CHAPTER IV: Local Behaviour Near a Hyperbolic Equilibrium Point

The main goal in this chapter is to discuss the following question:

Let $\bar{x}$ be an equilibrium point of a non-linear DE
$$\frac{dx}{dt} = f(x) \qquad \text{in } \mathbb{R}^n.$$
What information does the linearization
$$\frac{du}{dt} = Df(\bar{x})\,u\,, \qquad u = x - \bar{x}$$
provide about the orbits of the non-linear DE near the equilibrium point? Stability of an equilibrium point is one aspect of this question:

One of the main goals of the theory of dynamical systems is to determine the long-term behaviour of the solutions of an autonomous DE $\frac{dx}{dt} = f(x)$ in $\mathbb{R}^n$, i.e. to find the ω-limit set of an initial state a.
The results of this chapter are essential for

- determining which equilibrium points are the ω-limit set of typical initial states (these are the sinks)

- sketching the global portrait of the orbits, and hence determining all ω-limit sets.

Section IV.1: Liapunov's Stability Theorem

We first discuss the stability of an equilibrium point of a non-linear DE (review section II.5, 1° on linear stability). Our goal is to show that the stability of an equilibrium point can be ascertained, subject to a condition, by studying the linearization of the DE. We do this by making use of a Liapunov function (see page II.5.10) and the Liapunov Stability Theorem.

1° Non-linear stability

The basic definitions are the same as in the linear case, with the linear flow e^{tA} being replaced by the non-linear flow g^t.

<u>Defn</u>: 1) An equilibrium point $\bar{x}$ of a DE $\frac{dx}{dt} = f(x)$ in $\mathbb{R}^n$ is <u>stable</u> if for all neighbourhoods U of $\bar{x}$, there exists a neighbourhood V of $\bar{x}$ such that

$$g^t V \subseteq U, \quad \text{for all } t \geq 0,$$

where g^t is the flow of the DE

2) The equilibrium point $\bar{x}$ is said to be <u>asymptotically stable</u> if it is stable and if, in addition, for all $x \in V$,

$$\lim_{t \to \infty} \| g^t x - \bar{x} \| = 0$$

2° The Liapunov Stability Theorem:

The idea is motivated by the discussion on p II.5.7.

Consider a non-linear DE
$$\frac{dx}{dt} = f(x) \quad \text{in } \mathbb{R}^n.$$

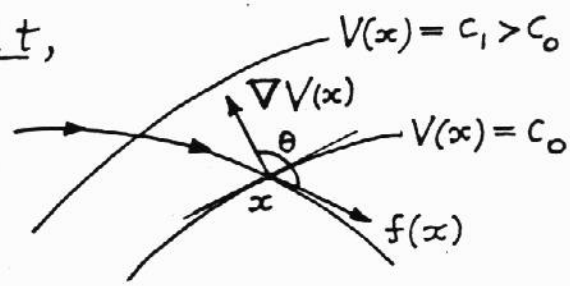
Let $V: \mathbb{R}^n \to \mathbb{R}$ be a C^1 function on state space (for example the energy, if the DE describes a mechanical system). We can calculate the rate of change of V along a solution of the DE:

$$\begin{aligned}
\frac{d}{dt} V(x(t)) &= \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial V}{\partial x_n}\frac{dx_n}{dt}, && \text{by the Chain Rule} \\
&= \frac{\partial V}{\partial x_1} f_1 + \cdots + \frac{\partial V}{\partial x_n} f_n, && \text{by the DE} \\
&= \nabla V(x(t)) \cdot f(x(t)), && \text{by definition of the scalar product in } \mathbb{R}^n.
\end{aligned}$$

Thus, <u>if</u> $\nabla V(x(t)) \cdot f(x(t)) < 0$ <u>for all t</u>,

<u>then $V(x)$ decreases with time along the corresponding orbit.</u>

From a geometrical point of view, the orbit cuts the level sets $V(x) = \text{constant}$ in the direction away from $\nabla V(x)$. (i.e. $\frac{\pi}{2} < \theta \le \pi$).



Level sets $V(x) = c$

<u>Recall</u>: 1. If $\nabla V(a) \neq 0$, then $\nabla V(a)$ is orthogonal to the level set $V(x) = \text{const.}$ through a.

