Relevant Reasoning and Implicit Belief

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Plan of work

0. Introduction
   What do we want?

1. Setting up the model
   What do we need?

2. Properties of our model
   What we can say about explicit and implicit beliefs.

3. Technical Results
   What we have.

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   2. Our model.

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(BM6) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

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($\Box_{(IC)}$.M) $\dfrac{\varphi \to \psi}{\Box_{(IC)}\varphi \to \Box_{(IC)}\psi}$

(BM8) $\neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi)$

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(BM13) $\dfrac{\varphi \to \chi \quad \psi \to \xi}{(\varphi \to \psi) \to (\chi \to \xi)}$

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Relevant logic preliminaries: Semantics

Definition (L-model)

- $\boxed{\mathfrak{M} = (S, L, \leq, {}^*, R, Q, Q_I, Q_L, V)}$ such that:
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  2. $(\mathfrak{M}, s) \models \neg\varphi \;\Leftrightarrow\; (\mathfrak{M}, s^*) \not\models \varphi$;
  3. $(\mathfrak{M}, s) \models \varphi \to \psi \;\Leftrightarrow\; \forall t, u \in S(Rstu, (\mathfrak{M}, t) \models \varphi \Rightarrow (\mathfrak{M}, u) \models \psi)$;
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- Validity: $\mathfrak{M} \models \varphi \Leftrightarrow L \subseteq [\![\varphi]\!]_{\mathfrak{M}}$.

Characterisation

Theorem 1

$\vdash_{\mathsf{L}} \varphi \iff$ *for all* L*-models* $\mathfrak{M}$, $\mathfrak{M} \models \varphi$.

Axiom/rule		**Frame condition**
(L1)	$\varphi \leftrightarrow \neg\neg\varphi$	$s^{**} = s$
(L2)	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	$Rstu \Rightarrow Rsu^{*}t^{*}$
(L3)	$((\varphi \to \psi) \wedge (\psi \to \chi)) \to (\varphi \to \chi)$	$Rstu \Rightarrow Rs(st)u$
(L4)	$\varphi \vee \neg\varphi$	$s \in L \Rightarrow s^{*} \leq s$
(L5)	$(\varphi \to \neg\varphi) \to \neg\varphi$	$Rss^{*}s$
(L6)	$(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$	$Rstuv \Rightarrow Rs(tu)v$
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(L10)	$\varphi \to (\varphi \to \varphi)$	$Rstu \Rightarrow (s \leq u \vee t \leq u)$
(L11)	$\varphi \Rrightarrow (\varphi \to \psi) \to \psi$	$\exists x(x \in L \;\; \& \;\; Rsxs)$
(L12)	$\varphi \Rrightarrow \Box\varphi$	$(x \in L \;\; \& \;\; Qxs) \Rightarrow s \in L$
(L13)	$\Box_{(I)}(\varphi \to \psi) \to (\Box_{(I)}\varphi \to \Box_{(I)}\psi)$	$RQ_{(I)}stu \Rightarrow \exists x(Q_{(I)}tx \;\; \& \;\; Q_{(I)}Rsxu)$
(L14)	$\Box_{(I)}\varphi \to \varphi$	$Q_{(I)}ss$
(L15)	$\Box_{(I)}\neg\varphi \to \neg\Box_{(I)}\varphi$	$\exists x(Q_{(I)}sx^{*} \;\; \& \;\; Q_{(I)}s^{*}x)$
(L16)	$\Box_{(I)}\varphi \to \Box_{(I)}\Box_{(I)}\varphi$	$(Q_{(I)}st \;\; \& \;\; Q_{(I)}tu) \Rightarrow Q_{(I)}su$
(L17)	$\neg\Box_{(I)}\varphi \to \Box_{(I)}\neg\Box_{(I)}\varphi$	$(Q_{(I)}s^{*}u \;\; \& \;\; Q_{(I)}st) \Rightarrow Q_{(I)}t^{*}u$

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  1. $1^* = 0 \ \ \& \ \ 0^* = 1$;
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  3. $Q_{(LI)}1s \Rightarrow s = 1$;
  4. $R010$;
  5. $R1st \Rightarrow (s = 0$ or $t = 1)$;

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  1. $1^* = 0 \ \& \ 0^* = 1$;
  2. $Q_{(LI)}00$;
  3. $Q_{(LI)}1s \Rightarrow s = 1$;
  4. $R010$;
  5. $R1st \Rightarrow (s = 0 \text{ or } t = 1)$;

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W-models

Definition (W-models)

- $\mathfrak{M} = (S, W, L, 0, 1, \leq, {}^*, R, Q, Q_I, Q_L, V)$ such that:
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Proof. It follows from the fact that $\bigwedge_{\gamma_i \in \Gamma} \gamma_i$ and φ may be true in the same worlds but not in the same situations.

More on omniscience and relevance

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- It would be interesting to regiment agents' reasoning with:
  1. Containment logics (Par89; Fer15);
  2. Topic-sensitive logics (H19).

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$\Gamma \models^{c}_{\mathfrak{M}} \varphi \Rightarrow \Box_I \Gamma \models^{c}_{\mathfrak{M}} \Box_I \varphi$.

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$\Box\varphi \models^{c}_{\mathfrak{M}} \Box_I\varphi$.

Proof. It follows from $Q_I(W) \subseteq Q(W)$.

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Lemma 6 (Classical closure)

$\Gamma \models^{c}_{\mathfrak{M}} \varphi \;\Leftrightarrow\; \Box\Gamma \models^{c}_{\mathfrak{M}} \Box_I\varphi$

Proof. It follows from Lemmas 4 and 5.

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2. Properties of our model
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3. Technical Results
   What we have.

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- The following axioms and rules:

$$
\begin{array}{rl}
(\Box\Box_I) & \Box\varphi \to \Box_I\varphi \\
(\Box_I.\mathrm{K}) & \Box_I(\varphi \to \psi) \to (\Box_I\varphi \to \Box_I\psi) \\
(\Box_I.\mathrm{N}) & \dfrac{\varphi}{\Box_I\varphi} \\
(\mathrm{BR}) & \dfrac{\Box_L(\varphi \to \psi)}{\varphi \to \psi}
\end{array}
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Soundness and Completeness - 1

Theorem 2 (Soundness)

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Theorem 2 (Soundness)

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$\vdash_{\mathsf{L}} \varphi \iff \vdash_{\mathsf{C.L}} \Box_L \varphi$.

Proof. $(\Rightarrow)$ by induction on the length of L-proofs. $(\Leftarrow)$ by semantic argument (model construction) and Soundness.

Soundness and Completeness - 2

Definition (Canonical C.L-model)

- $\boxed{\mathfrak{M}^c = (S^c, W^c, L^c, 0^c, 1^c, \leq^c, R^c, *^c, Q^c, Q_L^c, V^c)}$ such that:
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  8. $Q^c st$ iff $\Box\varphi \in s \Rightarrow \varphi \in t$;
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  11. $s \in V^c(p)$ iff $p \in s$.

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