# Maximally Multi-Focused Proofs for Skew Non-Commutative MILL

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# Permutative non-determinism

- In cut-free sequent calculi for various logical systems, there is lots of non-determinism in root-first proof search.

$$\dfrac{\dfrac{A, B, \Gamma \vdash C \quad \Delta \vdash D}{A, B, \Gamma, \Delta \vdash C \otimes D}\otimes\mathrm{R}}{A \otimes B, \Gamma, \Delta \vdash C \otimes D}\otimes\mathrm{L} \qquad \dfrac{\dfrac{A, B, \Gamma \vdash C}{A \otimes B, \Gamma \vdash C}\otimes\mathrm{L} \quad \Delta \vdash D}{A \otimes B, \Gamma, \Delta \vdash C \otimes D}\otimes\mathrm{R}$$

$$\dfrac{\Gamma \vdash A \quad \dfrac{B, \Delta \vdash C \quad \Omega \vdash C}{B, \Delta, \Omega \vdash C \otimes D}\otimes\mathrm{R}}{A \multimap B, \Gamma, \Delta, \Omega \vdash C \otimes D}\multimap\mathrm{L} \qquad \dfrac{\dfrac{\Gamma \vdash A \quad B, \Delta \vdash C}{A \multimap B, \Gamma, \Delta \vdash C}\multimap\mathrm{L} \quad \Omega \vdash D}{A \multimap B, \Gamma, \Delta, \Omega \vdash C \otimes D}\otimes\mathrm{R}$$

# Focusing

- Andreoli's focusing is an established approach for reducing it.
  Organize proof search in two phases:
  1. Eagerly apply invertible rules, like $\otimes$L.
  2. Pick a formula and apply non-invertible rules to it, like $\otimes$R, $\multimap$L.
- Multi-focusing: possibly focus on more than one formula in phase 2.

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- To eliminate all permutative non-determinism, usual solutions switch to a different formalism, e.g. proof nets.
- But permutative canonicity can be achieved in sequent calculus as well via maximally multi-focused proofs (Chaudhuri, Miller & Saurin'08).

# Content of the talk

The objective of the project:

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- Maximal multi-focusing for skew non-commutative multiplicative linear logic (`SkNMILL`).
- This logic is a semi-associative and semi-unital variant of Lambek calculus with one implication.
- Fully formalized in the Agda proof assistant.

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Why `SkNMILL`?

- In multi-focusing phase, at most two formulae can be under focus.
- Richer substructural logics are extensions.

# Sequent calculus for SkNMILL (Uustalu, V. & Wan'22)

- Formulae: $A, B ::= X \in At \mid I \mid A \otimes B \mid A \multimap B$
  $I, A \otimes B$ positive, $A \multimap B$ negative

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- Derivations constructed via inference rules:

$$\frac{}{X \mid \ \vdash X}\ \mathsf{ax} \qquad \frac{}{- \mid \ \vdash I}\ I\mathsf{R} \qquad \frac{S \mid \Gamma \vdash A \quad - \mid \Delta \vdash B}{S \mid \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R} \qquad \frac{S \mid \Gamma, A \vdash B}{S \mid \Gamma \vdash A \multimap B}\ \multimap\mathsf{R}$$

$$\frac{A \mid \Gamma \vdash C}{- \mid A, \Gamma \vdash C}\ \mathsf{pass} \qquad \frac{- \mid \Gamma \vdash C}{I \mid \Gamma \vdash C}\ I\mathsf{L} \qquad \frac{A \mid B, \Gamma \vdash C}{A \otimes B \mid \Gamma \vdash C}\ \otimes\mathsf{L} \qquad \frac{- \mid \Gamma \vdash A \quad B \mid \Delta \vdash C}{A \multimap B \mid \Gamma, \Delta \vdash C}\ \multimap\mathsf{L}$$

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- L-rules only apply to the formula in the stoup
- $\otimes$R sends the stoup formula (if present) to the first premise
- No structural rules of weakening, contraction, exchange
- New structural rule pass moving leftmost formula in context to the stoup, when latter is empty

