vdash \alpha \iff \vDash_{\mathbf{F}} [(E\gamma_1 \wedge \ldots \wedge E\gamma_m) \wedge U(\gamma_1, \ldots, \gamma_m; \top) \wedge \Diamond \mathbf{At}(\Gamma)] \to FE\alpha$$

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