Structural Completeness and Superintuitionistic Inquisitive Logics

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Structural completeness

- A logic L is structurally complete iff every L-admissible rule is L-derivable.
- Classical logic is structurally complete but intuitionistic logic is not.
- We will study structural completeness in the context of logics that are not necessarily closed under uniform substitution.

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The main steps

- We introduce a generalized notion of superintuitionistic logic.
- We define intuitionistic inquisitive logic as intuitionistic logic extended with a schema Split.
- We show a strong connection between structural completeness and the schema Split.
- We explore some consequences of this connection.

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Language

$$\varphi ::= p \mid \bot \mid \varphi \rightarrow \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$$

$$\neg\varphi =_{def} \varphi \rightarrow \bot$$

α, β, γ range over $\vee$-free formulas

φ, ψ, χ range over arbitrary formulas

Harrop formulas

In Harrop formulas disjunction can occur only in the antecedent of an implication:

$$(\ldots((\ldots \vee \ldots) \to \ldots)\ldots)$$

Restricted notions of substitution

- A substitution is regarded as a function $s : At \to Fle$.
- This function is extended to $s : Fle \to Fle$.

$$s(\bot) = \bot \text{ and } s(\varphi \circ \psi) = s(\varphi) \circ s(\psi) \text{ for each } \circ \in \{\to, \wedge, \vee\}$$

Definition

- An *H*-substitution is a substitution that assigns to each atomic formula a Harrop formula.
- A *D*-substitution is a substitution that assigns to each atomic formula a $\vee$-free formula.

Generalized superintuitionistic logics

Definition

A gsi-logic (generalized superintuitionistic logic) is any set of formulas L such that

(a) $\mathrm{IL} \subseteq L \subseteq \mathrm{CL}$;

(b) L is closed under modus ponens
(if $\varphi, \varphi \to \psi \in L$ then $\psi \in L$);

(c) L is closed under every D-substitution
(if $\varphi \in L$ then $s(\varphi) \in L$, for each D-substitution s).

A gsi-logic is standard if it is closed under every substitution.

Notation

$\vdash_L \varphi$	$\ldots$	$\varphi \in L$
$\varphi \vdash_L \psi$	$\ldots$	$\vdash_L \varphi \to \psi$
$\varphi \equiv_L \psi$	$\ldots$	$\varphi \vdash_L \psi$ and $\psi \vdash_L \varphi$

Inquisitive logics

The schema Split

$$(\alpha \to (\psi \vee \chi)) \to ((\alpha \to \psi) \vee (\alpha \to \chi))$$

where α ranges over $\vee$-free formulas

Definition

We say that a gsi-logic L is inquisitive if it

(a) contains all instances of Split

(b) has the disjunction property
($\varphi \vee \psi \in L$ implies $\varphi \in L$ or $\psi \in L$)

Intuitionistic and classical inquisitive logic

- Classical inquisitive logic InqCL = IL + *Split* + *RDN* ($\neg\neg\alpha \to \alpha$, for $\vee$-free α)

Ciardelli, I., Groenendijk, J., Roelofsen, F.: Inquisitive Semantics. Oxford University Press (2019)

- Intuitionistic inquisitive logic InqIL = IL + *Split*

Punčochář, V.: A generalization of inquisitive semantics. *Journal of Philosophical Logic* 45, 399–428 (2016)

Disjunctive normal form

Theorem

For every φ there are $\vee$-free formulas $\alpha_1, \ldots, \alpha_n$ such that

$$\varphi \equiv_{\mathsf{InqIL}} \alpha_1 \vee \ldots \vee \alpha_n.$$

Corollary

Every logic that includes Split is closed under all H-substitutions.

No inquisitive logic is standard

For every inquisitive L:

$$(r \rightarrow (p \vee q)) \rightarrow ((r \rightarrow p) \vee (r \rightarrow q)) \in L$$

For every inquisitive L:

$$((p \vee q) \rightarrow (p \vee q)) \rightarrow (((p \vee q) \rightarrow p) \vee ((p \vee q) \rightarrow q)) \notin L$$

Structural completeness for classical inquisitive logic

Iemhoff, R., Yang, F.: Structural completeness in propositional logics of dependence. *Archive for Mathematical Logic* **55**, 955–975 (2016)

The usual notion of structural completeness

Every admissible rule φ/ψ is derivable in L.

