Math 4/5190A -- Problem Set 3 (Chapter III)

1. Consider the 1-D DE

$$\frac{dx}{dt} = -x + x^3 .$$

a) Find the flow $g^t a$ explicitly. Sketch the solution curves in tx-space, and the orbits in x-space. How many orbits are there?

b) Verify directly that $g^{t+s} = g^t \circ g^s$.

c) Show that any solution with initial state a subject to $a^2 > 1$, has a finite escape time.

2. Consider the 2-D DE

$$\frac{dx_1}{dt} = x_1 - x_1^3$$

$$\frac{dx_2}{dt} = -x_2 .$$

a) Find the flow $g^t a$, and find an explicit equation for the orbits.

b) Sketch the portrait of the orbits showing typical orbits, exceptional orbits and the behaviour near the equilibrium points.

c) Give the linearizations of the DE at the equilibrium points. Are the corresponding linear flows topologically equivalent to the non-linear flows in a neighbourhood of the equilibrium point?

3. A one-parameter family of non-linear maps

$$g^t : \mathbb{R}^2 \to \mathbb{R}^2$$

is defined by

$$g^t \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \dfrac{a_1}{1-a_1 t} \\ e^{-t} a_2 \end{pmatrix} .$$

a) Verify that $\{g^t\}$ is a local flow, i.e. that it satisfies $g^0 = I$, $g^{t+s} = g^t \circ g^s$, for all appropriate t and s.

b) Show that the solution corresponding to any initial state (a_1, a_2) with $a_1 > 0$ has a finite escape time.

c) By eliminating t, show that the orbits are given by

$$x_2 e^{-1/x_1} = K \text{, a constant}$$

d) Find the autonomous DE which corresponds to the flow.

e) Sketch the portrait of the orbits. Hence give the ω-limit set $\omega(a)$ for each initial state $a \in \mathbb{R}^2$.

f) Find the linearization of the DE at the equilibrium point and sketch the portrait of its orbits. Is this linear flow topologically equivalent to the non-linear flow in a neighbourhood of the equilibrium point? Justify.

g) Is the non-linear flow topologically equivalent to <u>any</u> linear flow?

<u>Comment</u>: The equilibrium point in this example is important in bifurcation theory. Its form suggests its name.

4. A hypothetical physical system is described by the DE

$$\frac{dx}{dt} = x^2 + 3y^2 - 1$$

$$\frac{dy}{dt} = -2xy \ .$$

a) Show that the DE is Hamiltonian, and find $H(x,y)$.

b) Sketch the portrait of the orbits, and indicate the isoclines $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

Verify that $x^2 + y^2 = 1$ is an invariant set of the flow. How many orbits does it contain?

c) Let $\{g^t\}$ be the flow which corresponds to the DE. Why can one assert that if $\|a\| \le 1$, then $g^t a$ is defined for all $t \in \mathbb{R}$? By referring to b), evaluate

$$\lim_{t \to \infty} g^t a$$

for $a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, if the limit exists, i.e., determine the long-term behaviour corresponding to these initial states. Give the ω-limit set $\omega(a)$ for each point a.

5. Consider the DE

$$\frac{dx_1}{dt} = -x_2 - x_1^3 \ , \quad \frac{dx_2}{dt} = x_1 - x_2^3 \ .$$

a) Show that any disc $x_1^2 + x_2^2 \leq R^2$ is a trapping set (i.e., a closed bounded positively invariant set).

b) Apply the Global Liapunov Theorem to show that the ω-limit set for any initial state is the origin. Sketch the portrait of the orbits.

c) Find the linearization of the DE at (0,0). Is the linear flow topologically equivalent to the non-linear flow in a neighbourhood of (0,0)? Why is your answer not in conflict with the Hartman-Grobman Theorem?
[Note: the Hartman-Grobman Theorem will be discussed in Chapter IV.]

(This question is not to be handed in.)

(†) Consider the damped magneto-elastic beam, whose oscillations are governed by the DE:

$$\frac{dq}{dt} = p$$

$$\frac{dp}{dt} = -2\alpha p - q + \beta q(1-q^2)$$

where α and β are positive constants. Here $-2\alpha p$ is the damping force, $-q$ is the elastic restoring force and $\beta q(1-q^2)$ is the magnetic force, the magnets being located at $q = \pm 1$. If $\beta > 1$ the magnetic force "dominates", while if $\beta < 1$ the elastic force "dominates".
The goal is to sketch and justify the portrait of the orbits in the case of small damping ($a \gtrsim 0$) :

a) Use the Global Liapunov Theorem to prove that the ω-limit set for any initial state is an equilibrium point. Are there any periodic orbits?

b) Analyze the linearizations at the equilibrium points in the case $\beta > 1$, and find the eigenvectors where appropriate. Use the Stable Manifold Theorem to sketch the orbits near the saddle point.

c) Sketch the stable and unstable manifolds of the saddle point. The isocline $\frac{dp}{dt} = 0$ will be useful.

d) Complete the sketch of the portrait of the orbits. Indicate the basin of attraction of the sinks.

e) Use the portrait to show that for all $\varepsilon > 0$ there exist initial states a_1 and a_2 with $\|a_1 - a_2\| < \varepsilon$ and a real number $T > 0$ such that

$$\|g^t a_1 - g^t a_2\| > \sqrt{\frac{\beta-1}{\beta}} \quad , \text{ for all } t \geq T$$

where g^t is the flow associated with the DE. This means that there are arbitrarily close initial states whose orbits do not remain close.

f) Sketch the portrait of the orbits for the case of large damping.

(This question is not to be handed in.)

(††) Consider the DE

$$\frac{dx_1}{dt} = -2x_1 - (x_1^2 + x_2^2)$$

$$\frac{dx_2}{dt} = -(1-x_1)\, x_2$$

a) Analyze the linearizations of the DE at the equilibrium points. Use the Stable Manifold Theorem to sketch the orbits near the saddle point. Find a quadratic approximation for the stable manifold, in the form

$$x_1 \approx a + bx_2 + cx_2^2 \ , \quad a, b, c = \text{constant.}$$

b) Use the isoclines $\frac{dx_1}{dt} = 0$ and $\frac{dx_2}{dt} = 0$ to deduce that there are no periodic orbits.

c) Use the form of the DE to deduce that for all solutions with $x_2(0) \neq 0$,

$$\lim_{t\to-\infty} x_1(t) = +\infty \ , \quad \lim_{t\to-\infty} x_2(t) = 0 \ .$$

d) Use c) to sketch the global stable manifold, and hence sketch the portrait of the orbits. Use the symmetry $x_2 \to -x_2$. Indicate the basin of attraction of the sink.

e) Find a Liapunov function V for the sink, and determine the largest disc on which $\dot{V} < 0$.

Comment: Notice how the saddle directs the orbits towards the sink, by attracting orbits along its stable manifold, and repelling them along its unstable manifold.