UNIVERSITY OF CALGARY
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The Epsilon Calculus in Non-classical Logics
Recent Results and Open Questions

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Outline

The classical epsilon calculus

ε and τ in intermediate logics

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Epsilon theorems

Open questions

What is the epsilon calculus?

- Formalization of logic without quantifiers but with the ε-operator.
- If $A(x)$ is a formula, then $\varepsilon_x\, A(x)$ is an ε-term.
- Intuitively, $\varepsilon_x\, A(x)$ is an indefinite description:
  $\varepsilon_x\, A(x)$ is some x for which $A(x)$ is true.
- ε can replace $\exists$: $\exists x\, A(x) \Leftrightarrow A(\varepsilon_x\, A(x))$
- Axioms of ε-calculus:
  - Propositional tautologies
  - $A(t) \to A(\varepsilon_x\, A(x))$
- Predicate logic can be embedded in ε-calculus.

Why should you care?

- ▷ Alternative basis for proof-theoretic research: proof theory without sequents.
- ▷ Interesting logical formalism:
  - Trades logical structure for term structure.
  - Suitable for proof formalization.
- ▷ Other Applications:
  - Use of choice functions in provers (e.g., HOL, Isabelle).
  - Applications in linguistics (choice functions, anaphora).
  - Connections to Fine's "arbitrary object" theory.
  - Propositions-as-types for dynamic linking.

The classical epsilon calculus

Axiomatisation of the epsilon calculus

- $\mathbf{C}$ (axioms of the *elementary calculus*): all propositional tautologies
- $\mathbf{C}\varepsilon$ (the *pure epsilon calculus*): add to $\mathbf{C}$ all substitution instances of

$$A(t) \to A(\varepsilon_x\, A(x))\,. \tag{1}$$

  An axiom of the form (1) is called a *critical formula*.
- $\mathbf{QC}$ (the *predicate calculus*), $\mathbf{QC}\varepsilon$ (*extended predicate calculus*): $\mathbf{C}$ and $\mathbf{C}\varepsilon$, respectively, together with all instances of $A(t) \to \exists x\, A(x)$ and $\forall x\, A(x) \to A(t)$ in the respective language, and quantifier rules.

Embedding $\mathbf{QC}\varepsilon$ in $\mathbf{C}\varepsilon$

Map $^{\varepsilon}$ of expressions in $\mathbf{QC}\varepsilon$ to expressions in $\mathbf{C}\varepsilon$ as follows:

- ▷ $x^{\varepsilon} = x$
- ▷ $P(t_1, \ldots, t_n)^{\varepsilon} = P(t_1^{\varepsilon}, \ldots, t_n^{\varepsilon})$
- ▷ $(\neg A)^{\varepsilon} = \neg A^{\varepsilon}$
- ▷ $(A \vee B)^{\varepsilon} = A^{\varepsilon} \vee B^{\varepsilon}$
- ▷ $(A \wedge B)^{\varepsilon} = A^{\varepsilon} \wedge B^{\varepsilon}$
- ▷ $(A \to B)^{\varepsilon} = A^{\varepsilon} \to B^{\varepsilon}$
- ▷ $(\varepsilon_x\, A(x))^{\varepsilon} = \varepsilon_x\, A(x)^{\varepsilon}$
- ▷ $(\exists x\, A(x))^{\varepsilon} = A^{\varepsilon}(\varepsilon_x\, A(x)^{\varepsilon})$
- ▷ $(\forall x\, A(x))^{\varepsilon} = A^{\varepsilon}(\varepsilon_x\, \neg A(x)^{\varepsilon})$

The embedding lemma

- A^ε is of the form:

$$[A(t) \to \exists x\, A(x)]^\varepsilon \equiv A^\varepsilon(t^\varepsilon) \to A^\varepsilon(\varepsilon_x\, A(x)^\varepsilon)\ ,$$

  which is a critical formula.