(a) admissibility: for any substitution s, if $\vdash_L s(\varphi)$ then $\vdash_L s(\psi)$,

(b) derivability: $\varphi \vdash_L \psi$.

Four notions of structural completeness

$sub(L)$ is the set of substitutions under which L is closed

Definition

L is *SF*-complete if it holds:

$\varphi \vdash_L \psi$ iff for any substitution s, if $\vdash_L s(\varphi)$ then $\vdash_L s(\psi)$.

L is *SG*-complete if it holds:

$\varphi \vdash_L \psi$ iff for any $s \in sub(L)$, if $\vdash_L s(\varphi)$ then $\vdash_L s(\psi)$.

L is *SH*-complete if it holds:

$\varphi \vdash_L \psi$ iff for any H-substitution s, if $\vdash_L s(\varphi)$ then $\vdash_L s(\psi)$.

L is *SD*-complete if it holds:

$\varphi \vdash_L \psi$ iff for any D-substitution s, if $\vdash_L s(\varphi)$ then $\vdash_L s(\psi)$.

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Prucnal trick

Prucnal, T.: On the structural completeness of some pure implicational propositional calculi. *Studia Logica* 32, 45–50 (1973)

Minari, P., Wroński, A. (1988) The property (HD) in intermediate logics. A partial solution of a problem of H. Ono. *Reports on Mathematical Logic* 22, 21–25.

$$s^v_\alpha(p) = \begin{cases} \alpha \to p & \text{if } v(p) = 1 \\ \neg\neg\alpha \wedge (\alpha \to p) & \text{otherwise} \end{cases}$$

Theorem (Minari, Wroński)

For every standard gsi-logic L, ever Harrop formula α, every φ, ψ:

$$\textit{if} \vdash_L \alpha \to (\varphi \vee \psi) \textit{ then} \vdash_L (\alpha \to \varphi) \vee (\alpha \to \psi).$$

The main result

Theorem

For every gsi-logic L the following claims are equivalent:

(a) *L is SH-complete,*

(b) *L is SD-complete,*

(c) *Split is valid in L.*

Relations among the notions of structural completeness

$$\begin{array}{ccc} SF & \begin{array}{c}\Longrightarrow\\ \not\Longleftarrow\end{array} & SG \\ \not\Downarrow\,\not\Uparrow & & \Uparrow\,\not\Downarrow \\ SD & \Longleftrightarrow & SH \end{array}$$

- ML (logic of finite problems) is a counterexample to $SF \Longrightarrow SD(H)$ and $SG \Longrightarrow SD(H)$,
- any inquisitive gsi-logic is a counterexample to $SD(H) \Longrightarrow SF$ and $SG \Longrightarrow SF$.

(ML is the logic of Kripke frames that have the structure of finite Boolean algebras without the top element.)

Some consequences

Corollary

Every $SD(H)$-complete gsi-logic is hereditarily $SD(H)$-complete.

Corollary

InqIL *is hereditarily SG-complete.*

Definition

A gsi-logic is *optimal* if it is SG-complete and has the disjunction property.

Corollary

Every inquisitive gsi-logic is optimal.

A property of inquisitive gsi-logics

Proposition

Let L be an inquisitive gsi-logic. Let α *be a consistent* $\vee$*-free formula (*$\nvdash_L \neg\alpha$ *and* φ, ψ *arbitrary formulas. Then*

(a) $\alpha \nvdash_L \bot$,

(b) $\alpha \vdash_L \varphi \to \psi$ *iff for all* $\vee$*-free* $\beta \vdash_L \alpha$, *if* $\beta \vdash_L \varphi$ *then* $\beta \vdash_L \psi$,

(c) $\alpha \vdash_L \varphi \wedge \psi$ *iff* $\alpha \vdash_L \varphi$ *and* $\alpha \vdash_L \psi$,

(d) $\alpha \vdash_L \varphi \vee \psi$ *iff* $\alpha \vdash_L \varphi$ *or* $\alpha \vdash_L \psi$.

