Math 4/5190A Problem Sets 1 and 2 (Chapter II)

1. Consider the linear DE

$$\frac{dx}{dt} = Ax$$

in $\mathbb{R}^2$, where the matrix A is given by

i) $\begin{pmatrix} 3 & 1 \\ -4 & -2 \end{pmatrix}$ ii) $\begin{pmatrix} -5 & -2 \\ 8 & 3 \end{pmatrix}$ iii) $\begin{pmatrix} -1 & -2 \\ 1 & -3 \end{pmatrix}$ iv) $\begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix}$

a) Transform the matrix A to Jordan canonical form and hence find the linear flow e^{At}.

b) Sketch the phase portrait in the canonical basis, and in the standard basis.

c) Use a) to find the unique solution $x(t)$ of the DE which

satisfies $x(1) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

d) Find all points $x \in \mathbb{R}^2$ such that $\lim_{t \to \infty} e^{At} x = 0$.

2. a) Verify that the linear flows defined by the DEs

$$x' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} x, \quad y' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} y$$

are topologically equivalent under the homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$h(y_1, y_2) = \begin{cases} (y_1 + y_2 \log|y_2|, y_2), & \text{if } y_2 \neq 0 \\ (y_1, 0), & \text{if } y_2 = 0 \end{cases}$$

Sketch the phase portraits and illustrate the action of h. Are these flows linearly equivalent?

b) Modify the homeomorphism h in a) to prove that the linear flows defined by

$$x' = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} x \quad \text{and} \quad y' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} y$$

with $\lambda > 0$, are topologically equivalent.

3. Consider the DE

$$x' = A(\varepsilon)x, \qquad A(\varepsilon) = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$$

a) Calculate the linear flow $e^{tA(\varepsilon)}$ in the three cases $\varepsilon > 0$, $\varepsilon = 0$ and $\varepsilon < 0$. Is the flow a continuous function of ε?

b) Sketch the three phase portraits, for $|\varepsilon|$ close to zero, illustrating the transition through $\varepsilon = 0$.

c) Are the flows topologically equivalent?

4. Consider the DE

$$x' = Ax, \quad A = \begin{pmatrix} -\frac{1}{100} & 1 \\ 0 & -\frac{1}{100} \end{pmatrix}$$

a) Find constants M and k such that

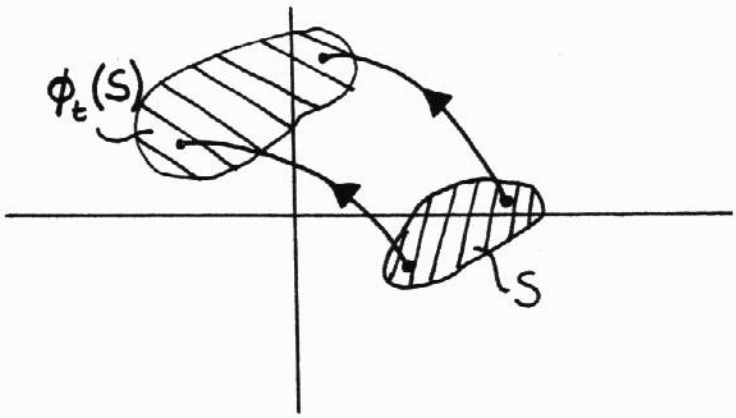
$$\|e^{At}x\| \le Me^{-kt}\,\|x\|, \quad \text{for all } t \ge 0 .$$

b) Find a Lyapunov function $V(x)$ for the DE. Sketch the level curves of V superimposed on the phase portrait.

5. a) By considering the Jordan canonical forms, discover a simple expression for $\det(e^A)$, where A is a 2 x 2 matrix.

b) Consider the action of a linear flow $g^t = e^{At}$ on a set $S \subset \mathbb{R}^2$ of finite area.