- A^ε is of the form:

$$[\forall x\, A(x) \to A(t)]^\varepsilon \equiv A^\varepsilon(\varepsilon_x \neg A(x)) \to A^\varepsilon(t^\varepsilon)$$

  This is the contrapositive of, and hence provable from, the critical formula

$$\neg A^\varepsilon(t^\varepsilon) \to \neg A^\varepsilon(\varepsilon_x \neg A(x))$$

The embedding lemma

- Translations of axioms provable
- Modus ponens preserved under $^\varepsilon$:

$$\frac{A \quad A \to B}{B} \quad \mapsto \quad \frac{A^\varepsilon \quad A^\varepsilon \to B^\varepsilon}{B^\varepsilon}$$

- Applications of generalization rule redundant:

$$\frac{\begin{array}{c}\vdots\ \pi \\ A \to B(x)\end{array}}{A \to \forall x\, B(x)} \quad \mapsto \quad \begin{array}{c}\vdots\ \pi[\varepsilon_x\, B^\varepsilon(x)/x] \\ A^\varepsilon \to B^\varepsilon(\varepsilon_x\, B^\varepsilon(x))\end{array}$$

The First Epsilon Theorem

First Epsilon Theorem

If A is a formula without bound variables (no quantifiers, no epsilons) and $\mathbf{QC}\varepsilon \vdash A$ then $\mathbf{C} \vdash A$.

Second Epsilon Theorem

If A is a formula without epsilons and $\mathbf{C}^{\varepsilon} \vdash A^{\varepsilon}$ then $\mathbf{QC} \vdash A$.

Herbrand Theorem for $\exists_1$

If $\exists x_1 \ldots \exists x_n A(x_1, \ldots, x_n)$ is a purely existential formula

$$\mathbf{QC} \vdash \exists x_1 \ldots \exists x_n A(x_1, \ldots, x_n),$$

then there are terms t_{ij} such that

$$\mathbf{C} \vdash \bigvee_i A(t_{i1}, \ldots, t_{in}).$$

From the last formula, the original formula can be proved in $\mathbf{QC}$.

- Can be extended to prenex formulas (by "Herbrandization")
- Can be extended to all formulas, since $\mathbf{QC}$ proves every formula equivalent to prenex form.
- Herbrand Theorem is a consequence of Extended Epsilon Theorem

Extended First Epsilon Theorem

Extended First Epsilon Theorem

Suppose $D(e_1, \ldots, e_m)$ is a quantifier-free formula containing only the ε-terms e_1, $\ldots$, e_m, and

$$\mathbf{C}\varepsilon \vdash_\pi D(e_1, \ldots, e_m) \;,$$

then there are ε-free terms t^i_j such that

$$\mathbf{C} \vdash \bigvee_{i=1}^{n} D(t^i_1, \ldots, t^i_m)$$

(Moser & Z 2006: $n \leq \left.2^{2^{\cdot^{\cdot^{\cdot^{2^{3\cdot \mathrm{cc}(\pi)}}}}}}\right\}$ stack of $3 \cdot \mathrm{cc}(\pi)$ 2's.)

ε and τ in intermediate logics

Intermediate Logics

- In classical logic, $\exists$ and $\forall$ are interdefinable
- Not true in *intuitionistic logic* and its extensions (*intermediate logics*)
- Epsilon operator seems intuitively related to *choice*, so intuitionistically suspect
- So: what happens when ε added to a intermediate logic?

Interdefinability of $\forall$ and $\exists$

- In classical logic:

$$\neg\exists x\, \neg A(x) \leftrightarrow \forall x\, A(x)$$

$$\neg\neg A(\varepsilon_x\, \neg A(x)) \leftrightarrow A(\varepsilon_x\, \neg A(x))$$

- $\rightarrow$ fails in intuitionistic logic
- Cannot define $\forall$ as $\neg\exists\neg$
- Cannot faithfully translate $\forall x\, A(x)$ to $A(\varepsilon_x\, \neg A(x))$

Intermediate logics

$\mathbf{H}$	Intuitionistic logic
$\mathbf{KC}$	Logic of weak excluded middle: $\mathbf{H} + J = \neg A \vee \neg\neg A$
$\mathbf{LC}$	infinite-valued Gödel logic, linear Kripke frames
	$\mathbf{H} + Lin = (A \to B) \vee (B \to A)$
$\mathbf{LC}_m$	m-valued Gödel logic, linear Kripke frames of length $< m$
	$\mathbf{H} + B_m = (A_1 \to A_2) \vee \cdots \vee (A_m \to A_{m+1})$
$\mathbf{C}$	Classical logic: $\mathbf{H} + A \vee \neg A$, $\mathbf{H} + B_2$