2. $A \cdot B = \|A\| \, \|B\| \cos\theta$, where θ is the angle between A + B.

Suppose that $\bar{x}$ is an equilibrium point of the DE. We are interested in the situation when

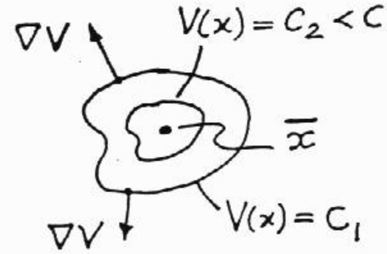
$$V(\bar{x}) = 0$$

and

$$V(x) > 0 \quad \text{for all } x \in U - \{\bar{x}\},$$

where U is a neighbourhood of $\bar{x}$, i.e $\bar{x}$ is a strict local minimum point of V.

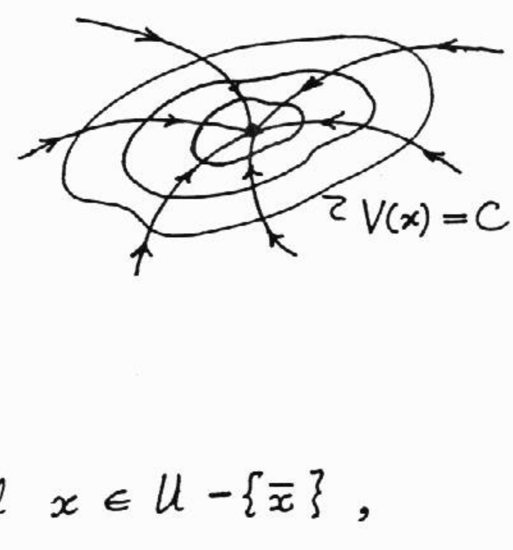
We expect the level sets of V in U to be concentric curves ($n=2$) or concentric surfaces ($n=3$).



Suppose that in this situation

$$\nabla V(x) \cdot f(x) < 0 \quad \text{for all } x \in U - \{\bar{x}\} \qquad (1)$$

So that V decreases along all orbits in $U - \{\bar{x}\}$. Any orbit in $U - \{\bar{x}\}$ will cut the level sets of V in the inward direction, & we expect that this will continue until the orbit is forced to approach the equilibrium point $\bar{x}$ as $t \to \infty$, showing that the equilibrium point is <u>asymptotically stable</u>.



If (1) is replaced by

$$\nabla V(x) \cdot f(x) \leq 0 \quad \text{for all } x \in U - \{\bar{x}\},$$

then U may contain periodic orbits, & we only obtain the weaker conclusion that $\bar{x}$ is <u>stable</u>.

Finally, if
$$\nabla V(x) \cdot f(x) > 0 \quad \text{for all } x \in \mathcal{U} - \{\bar{x}\},$$
the orbits are forced away from $\bar{x}$, which is thus an unstable equilibrium point.

It should be noted that the preceding arguments are heuristic. Nevertheless, they are reasonably convincing, * can be made rigorous. The conclusions are summarized in the Theorem to follow.

Theorem IV.1.1 (Liapunov Stability Theorem)

Let $\bar{x}$ be an equilibrium point of the DE
$$\frac{dx}{dt} = f(x) \quad \text{in } \mathbb{R}^n.$$

Let $V: \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that
$$V(\bar{x}) = 0, \qquad V(x) > 0 \text{ for all } x \in \mathcal{U} - \{\bar{x}\},$$
where $\mathcal{U}$ is a neighbourhood of $\bar{x}$.

Let $\dot{V}(x) = \nabla V(x) \cdot f(x)$.

1) If $\dot{V}(x) < 0$ for all $x \in \mathcal{U} - \{\bar{x}\}$, then $\bar{x}$ is asymptotically stable.
2) If $\dot{V}(x) \leq 0$ for all $x \in \mathcal{U} - \{\bar{x}\}$, then $\bar{x}$ is stable
3) If $\dot{V}(x) > 0$ for all $x \in \mathcal{U} - \{\bar{x}\}$, then $\bar{x}$ is unstable.

Proof: This will be proved as a corollary of the Global Liapunov Theorem.