# Equivalence of derivations

- $f \doteq g$ iff they are equal modulo permutative conversions
- Congruence $\doteq$ generated by equations:

$$
\begin{aligned}
\otimes\mathsf{R}\ (I\mathsf{L}\ f, g) &\doteq I\mathsf{L}\ (\otimes\mathsf{R}\ (f, g)) \\
\otimes\mathsf{R}\ (\otimes\mathsf{L}\ f, g) &\doteq \otimes\mathsf{L}\ (\otimes\mathsf{R}\ (f, g)) \\
\mathsf{pass}\ (\multimap\mathsf{R}\ f) &\doteq \multimap\mathsf{R}\ (\mathsf{pass}\ f) \\
I\mathsf{L}\ (\multimap\mathsf{R}\ f) &\doteq \multimap\mathsf{R}\ (I\mathsf{L}\ f) \\
\otimes\mathsf{L}\ (\multimap\mathsf{R}\ f) &\doteq \multimap\mathsf{R}\ (\otimes\mathsf{L}\ f) \\
\multimap\mathsf{L}\ (f, \multimap\mathsf{R}\ g) &\doteq \multimap\mathsf{R}\ (\multimap\mathsf{L}\ (f, g)) \\
\otimes\mathsf{R}\ (\mathsf{pass}\ f, g) &\doteq \mathsf{pass}\ (\otimes\mathsf{R}\ (f, g)) \\
\otimes\mathsf{R}\ (\multimap\mathsf{L}\ (f, g), h) &\doteq \multimap\mathsf{L}\ (f, \otimes\mathsf{R}\ (g, h))
\end{aligned}
$$

# Semi-associativity, semi-unitality

- These sequents are derivable:

$$
\begin{aligned}
I \otimes A \mid\ &\vdash A \\
A \mid\ &\vdash A \otimes I \\
(A \otimes B) \otimes C \mid\ &\vdash A \otimes (B \otimes C)
\end{aligned}
$$

- These sequents are <u>not</u>:

$$
\begin{aligned}
X \mid\ &\nvdash I \otimes X \\
X \otimes I \mid\ &\nvdash X \\
X \otimes (Y \otimes Z) \mid\ &\nvdash (X \otimes Y) \otimes Z
\end{aligned}
$$

- Categorical semantics in <u>skew monoidal closed category</u> (Street'13).

# Multi-focused sequent calculus

Invertible phase $\Uparrow$

$$\frac{S \mid \Gamma, A \Uparrow B}{S \mid \Gamma \Uparrow A \multimap B} \multimap\mathsf{R} \qquad \frac{A \mid B, \Gamma \Uparrow Q}{A \otimes B \mid \Gamma \Uparrow Q} \otimes\mathsf{L} \qquad \frac{- \mid \Gamma \Uparrow Q}{I \mid \Gamma \Uparrow Q} I\mathsf{L} \qquad \frac{T \mid \Gamma \Downarrow Q}{T \mid \Gamma \Uparrow Q} \mathsf{foc}$$

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Focusing phase $\Downarrow$

$$\frac{T \mid \Gamma \Downarrow_{\mathsf{lf}} Q \quad \boxed{Q} \mid \Delta \Downarrow \boxed{A}_b}{T \mid \Gamma, \Delta \Downarrow \boxed{A}_b}\ \mathsf{foc}_{\mathsf{L}} \qquad \frac{\boxed{S}_b \mid \Gamma \Downarrow \boxed{T} \quad T \mid \Delta \Downarrow_{\mathsf{rf}} Q}{\boxed{S}_b \mid \Gamma, \Delta \Downarrow Q}\ \mathsf{foc}_{\mathsf{R}}$$

$$\frac{}{\boxed{X} \mid \ \Downarrow \boxed{X}}\ \mathsf{ax} \qquad \frac{S \mid \Gamma \Uparrow A \quad \mathsf{UT}(b, c, S, A)}{\boxed{S}_b \mid \Gamma \Downarrow \boxed{A}_c}\ \mathsf{unfoc}$$

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Left-focusing phase $\Downarrow_{\mathsf{lf}}$

$$\frac{A \mid \Gamma \Downarrow_{\mathsf{lf}} Q}{- \mid A, \Gamma \Downarrow_{\mathsf{lf}} Q}\ \mathsf{pass} \qquad \frac{- \mid \Gamma \Uparrow A \qquad B \mid \Delta \Downarrow_{\mathsf{lf}} Q}{A \multimap B \mid \Gamma, \Delta \Downarrow_{\mathsf{lf}} Q}\ {\multimap}\mathsf{L} \qquad \frac{}{Q \mid \quad \Downarrow_{\mathsf{lf}} Q}\ \mathsf{blur}_{\mathsf{L}}$$

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$$\frac{S \mid \Gamma, A \Uparrow B}{S \mid \Gamma \Uparrow A \multimap B}\ {\multimap}\mathsf{R} \qquad \frac{A \mid B, \Gamma \Uparrow Q}{A \otimes B \mid \Gamma \Uparrow Q}\ {\otimes}\mathsf{L} \qquad \frac{- \mid \Gamma \Uparrow Q}{I \mid \Gamma \Uparrow Q}\ I\mathsf{L} \qquad \frac{T \mid \Gamma \Downarrow Q}{T \mid \Gamma \Uparrow Q}\ \mathsf{foc}$$