Let $A(t) = \text{Area}\,[g^t(S)]$, $t \in \mathbb{R}$. Show that $A(t) = e^{(\text{tr}A)t}\,A(0)$, where $\text{tr}(A)$ is the trace of the matrix A.
[Recall: How do areas change under a linear map?]



c) Are all area-preserving linear flows in $\mathbb{R}^2$ topologically equivalent?

6. List the Jordan canonical forms for real 3 x 3 and 4 x 4 matrices. In each case give the eigenvectors (if any) and the irreducible invariant subspaces of A.

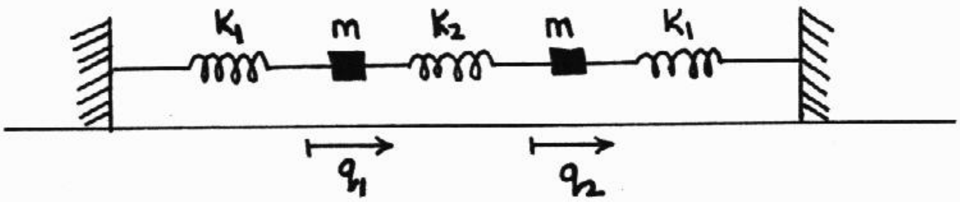
Note: A subspace $E \subset \mathbb{R}^n$ is an invariant subspace of a matrix A if $x \in E$ implies $Ax \in E$. The subspace is irreducible if it contains no non-trivial invariant subspaces.

7. If $A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$, where A_1 and A_2 are square matrices, calculate e^A.

8. For each 3 x 3 canonical form in Q6, calculate the linear flow e^{tA}.

(This question is not to be handed in).

(*) Consider the undamped symmetric two-mass oscillator as shown:



a) Let q_1 and q_2 denote the displacement of the masses from the equilibrium position, and let

$$x = \left(q_1, q_2, \frac{dq_1}{dt}, \frac{dq_2}{dt}\right) \in \mathbb{R}^4$$

describe the state of the system. Show that the motion is governed by the DE

$$\frac{dx}{dt} = Ax, \quad A = \begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -\frac{(k_1 + k_2)}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & -\frac{(k_1 + k_2)}{m} \end{pmatrix}$$

b) The transformation to canonical form can be achieved by forming the sum and difference of the original second order DEs. Letting

$$y = \left(q_1 + q_2, \ (m/k_1)^{1/2}\left(\frac{dq_1}{dt} + \frac{dq_2}{dt}\right), \ q_1 - q_2, \ (m/k_1+2k_2)^{1/2}\left(\frac{dq_1}{dt} - \frac{dq_2}{dt}\right)\right),$$

Show that the DE assumes the form

$$\frac{dy}{dt} = By, \quad B = \left(\begin{array}{cc|cc} 0 & \beta_1 & & \\ -\beta_1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & \beta_2 \\ \multicolumn{2}{c|}{0} & -\beta_2 & 0 \end{array}\right), \quad \beta_1 > 0, \ \beta_2 > 0$$

c) Find the flow e^{tB}, and show that for any solution $y = e^{tB} a$, there are constants c_1 and c_2 such that

$$y_1^2 + y_2^2 = c_1 \quad \text{and} \quad y_3^2 + y_4^2 = c_2$$

d) Interpret physically the special orbits which lie in the invariant 2-spaces $y_1 = y_2 = 0$, and $y_3 = y_4 = 0$. Can you suggest a physical interpretation for the constants c_1, c_2 in part c)?

e) Find a restriction on the spring constants k_1 and k_2 which will guarantee that all orbits, except the equilibrium point 0, are <u>periodic</u>. This means that for each initial state, the system successively returns to that state as time evolves.

f) Suppose the orbit $\gamma(a)$ through $a \in \mathbb{R}^4$, is <u>not</u> <u>periodic</u>. How does the system evolve, if its initial state is a? Can you describe the non-periodic orbits geometrically in $\mathbb{R}^4$? Can you describe their projection into the $y_1 - y_3$ plane?

<u>Comments</u>: Part f) is an open-ended and difficult question, and it leads to a number of important ideas in the theory of dynamical systems.