Terminology: A function $V: \mathbb{R}^n \to \mathbb{R}$ which satisfies
$$V(\bar{x}) = 0, \qquad V(x) > 0 \quad \text{for all } x \in U - \{\bar{x}\}$$
and
$$\dot{V}(x) \leq 0 \quad (\text{respectively } < 0) \quad \text{for all } x \in U - \{\bar{x}\},$$
is called a Liapunov function for the equilibrium point $\bar{x}$. (respectively, strict Liapunov function ⋯).

Exercise: Consider the DE
$$\frac{dx_1}{dt} = -x_2 - x_1 \sin^2 x_1$$
$$\frac{dx_2}{dt} = x_1 - x_2 \sin^2 x_2$$

Show that the equilibrium point $(0,0)$ is asymptotically stable.

Answer: Let $V(x_1, x_2) = x_1^2 + x_2^2$. What is the largest circular neighbourhood U that can be used?

$\dot{V} = -2x_1^2 \sin^2 x_1 - 2x_2^2 \sin^2 x_2$ $\qquad$ $x_1^2 + x_2^2 < \pi$

Exercise: Consider a classical Hamiltonian DE
$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

where $H(q, p) = \frac{1}{2m} p^2 + V(q)$. Prove that if $\bar{q}$ is a strict local minimum point of V, then $(0, \bar{q})$ is a stable equilibrium point of the DE (thence a stable equilibrium state of the mechanical system)

Answer: Show that $L(q, p) = H(q, p) - V(\bar{q})$ is a Liapunov function.

3° Linearization and the Stability of an Equilibrium Point

We now apply the Liapunov Stability Theorem to give a sufficient condition for asymptotic stability of an equilibrium point $\bar{x}$, based on the eigenvalues of the derivative matrix $Df(\bar{x})$.

Theorem IV.1.2:
Let $\bar{x}$ be an equilibrium point of the DE $\frac{dx}{dt} = f(x)$ in $\mathbb{R}^n$.
If all eigenvalues of the derivative matrix $Df(\bar{x})$ satisfy

$$Re(\lambda) < 0,$$

then the equilibrium point $\bar{x}$ is asymptotically stable.

Proof: Without loss of generality, choose $\bar{x} = 0$.
Consider the linear approximation of $f(x)$ at 0:

$$f(x) = f(0) + Df(0)\,x + g(x),$$

where $g(x)$ is the error. Since 0 is an equilibrium point, the DE assumes the form:

$$\frac{dx}{dt} = Ax + g(x), \tag{2}$$

where $A = Df(0)$. Since $Re(\lambda) < 0$ for all eigenvalues of A, there exists a symmetric, positive definite matrix Q such that

$$A^T Q + QA = -I \tag{3}$$

(by the lemma on p II.5.9). To complete the proof, we show that

$$V(x) = x^T Q x$$

is a strict Liapunov function for $\bar{x} = 0$.

Firstly, $V(0)=0$ and $V(x)>0$ for all $x \neq 0$. Secondly, for any solution $x(t)$, it follows from equations (2) and (3) that

$$\frac{dV(x(t))}{dt} = -x^T x + R(x), \tag{4}$$

where

$$R(x) = 2x^T Q\, g(x),$$

(details as an exercise). It follows that

$$|R(x)| \leq 2M \, \|g(x)\| \, \|x\| \, , \quad \text{for all } x \in \mathbb{R}^n; \tag{5}$$

where M is the largest eigenvalue of Q (details). Since f is of class C^1,

$$\lim_{x \to 0} \frac{\|g(x)\|}{\|x\|} = 0$$

(ie the error tends to zero faster than the displacement; see the theorem in the review notes). Thus for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| < \delta \quad \text{implies} \quad \|g(x)\| < \varepsilon \|x\| .$$

Using equation (5)

$$\|x\| < \delta \quad \text{implies} \quad |R(x)| < 2\varepsilon M \|x\|^2 .$$

Finally, by equation (4)

$$\|x\| < \delta \quad \text{implies} \quad \frac{dV}{dt}(x(t)) < -(1-2\varepsilon M) \|x\|^2$$

Thus if we choose ε so that $\varepsilon < \frac{1}{2M}$, then V is a strict Liapunov function for $\bar{x}=0$, + the Liapunov Stability Theorem implies that $\bar{x}=0$ is an asymptotically stable equilibrium point. □

Section IV.2: Linearization and the Hartman-Grobman Theorem

1° An Example of Linearization in 2-D

Consider the DE in section III.1, 2°:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1(1-x_1) \\[2ex] \dfrac{dx_2}{dt} = -2x_2 \end{cases}$$

We consider the orbits of the linearizations at the equilibrium points $(0,0)$ and $(1,0)$.