Quantified intermediate logics

$\mathbf{QH}$	Intuitionistic logic
$\mathbf{QKC}$	Weak excluded middle: $\mathbf{QH} + J$
$\mathbf{QLC}$	Linear Kripke frames: $\mathbf{QH} + Lin$
$\mathbf{QLC}_m$	$\mathbf{QH} + B_m$
$\mathbf{G}_{\mathbb{R}}$	Gödel logic on $[0, 1]$, constant-domain linear Kripke frames
	$\mathbf{QLC} + CD = \forall x(A(x) \vee B) \to (\forall x\, A(x) \vee B)$
$\mathbf{G}_0$	Gödel logic on $\{0\} \cup [1/2, 1]$
	$\mathbf{QLC} + CD + K = \forall x \neg\neg A(x) \to \neg\neg\forall x\, A(x)$
$\mathbf{G}_m$	m-valued Gödel logic: $\mathbf{QH} + B_m + CD$
$\mathbf{QC}$	Classical logic

ε and τ

- Introduce dual operator τ: $\tau_x\, A(x)$
- Critical formulas now:
  - $A(t) \to A(\varepsilon_x\, A(x))$ and
  - $A(\tau_x\, A(x)) \to A(t)$
- $\varepsilon\tau$-translation just like ε-translation, except for:
  - $(\exists x\, A(x))^{\varepsilon\tau} = A^{\varepsilon\tau}(\varepsilon_x\, A(x)^{\varepsilon\tau})$
  - $(\forall x\, A(x))^{\varepsilon\tau} = A^{\varepsilon\tau}(\tau_x\, A(x)^{\varepsilon\tau})$

Conservativity and non-conservativity

Conservativity questions

- In classical logic, addition of ε is conservative.
- Question: Does addition of ε and τ to intermediate logic have effect on theorems?
- Results by Bell and DeVidi suggest yes: under certain assumptions, even excluded middle $A \vee \neg A$ becomes provable.
- However, these results rely on presence of $=$ and need axioms.
- What about *pure logic?*
  - No effect on propositional level.
  - All quantifier shifts become provable.

Shadow

The *shadow* A^s of a formula is defined as follows:

$$P(t_1, \ldots, t_n)^s = X_P$$

$$(A \wedge B)^s = A^s \wedge B^s \qquad (A \vee B)^s = A^s \vee B^s$$

$$(A \rightarrow B)^s = A^s \rightarrow B^s \qquad (\neg A)^s = \neg A^s$$

$$(\exists x\, A)^s = A^s \qquad (\forall x\, A)^s = A^s$$

where X_P is a propositional variable.

The *shadow of a proof* $\pi = A_1, \ldots, A_n$ is $A_1^s, \ldots, A_n^s$.

Logics preserved under shadow

Conservativity of $\varepsilon\tau$

If $A_1, \dots, A_n \vdash_{\mathbf{L}\varepsilon\tau} B$, then $A_1^s, \dots, A_n^s \vdash B^s$.

$\mathbf{L}\varepsilon\tau$ is conservative over $\mathbf{L}$ for propositional formulas.

- ▷ The shadows of critical formulas are of the form $A \to A$.
- ▷ Intermediate logics prove $A \to A$.
- ▷ The shadow of modus ponens is modus ponens.
- ▷ (The shadows of premise and conclusion of universal quantification rules are identical so also holds for $\mathbf{QL}\varepsilon\tau$.)

Quantifier shifts

$$\forall x(A(x) \lor B) \to (\forall x\, A(x) \lor B) \qquad (CD)$$

$$(A(\tau_x(A(x) \lor B)) \lor B) \to (A(\tau_x\, A(x)) \lor B) \qquad (CD^{\varepsilon\tau})$$

$$(B \to \exists x\, A(x)) \to \exists x(B \to A(x)) \qquad (Q_\exists)$$

$$(B \to A(\varepsilon_x\, A(x))) \to (B \to A(\varepsilon_x(B \to A(x)))) \qquad (Q_\exists^{\varepsilon\tau})$$

$$(\forall x\, A(x) \to B) \to \exists x(A(x) \to B) \qquad (Q_\forall)$$

$$(A(\tau_x\, A(x)) \to B) \to (A(\varepsilon_x(A(x) \to B)) \to B) \qquad (Q_\forall^{\varepsilon\tau})$$

- In each case, x is not free in B.
- Note: $(Q_\forall^{\varepsilon\tau})$ is a critical formula.