A canonical model construction for inquisitive gsi-logics

$\mathcal{M}_L = \langle S_L, \leqslant_L, V_L \rangle$, where

- S_L is the set of $\vee$-free, consistent formulas,
- $\alpha \leqslant_L \beta$ iff $\beta \vdash_L \alpha$,
- $\alpha \in V_L(p)$ iff $\alpha \vdash_L p$.

Theorem

For each φ and each consistent $\vee$-free α,

$$\alpha \Vdash \varphi \text{ in } \mathcal{M}_L \text{ if and only if } \alpha \vdash_L \varphi.$$

As a consequence, $\varphi \in L$ if and only if φ is valid in $\mathcal{M}_L$.

A property of inquisitive gsi-logics

We write

- $s \succ \varphi$ iff $\vdash_L s(\varphi)$, for a fixed inquisitive gsi-logic,
- $s \leqslant t$ iff there is a D-substitution u such that $t = u \circ s$.

Proposition

Let s be a D-substitution and φ, ψ arbitrary formulas. Then

(a) $s \not\succ \bot$,

(b) $s \succ \varphi \to \psi$ *iff for any D-sub. $t \geqslant s$, if $t \succ \varphi$ then $t \succ \psi$,*

(c) $s \succ \varphi \wedge \psi$ *iff $s \succ \varphi$ and $s \succ \psi$,*

(d) $s \succ \varphi \vee \psi$ *iff $s \succ \varphi$ or $s \succ \psi$.*

A canonical model construction for inquisitive gsi-logics

$\mathcal{M}^L = \langle S^L, \leqslant^L, V^L \rangle$, where

- S^L is the set of all D-substitutions,
- $s \leqslant^L t$ iff there is a D-substitution u such that $t = u \circ s$,
- $s \in V^L(p)$ iff $s \succ p$.

Theorem

For each φ and each D-substitution s,

$$s \Vdash \varphi \text{ in } \mathcal{M}^L \text{ if and only if } s \succ \varphi.$$

As a consequence, $\varphi \in L$ if and only if φ is valid in $\mathcal{M}^L$.

Schematic fragments and schematic closures

For any gsi-logic L we can consider its schematic fragment $S(L)$ and schematic closure $C(L)$.

$$S(L) \subseteq L \subseteq C(L)$$

- $S(L)$ is the greatest standard gsi-logic included in L;
- $C(L)$ is the least standard gsi-logic extending L.

Schematic fragments of inquisitive gsi-logics

Theorem

Let L be an inquisitive gsi-logic. Then $S(L) = \mathsf{ML}$.

Grilletti, G.: Medvedev logic is the logic of finite distributive lattices without top element. *Advances in Modal Logic* (2022)

Gödel-Dummett logic

Usually, LC = IL $\oplus$ *PreLin*

$$(\varphi \to \psi) \vee (\psi \to \varphi).$$

Equivalently, LC = IL $\oplus$ *FullSplit*

$$(\chi \to (\varphi \vee \psi)) \to ((\chi \to \varphi) \vee (\chi \to \psi)).$$

Gödel-Dummett logic

Lemma

$\varphi \vee \psi \equiv_{\mathsf{LC}} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

Lemma

Every gsi-logic that includes LC *is standard.*

Theorem

LC *is hereditarily SF-complete over all gsi-logics.*

Schematic closures of inquisitive gsi-logics

The gsi-logics that include LC form a chain:

$$\mathsf{LC} = \mathsf{G}_\omega \subseteq \ldots \subseteq \mathsf{G}_5 \subseteq \mathsf{G}_4 \subseteq \mathsf{G}_3 \subseteq \mathsf{G}_2 = \mathsf{CL}.$$

Theorem

Let L be an inquisitive gsi-logic. Then

$$C(L) = \mathsf{LC} \oplus L^{df} = \mathsf{G}_n, \;\; \textit{for } n = \max\{m \mid L^{df} \subseteq \mathsf{G}_m^{df}\}.$$

Some questions for future research

- Are there any applications of our results to ML?
- Are there any other optimal gsi-logics besides ML and inquisitive gsi-logics?
- Is there any model-theoretic explanation of the equivalence between $\mathcal{M}_L$ and $\mathcal{M}^L$?
- Could our approach be adapted to substructural inquisitive logics?