Linearization at $(0,0)$:

$$\frac{du}{dt} = Au,$$

$$A = Df(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

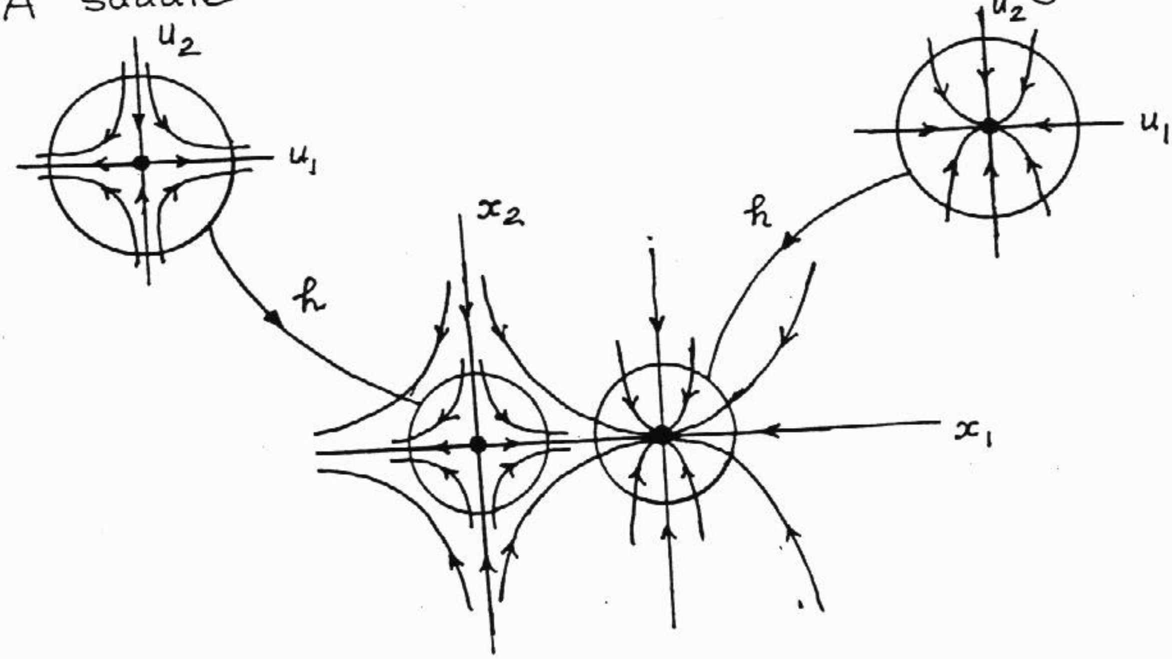
A saddle

Linearization at $(1,0)$:

$$\frac{du}{dt} = Au,$$

$$A = Df(1,0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

An attracting node



For each equilibrium point, there is a homeomorphism h which maps the orbits of the linearized flow in a neighbourhood of O onto the orbits of the non-linear flow in a neighbourhood of the equilibrium point. In other words, the linearizations give a reliable description of the non-linear orbits <u>near the equilibrium points</u>.

2° Examples of the Failure of Linearization

Consider a non-linear DE
$$\frac{dx}{dt} = f(x),$$
where $f: \mathbb{R}^n \to \mathbb{R}^n$. At an equilibrium point $\bar{x}$, the linear approximation is

$$f(x) = Df(\bar{x})(x-\bar{x}) + R_1(x,\bar{x}). \qquad (1)$$

If f is of class C', then
$$\lim_{x\to\bar{x}} \frac{\|R_1(x,\bar{x})\|}{\|x-\bar{x}\|} = 0 \qquad (2)$$
i.e. the error $\|R(x,\bar{x})\|$ tends to zero faster than the displacement
One thus expects that near $\bar{x}$ the orbits of the non-linear DE will be approximated by the orbits of the linearization
$$\frac{du}{dt} = Df(\bar{x})\,u, \qquad u = x-\bar{x},$$

as in the previous example.