Proving quantifier shifts

$$CD^{\varepsilon\tau} \qquad (A(\tau_x\, A) \vee B) \to (A(\tau_x(A \vee B)) \vee B)$$

$$A_1 = A(\tau_x\, A) \qquad A_2 = A(\tau_x(A \vee B))$$

$$A(\tau_x\, A) \to A(\tau_x(A \vee B))$$

$$(A_1 \to A_2) \to ((A_1 \vee B) \to (A_2 \vee B))$$

$$Q_\exists^{\varepsilon\tau} \qquad (B \to A(\varepsilon_x(B \to A))) \to (B \to A(\varepsilon_x\, A))$$

$$A_1 = A(\varepsilon_x(B \to A)) \qquad A_2 = A(\varepsilon_x\, A)$$

$$A(\varepsilon_x(B \to A)) \to A(\varepsilon_x\, A)$$

$$(A_1 \to A_2) \to ((B \to A_1) \to (B \to A_2))$$

$A_1 \to A_2$ is a critical formula

Epsilon theorems

Extended First $\varepsilon\tau$-Theorem

If $D(e_1, \ldots, e_n)$ is an $\varepsilon\tau$-formula with $e_1, \ldots, e_n$, its only $\varepsilon\tau$-terms and

$$\mathbf{L}\varepsilon\tau \vdash A(e_1, \ldots, e_n),$$

then there are terms t_{ij} such that

$$\mathbf{L} \vdash \bigvee_i D(t_{i1}, \ldots, t_{in}).$$

No Extended First $\varepsilon\tau$-Theorem in intermediate logics

Theorem

Suppose $\mathbf{L}^{\varepsilon\tau}$ has the extended first epsilon theorem. Then

$$\mathbf{L} \vdash B_m = (A_1 \to A_2) \vee \ldots \vee (A_m \to A_{m+1})$$

for some m.

Proof sketch

- Consider $\exists x \forall y (P(y) \to P(x))$.
- (Equivalent over $\mathbf{QH}$ to $Q_\exists$ so $\varepsilon\tau$-translation provable.)
- Herbrand form: $\exists x (P(f(x) \to P(x)))$.
- $\varepsilon\tau$-Translation: $P(f(e)) \to P(e)$ where $e = \varepsilon_x (P(f(x) \to P(x))$.
- Herbrand disjunction of formulas of the form $P(f^n(s)) \to P(f^{n-1}(s))$
- Rearrange, substitute to get B_m

Epsilon Theorems and intermediate logics

- ▷ Extended first theorem holds for $\mathbf{L}\varepsilon\tau$ only if $\mathbf{L} \vdash B_m$ for some m.
- ▷ $\mathbf{L} \vdash B_m$ exactly for the finite valued Gödel logics $\mathbf{LC}_m$.
- ▷ In particular, no extended first $\varepsilon\tau$-theorem for
  - intuitionistic logic $\mathbf{H}$,
  - (infinite valued) Gödel logic $\mathbf{LC}$,
  - logic of weak excluded middle $\mathbf{KC}$.

Epsilon elimination

- Classical epsilon theorem proceeds by iterated elimination of critical formulas
- A proof of $D(e)$ with critical formulas

$$A(t_1) \to A(e), \ldots, A(t_n) \to A(e)$$

belonging to ε-term e yields proofs of

$$A(t_1) \to D(t_1), \ldots, A(t_n) \to D(t_n), \text{ and } (\neg A(t_1) \wedge \ldots \neg A(t_n)) \to D(e)$$

which combine to a proof of

$$D(t_1) \vee \ldots \vee D(t_n) \vee D(e)$$

by excluded middle.
- But of course, excluded middle can't be used in intermediate logics.

Epsilon elimination sets

- We analyze and refine Hilbert's approach:
- In a proof of $D(e)$ we divide the critical formulas involved into:
  - Γ where e is not the critical $\varepsilon\tau$-term;
  - $\Lambda(e) \cup \Lambda'(e)$ where e is the critical $\varepsilon\tau$-term
- $s_1, \ldots, s_k$ is an e-elimination set for $\Lambda(e)$ if

$$\Gamma[s_1/e], \ldots, \Gamma[s_k/e], \Lambda'(e) \vdash D(s_1) \vee \cdots \vee D(s_k)$$

- In $\mathbf{C}$, if
  - $\Lambda(e) = \{A(t_1) \rightarrow A(e)\}$ are all the critical formulas belonging to e and
  - Γ are all other critical formulas

  then $t_1, \ldots, t_n, e$ is an e-elimination set for $\Lambda(e)$.

Elimination sets for Lin and B_m

- Let $\Lambda(e)$ be all the critical formulas belongin to e of the form $A(t) \to A(e)$ where t does not contain e.
- Let $\Lambda'(e)$ be all those where t does contain e.
- Then:
  - $\Lambda(e)$ has e-elimination sets if $\mathbf{L} \vdash Lin$.
  - $\Lambda'(e)$ has e-elimination sets if $\mathbf{L} \vdash B_m$ for some m.