Focusing phase $\Downarrow$

$$\frac{T \mid \Gamma \Downarrow_{\mathsf{lf}} Q \quad \boxed{Q} \mid \Delta \Downarrow \boxed{A}_b}{T \mid \Gamma, \Delta \Downarrow \boxed{A}_b}\ \mathsf{foc_L} \qquad \frac{\boxed{S}_b \mid \Gamma \Downarrow \boxed{T} \quad T \mid \Delta \Downarrow_{\mathsf{rf}} Q}{\boxed{S}_b \mid \Gamma, \Delta \Downarrow Q}\ \mathsf{foc_R}$$

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Right-focusing phase $\Downarrow_{\mathsf{rf}}$

$$\frac{}{- \mid \ \Downarrow_{\mathsf{rf}} I}\ I\mathsf{R} \qquad \frac{T \mid \Gamma \Downarrow_{\mathsf{rf}} A \quad - \mid \Delta \Uparrow B}{T \mid \Gamma, \Delta \Downarrow_{\mathsf{rf}} A \otimes B}\ {\otimes}\mathsf{R} \qquad \frac{}{M \mid \ \Downarrow_{\mathsf{rf}} M}\ \mathsf{blur_R}$$

# Effective multi-focusing

## Soundness and completeness

- An embedding function $\mathtt{emb} : S \mid \Gamma \Uparrow A \;\rightarrow\; S \mid \Gamma \vdash A$
- A focusing function $\mathtt{focus} : S \mid \Gamma \vdash A \;\rightarrow\; S \mid \Gamma \Uparrow A$

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- There exist $\circeq$-related proofs that are not identified by `focus`
- E.g. 4 distinct proofs of $X \multimap I \mid X, Y \Uparrow (Z \multimap Z) \otimes Y$

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- E.g. 4 distinct proofs of $X \multimap I \mid X, Y \Uparrow (Z \multimap Z) \otimes Y$

## Equational soundness and completeness

- Capture the remaining non-determinism in a congruence $\circeq_{\Uparrow}$ on proofs of $S \mid \Gamma \Uparrow A$

**Theorem**

*Maps* focus *and* emb *underlie an isomorphism:*

$$(S \mid \Gamma \vdash A)/\circeq \quad \cong \quad (S \mid \Gamma \Uparrow A)/\circeq_{\Uparrow}$$

# Right-focusing before left-focusing?

- Question: When does a right-focusing phase need to be performed strictly before a left-focusing one?
  Strictly = separated by (at least one) invertible phase $\Uparrow$.

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- Question: When does a right-focusing phase need to be performed strictly before a left-focusing one?
  Strictly = separated by (at least one) invertible phase $\Uparrow$.

- A proof of $X \multimap Y \mid Z \Uparrow (X \multimap Y) \otimes Z$ is obtained by focusing on the succedent, focusing on the stoup does not work.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{\vdots \qquad}{- \mid X \Uparrow X} \quad \dfrac{}{Y \mid \; \Downarrow_{\mathsf{lf}} Y}\,\mathsf{blur_L}
        }{X \multimap Y \mid X \Downarrow_{\mathsf{lf}} Y}\,\multimap\mathsf{L}
        \quad
        \dfrac{\vdots}{\boxed{Y} \mid \; \Downarrow Y}
      }{X \multimap Y \mid X \Downarrow Y}\,\mathsf{foc_L}
    }{X \multimap Y \mid X \Uparrow Y}\,\mathsf{foc}
  }{X \multimap Y \mid \; \Uparrow X \multimap Y}\,\multimap\mathsf{R}
}{X \multimap Y \mid \; \Downarrow \boxed{X \multimap Y}}\,\mathsf{unfoc}
$$

$$
\dfrac{
  \dfrac{
    X \multimap Y \mid \; \Downarrow \boxed{X \multimap Y}
    \quad
    \dfrac{
      \dfrac{}{X \multimap Y \mid \; \Downarrow_{\mathsf{rf}} X \multimap Y}\,\mathsf{blur_R}
      \quad
      \dfrac{\vdots}{- \mid Z \Uparrow Z}
    }{X \multimap Y \mid Z \Downarrow_{\mathsf{rf}} (X \multimap Y) \otimes Z}\,\otimes\mathsf{R}
  }{X \multimap Y \mid Z \Downarrow (X \multimap Y) \otimes Z}\,\mathsf{foc_R}
}{X \multimap Y \mid Z \Uparrow (X \multimap Y) \otimes Z}\,\mathsf{foc}
$$

# Introducing tags

- Answer: when the subsequent left-focusing phase meaningfully employs new formulae appearing in context during invertible phase.