Q: Can this expectation be unfulfilled in some situations?

A: YES. Consider the following <u>analogy</u>:

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Suppose that $\bar{x}$ is a critical point of g, i.e. $\nabla g(\bar{x}) = 0$.

Consider the Taylor polynomial approximation of g at $\bar{x}$:

$$g(x) \approx P_{2,\bar{x}}(x) \qquad \text{——} \quad (3)$$

where

$$P_{2,\bar{x}}(x) = g(\bar{x}) + \tfrac{1}{2}(x-\bar{x})^T Hg(\bar{x})(x-\bar{x})$$

and

$$Hg(\bar{x}) = \begin{pmatrix} g_{11}(\bar{x}) & g_{12}(\bar{x}) \\ g_{12}(\bar{x}) & g_{22}(\bar{x}) \end{pmatrix}$$

is the Hessian matrix of g at $\bar{x}$. The second derivative test states that $\bar{x}$ is a local maximum (respectively local minimum, saddle point) of g if and only if $\bar{x}$ is a local maximum (resp. local minimum, saddle point) of $P_{2,\bar{x}}$, subject to a certain restriction, namely

$$\det Hg(\bar{x}) \neq 0$$

However, <u>if $\det Hg(\bar{x}) = 0$, the approximation (3) may fail to describe the behaviour of g near $\bar{x}$</u>.

eg Consider $g(x_1, x_2) = (x_1 + x_2)^2 - x_1^4$.

The critical point $(0,0)$ is a <u>saddle point of g</u>, since $g(0, x_2) > 0$ but $g(-x_2, x_2) < 0$ for $x_2 \neq 0$.
However the Taylor polynomial at $(0,0)$ is

$$P_2(x_1, x_2) = (x_1 + x_2)^2$$

and $(0,0)$ is a <u>local minimum point of P_2</u>.
The approximation fails because

$$\det[Hg(0,0)] = \det\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 0. \qquad \square$$

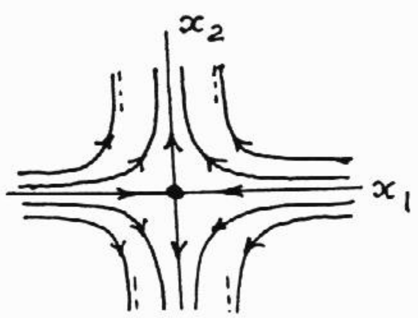
In a similar manner, the linearization of a nonlinear DE can fail to give reliable information about the orbits, if a certain restriction does not hold.
Here are some examples.

<u>Example 1:</u>

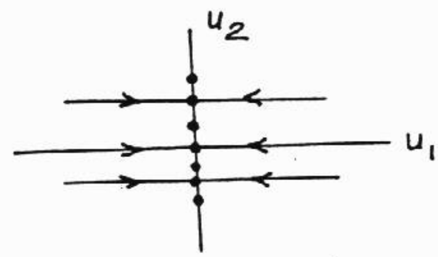
$$\begin{cases} \dfrac{dx_1}{dt} = -x_1 \\ \dfrac{dx_2}{dt} = x_2^3 \end{cases}$$

<u>Linearization at (0,0):</u>

$$\frac{du}{dt} = A, \qquad A = Df(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$



A non-linear saddle
Orbits: $x_1 = K e^{\frac{1}{2x_2^2}}$



An attracting line

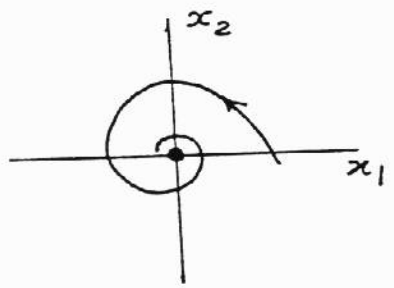
The linear and non-linear flows are not topologically equivalent in a neighbourhood of the equilibrium point, and hence the linearization fails. The source of the failure is that <u>the matrix $Df(0,0)$ has a zero eigenvalue</u>.