Extended First $\varepsilon\tau$-Theorem for G_m

- Pick e of maximal rank and maximal degree.
- Eliminate $\Lambda(e)$ using Lin.
- Eliminate remaining $\Lambda'(e)$ using B_m.
- Repeat.
- Proper order (rank, degree) ensures termination.

Summary of results

Conservativity over propositional logic	Yes	Any
Extended First ε-Theorem	No	Any except $\mathbf{LC}_m$
	Yes	$\mathbf{LC}$, $\mathbf{KC}$ (for negated formulas)
	Yes	$\mathbf{LC}_m$
Second $\varepsilon\tau$-Theorem	No	$\mathbf{G}_{\mathbb{R}}$, $\mathbf{QLC}_m$
	Yes	$\mathbf{G}_m$

Open questions

Open questions

- The second $\varepsilon\tau$-theorem: when is the extended $\varepsilon\tau$-calculus conservative over a quantified logic?
- Characterize proofs for which $\varepsilon\tau$-elimination works with just Lin.
- Semantics of intuitionistic ε's and τ's.

Intuitionistic Kripke semantics

- Worlds W with (reflexive, transitive) accessibility relation R
- Domain $D(w)$ for each $w \in W$
- Domains must be monotonic: $wRv \Rightarrow D(w) \subseteq D(v)$
- Predicates must be monotonic:

$$wRv \wedge w \Vdash P(\vec{d}) \qquad \Rightarrow \qquad v \Vdash P(\vec{d})$$

Intuitionistic satisfaction

$$
\begin{aligned}
w \Vdash A \wedge B &\Leftrightarrow && w \Vdash A \text{ and } w \Vdash B \\
w \Vdash A \vee B &\Leftrightarrow && w \Vdash A \text{ or } w \Vdash B \\
w \Vdash A \rightarrow B &\Leftrightarrow && w \nVdash A \text{ or } w \Vdash B \\
w \Vdash \exists x\, A(x) &\Leftrightarrow && \text{for some } d \in D, w \Vdash A(d) \\
w \Vdash \forall x\, A(x) &\Leftrightarrow && \text{for all } d \in D, w \Vdash A(d)
\end{aligned}
$$

Intuitionistic satisfaction

$$
\begin{aligned}
w \Vdash A \wedge B \Leftrightarrow & & w \Vdash A \text{ and } w \Vdash B \\
w \Vdash A \vee B \Leftrightarrow & & w \Vdash A \text{ or } w \Vdash B \\
w \Vdash A \rightarrow B \Leftrightarrow & \text{ for all } v \text{ st } wRv, & v \nVdash A \text{ or } v \Vdash B \\
w \Vdash \exists x\, A(x) \Leftrightarrow & & \text{for some } d \in D, w \Vdash A(d) \\
w \Vdash \forall x\, A(x) \Leftrightarrow & \text{ for all } v \text{ st } wRv, & \text{for all } d \in D, v \Vdash A(d)
\end{aligned}
$$

Epsilons in Kripke semantics

$D(\infty)$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\dots$
$D(\infty - 1)$	0	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\dots$
$D(\infty - 1)$	0	1	$\mathbf{2}$	$\mathbf{3}$	$\dots$
$D(\infty - 3)$	0	1	2	$\mathbf{3}$	$\dots$
$\vdots$					
$D(0)$	0	1	2	3	$\dots$

- ▷ $0 \nVdash \exists x\, P(x)$
- ▷ $\infty - i \Vdash \exists x\, P(x)$
- ▷ no n can be $\varepsilon_x\, P(x)$ in *all* worlds $\infty - i$
- ▷ $\nVdash \exists x\, P(x) \to P(\varepsilon_x\, P(x))$
- ▷ $\nVdash \exists y(\exists x\, P(x) \to P(y))$

Taus in Kripke semantics

$$\begin{array}{cccccc} \vdots \\ D(3) & \mathbf{0} & \mathbf{1} & \mathbf{2} & 3 & \dots \\ D(2) & \mathbf{0} & \mathbf{1} & 2 & 3 & \dots \\ D(1) & \mathbf{0} & 1 & 2 & 3 & \dots \\ D(0) & 0 & 1 & 2 & 3 & \dots \end{array}$$