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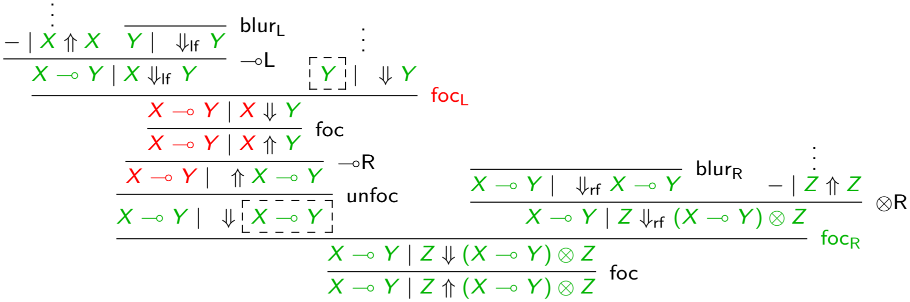
- Our solution:
  - Tag formulae in context: $A$ new, $A$ old
  - Tag stoup: $S$ no left-focusing in previous foc. phase, $S$ o/w
  - Tag succedent: $C$ no right-focusing in previous foc. phase, $C$ o/w



# Maximally multi-focused sequent calculus (some rules)

Invertible phase $\Uparrow_{\mathsf{m}}$

$$\frac{S \mid \Gamma, A \Uparrow_{\mathsf{m}} B}{S \mid \Gamma \Uparrow_{\mathsf{m}} A \multimap B} \multimap\mathsf{R} \qquad \frac{S \mid \Gamma, A \Uparrow_{\mathsf{m}} B}{S \mid \Gamma \Uparrow_{\mathsf{m}} A \multimap B} \multimap\mathsf{R} \qquad \ldots$$

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Focusing phase $\Downarrow_{\mathsf{m}}$

$$\frac{T \mid \Gamma \Downarrow_{\mathsf{lfm}} Q \quad \boxed{Q} \mid \Delta \Downarrow_{\mathsf{m}} A}{T \mid \Gamma, \Delta \Downarrow_{\mathsf{m}} A} \mathsf{foc_L} \qquad \frac{T \mid \Gamma \Downarrow_{\mathsf{lfm}} Q \quad \boxed{Q} \mid \Delta \Downarrow_{\mathsf{m}} A \quad \bullet \in \Gamma}{T \mid \Gamma, \Delta \Downarrow_{\mathsf{m}} A} \mathsf{foc_L}$$

$$\frac{S^{\neg b} \mid \Gamma \Uparrow_{\mathsf{m}} A^{\neg c} \quad \mathsf{UT}(b, c, S, A)}{\boxed{S}_b \mid \Gamma \Downarrow_{\mathsf{m}} \boxed{A}_c} \mathsf{unfoc} \qquad \dots$$

$(S^{\mathsf{true}} = S,\ S^{\mathsf{false}} = S,\ A^{\mathsf{true}} = A,\ A^{\mathsf{false}} = A)$

Left-focusing $\Downarrow_{\mathsf{lfm}}$ and right-focusing phase $\Downarrow_{\mathsf{rfm}}$ are as before.

# Effective maximization

Soundness and completeness

- An untagging function untag : $S \mid \Gamma \Uparrow_{\mathsf{m}} A \;\rightarrow\; S \mid \Gamma \Uparrow A$
- A maximization function max : $S \mid \Gamma \Uparrow A \;\rightarrow\; S \mid \Gamma \Uparrow_{\mathsf{m}} A$

**Theorem**

*Maps* max *and* untag *underlie an isomorphism:*

$$(S \mid \Gamma \Uparrow A)/\doteq_{\Uparrow} \quad\cong\quad S \mid \Gamma \Uparrow_{\mathsf{m}} A$$

**Corollary**

$$(S \mid \Gamma \vdash A)/\doteq \quad\cong\quad (S \mid \Gamma \Uparrow A)/\doteq_{\Uparrow} \quad\cong\quad S \mid \Gamma \Uparrow_{\mathsf{m}} A$$

# Conclusions

- Results fully formalized in Agda:

  `https://github.com/niccoloveltri/multifocus-sknmill`
- First steps towards formalization of maximal multi-focusing for richer substructural logics.

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- We have some ideas on how to extend the technique to other logics:
  - Full associativity and unitality for $(\otimes, I)$, exchange rule
  - Current work with Wan on adding additives (at LSFA'23)

- With Uustalu and Wan, we studied another class of normal forms for `SkNMILL`, where left-foc. rules are prioritized over right-foc. ones. Study the relationship between the two classes of normal forms.