<u>Example 2:</u>

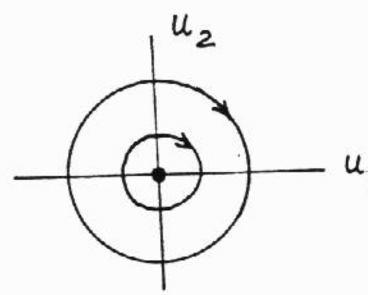
$$\begin{cases} \dfrac{dx_1}{dt} = -x_2 - x_1(x_1^2+x_2^2) \\ \dfrac{dx_2}{dt} = x_1 - x_2(x_1^2+x_2^2) \end{cases}$$

<u>Linearization at (0,0):</u>

$$\frac{du}{dt} = Au, \quad A = Df(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



a non-linear spiral



a centre

Note that $V(x_1,x_2) = x_1^2+x_2^2$ is a strict Liapunov function for the equilibrium point (0,0) which is thus asymptotically stable (non-linear DE).

Again, the linear and non-linear flows are <u>not</u> topologically equivalent in a neighbourhood of the equilibrium points, hence the linearization fails. The source of the failure is that <u>the matrix $Df(0,0)$ has eigenvalues with zero real parts</u>.

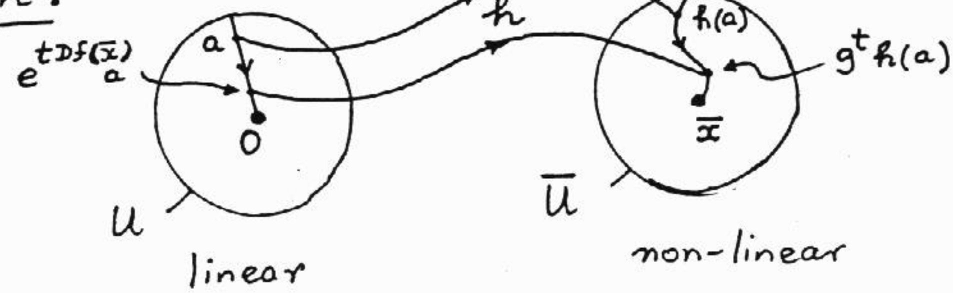
3° The Hartman-Grobman Theorem

The preceding examples motivate the theorem to follow.

Theorem (Hartman-Grobman):
Let $\bar{x}$ be an equilibrium point of the DE
$$\frac{dx}{dt} = f(x) \quad \text{in } \mathbb{R}^n,$$
where $f: \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1. If all eigenvalues of the matrix $Df(\bar{x})$ satisfy $\mathrm{Re}\,\lambda \neq 0$, then there is a homeomorphism $h : \mathcal{U} \to \bar{\mathcal{U}}$ of a neighbourhood $\mathcal{U}$ of 0 onto a neighbourhood $\bar{\mathcal{U}}$ of $\bar{x}$ which maps orbits of the linear flow $e^{tDf(\bar{x})}$ onto orbits of the non-linear flow g^t of the DE, preserving the parameter t.

Proof: Theorem IV.1.2 essentially provides the proof in the case where all eigenvalues satisfy $\mathrm{Re}\,\lambda < 0$, since it gives a neighbourhood $\bar{\mathcal{U}}$ of $\bar{x}$ such that $\omega(a) = \{\bar{x}\}$ for all $a \in \bar{\mathcal{U}}$. The general proof is difficult; and was first given in 1960.
Ref: P. Hartman, Ordinary DEs, 2nd ed., p244-250.

Illustration:



Orbit preservation condition:
$$h \circ e^{tDf(\bar{x})} = g^t \circ h$$

The Hartman-Grosman theorem can be stated more concisely using the concept of topological equivalence, which can be generalized to non-linear flows (see section II.4)

<u>Defn</u>: Two flows g^t and $\tilde{g}^t$ on $\mathbb{R}^n$ are said to be <u>topologically equivalent</u> if there is a homeomorphism $h: \mathbb{R}^n \to \mathbb{R}^n$ which maps orbits of g^t onto orbits of $\tilde{g}^t$, and preserves the direction of the parameter t.

Then we can state:

If $\bar{x}$ is a <u>hyperbolic equilibrium point</u>, then the flow of the DE

$$\frac{dx}{dt} = f(x)$$

and the flow of its linearization

$$\frac{du}{dt} = Df(\bar{x})\, u\,,$$

are <u>locally</u> topologically equivalent.

