Bisimulations between Verbrugge models and Veltman models

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Interpretability logic

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- **syntax**: basic modal logic + binary modal operator $\rhd$

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Satisfaction: $w \Vdash A \rhd B$ if for all u s.t. wRu and $u \Vdash A$ there is v s.t. $uS_w v$ and $v \Vdash B$

Generalized semantics

Verbrugge models:

- $W \neq \emptyset$
- $R \subseteq W \times W$ transitive and reverse well-founded
- for each $w \in W$, $S_w \subseteq R[w] \times \mathcal{P}(R[w])$
  - if wRu then $uS_w\{u\}$
  - if uS_wV and vS_wZ_v for all $v \in V$ then $uS_w(\cup Z_v)$
  - if $wRuRv$ then $uS_w\{v\}$

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- the converse does not hold generally, but it holds in case of image-finite Veltman models (an analogue of Hennessy-Milner theorem)

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- Hennessy-Milner analogue does not hold

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Now, as desired:

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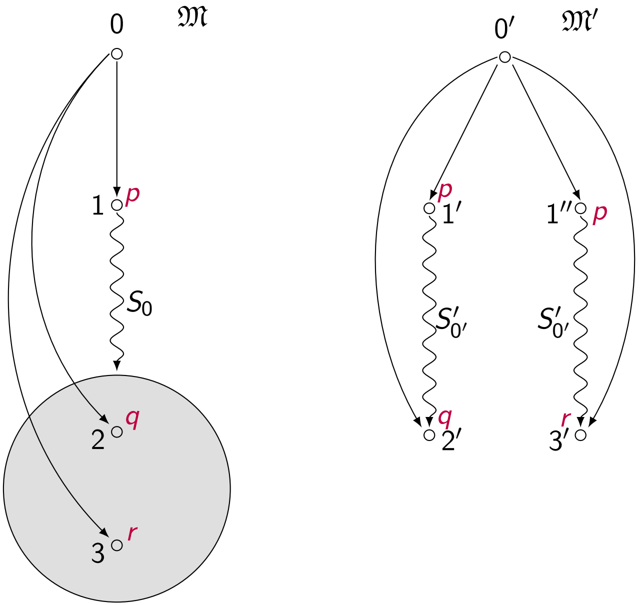
- Hennessy-Milner analogue holds

Example



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- everything else is a number of technicalities to ensure the obtained model is indeed a Veltman model and that the natural identification between worlds in Verbrugge and Veltman model is indeed a bisimilation