- $i \nVdash \forall x\, P(x)$
- for every n there is an i so that $i \Vdash P(n)$
- no n can be $\tau_x\, P(x)$
- $\nVdash P(\tau_x\, P(x)) \to \forall x\, P(x))$
- $\nVdash \exists x(P(x) \to \forall y\, P(y))$

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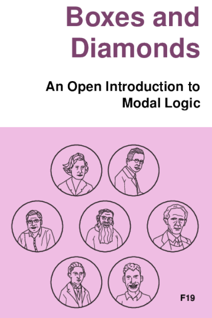

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Ff	Ff	Ff	Ff	Ff	Fb	Fb	Fb	Fb	Fb	Fb	Fb	Fb	Ff	Ff	Ff	Ff
Nf	Nf	Ff	Nf	Ff	Nb	Nb	Fb	Nb	Fb	Nb	Fb	Fb	Nf	Ff	Nf	Ff
Fn	Fn	Ff	Ff	Fn	Ft	Ft	Fb	Fb	Fb	Fb	Ft	Ft	Fn	Ff	Ff	Fn
Nt	Nt	Fb	Nb	Ft	Nt	Nt	Fb	Nb	Fb	Nb	Ft	Ft	Nt	Fb	Nb	Ft
Tt	Nt	Fb	Nb	Ft	Nt	Tt	Fb	Nb	Bb	Tb	Ft	Bt	Tt	Bb	Tb	Bt
Fb	Fb	Fb	Fb	Fb	Fb	Fb	Fb	Fb	Fb	Fb	Fb	Fb	Fb	Fb	Fb	Fb
Nb	Nb	Fb	Nb	Fb	Nb	Nb	Fb	Nb	Fb	Nb	Fb	Fb	Nb	Fb	Nb	Fb
Bb	Fb	Fb	Fb	Fb	Fb	Bb	Fb	Fb	Bb	Bb	Fb	Bb	Bb	Bb	Bb	Bb
Tb	Nb	Fb	Nb	Fb	Nb	Tb	Fb	Nb	Bb	Tb	Fb	Bb	Tb	Bb	Tb	Bb
Ft	Ft	Fb	Fb	Ft	Ft	Ft	Fb	Fb	Fb	Fb	Ft	Ft	Ft	Fb	Fb	Ft
Bt	Ft	Fb	Fb	Ft	Ft	Bt	Fb	Fb	Bb	Bb	Ft	Bt	Bt	Bb	Bb	Bt
Tn	Nn	Ff	Nf	Fn	Nt	Tt	Fb	Nb	Bb	Tb	Ft	Bt	Tn	Bf	Tf	Bn
Bf	Ff	Ff	Ff	Ff	Fb	Bb	Fb	Fb	Bb	Bb	Fb	Bb	Bf	Bf	Bf	Bf
Tf	Nf	Ff	Nf	Ff	Nb	Tb	Fb	Nb	Bb	Tb	Fb	Bb	Tf	Bf	Tf	Bf
Bn	Fn	Ff	Ff	Fn	Ft	Bt	Fb	Fb	Bb	Bb	Ft	Bt	Bn	Bf	Bf	Bn

- Ever deal with n-valued logics? Multlog (**logic.at/multlog**) can help!
- Computes sequent and tableaux rules from truth tables
- Now with interactive mode: find homomorphisms between logics, ways to express a connective using others, show that things are or aren't tautologies, etc.
- Richard Zach (2023). "An Epimorphism between Fine and Ferguson's Matrices for Angell's AC". *Logic and Logical Philosophy* 32, pp. 161–179. DOI: `10.12775/LLP.2022.025`

Thanks!

Jeremy Avigad and Richard Zach (2020). "The Epsilon Calculus". In: *Stanford Encyclopedia of Philosophy*. Fall 2020. URL: `https://plato.stanford.edu/archives/fall2020/entries/epsilon-calculus/`

Georg Moser and Richard Zach (2006). "The Epsilon Calculus and Herbrand Complexity". *Studia Logica* 82, pp. 133–155. DOI: `10.1007/s11225-006-6610-7`

Richard Zach (2017). "Semantics and Proof Theory of the Epsilon Calculus". In: *Logic and Its Applications. ICLA 2017*. Berlin: Springer, pp. 27–47. DOI: `10.1007/978-3-662-54069-5_4`

Matthias Baaz and Richard Zach (2022). "Epsilon Theorems in Intermediate Logics". *The Journal of Symbolic Logic* 87, pp. 682–720. DOI: `10.1017/jsl.2021.103`

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