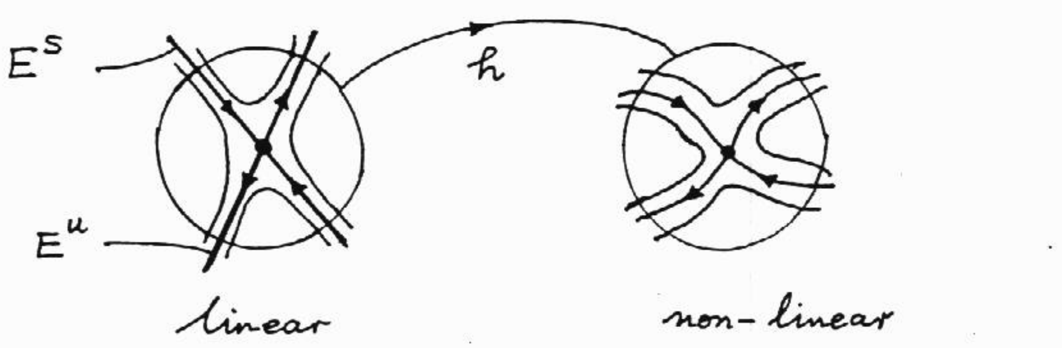
<u>Note</u>: An equilibrium point $\bar{x}$ of a non-linear DE is said to be <u>hyperbolic</u> if all eigenvalues of the matrix $Df(\bar{x})$ satisfy $Re(\lambda) \neq 0$.

Section IV.3: Saddle Points & the Stable Manifold Theorem

Defn: An equilibrium point $\bar{x}$ of a DE $\frac{dx}{dt} = f(x)$ in $\mathbb{R}^n$ is a saddle point if the real parts of the eigenvalues of the matrix $Df(\bar{x})$ are all non-zero, and not all of one sign.

(ie. a saddle point is a hyperbolic (all $\mathrm{Re}\,\lambda \neq 0$) equilibrium point which is neither a sink (all $\mathrm{Re}\,\lambda < 0$) nor a source (all $\mathrm{Re}\,\lambda > 0$).)

The Hartman-Grobman theorem gives a qualitative local description of a (non-linear) saddle. In particular, in $\mathbb{R}^2$ we have the picture below:



The 1-D subspace E^s that is spanned by the eigenvector which corresponds to the eigenvalue $\lambda_1 < 0$ is called the stable subspace of the equilibrium point, and the 1-D subspace that corresponds to $\lambda_2 > 0$ is called the unstable subspace.

The Stable Manifold Theorem, which we discuss in this section gives a more precise description of the orbits near a saddle point.

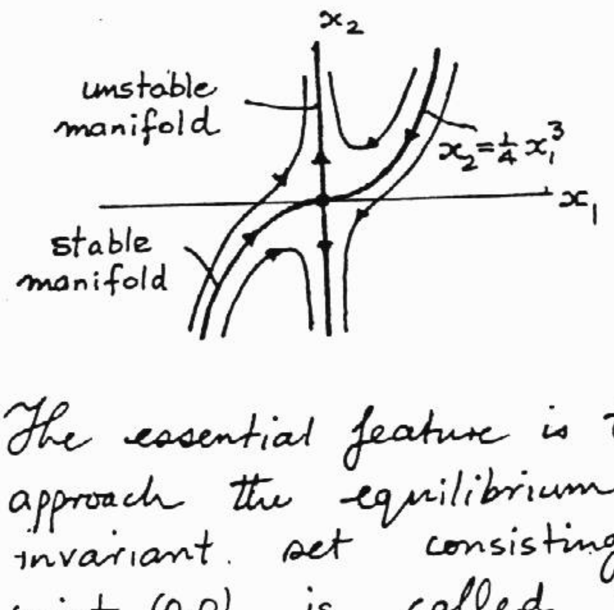
1° An Example of a Non-Linear Saddle in 2-D

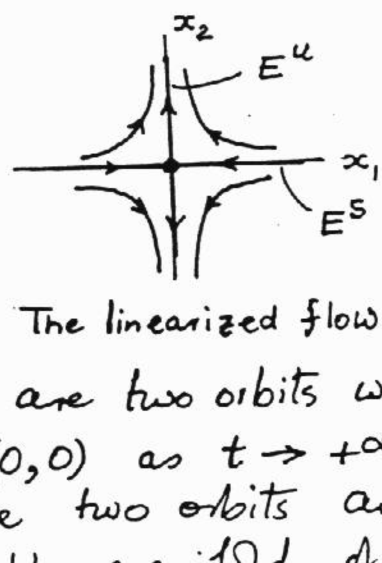
Consider the DE
$$\begin{cases} \dfrac{dx_1}{dt} = -x_1 \\ \dfrac{dx_2}{dt} = x_2 - x_1^3 \end{cases}$$

One can integrate explicitly (exercise) to obtain the flow:
$$g^t \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} e^{-t} a_1 \\ e^t a_2 + \frac{1}{4}(e^{-3t} - e^t) a_1^3 \end{pmatrix}$$

and the orbits:
$$-x_1 x_2 + \frac{1}{4} x_1^4 = C, \qquad C = \text{constant}.$$

The portrait of the orbits is obtained by first sketching the level set $C = 0$.





The linearized flow

The essential feature is that there are two orbits which approach the equilibrium point $(0,0)$ as $t \to +\infty$. The invariant set consisting of these two orbits and the point $(0,0)$ is called the stable manifold of the saddle point $(0,0)$. Similarly, the two orbits which approach $(0,0)$ as $t \to -\infty$ define the unstable manifold of the saddle point.

The essential point is that the stable manifold of the non-linear saddle is a smooth curve which is tangent to the stable subspace of the linear saddle.

2° The Stable Manifold Theorem

Definition: Let $\bar{x}$ be a saddle point of the DE $\frac{dx}{dt} = f(x)$ in $\mathbb{R}$, and let U be a neighbourhood of $\bar{x}$. The local stable manifold of $\bar{x}$ in U is defined by

$$W^s(\bar{x}, U) = \{ x \in U \mid g^t x \xrightarrow[\infty]{t} \bar{x},\ g^t x \in U \text{ for all } t \geq 0 \}$$

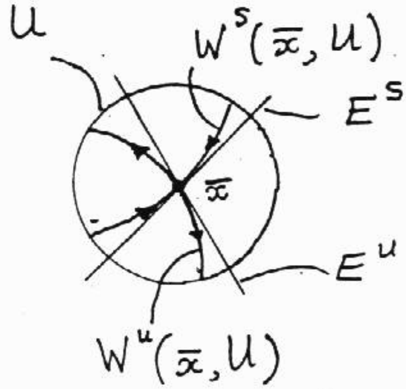
We can now state

Theorem (Stable Manifold Theorem):
Let $\bar{x}$ be a saddle point of $\frac{dx}{dt} = f(x)$ in $\mathbb{R}^2$, where f is of class C^1, and let E^s be the stable subspace of the linearization at $\bar{x}$. Then there exists a neighbourhood U of $\bar{x}$ such that the local stable manifold $W^s(\bar{x}, U)$ is a smooth (C^1) curve which is tangent to E^s at $\bar{x}$.

Proof:

Comment: One can define, in an analogous way, the local unstable manifold of $\bar{x}$ in U (denoted $W^u(\bar{x}, U)$), & there is a similar "Unstable Manifold Theorem".

Diagram:



Section IV.4: Local Behaviour Near a Non-linear Sink

Suppose that $\bar{x}$ is an equilibrium point of a nonlinear DE

$$\frac{dx}{dt} = f(x) \qquad (1)$$

in $\mathbb{R}^2$. Suppose that all eigenvalues of the matrix

$$Df(\bar{x})$$

satisfy $Re(\lambda) < 0$, ie $\bar{x}$ is a sink. The Hartman–Grobman theorem asserts that in some neighbourhood of $\bar{x}$, the flow of the DE (1) is topologically equivalent to the flow of the linearization

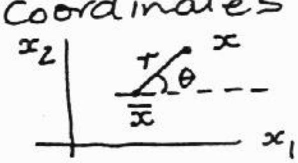
$$\frac{du}{dt} = Df(\bar{x})\,u, \qquad u = x - \bar{x}.$$

In this section we give a more detailed description of the non-linear orbits near a sink.

1° Non-linear Spirals.

Let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be an asymptotically stable equilibrium point of the DE (1). In order to describe the orbits near $\bar{x}$, introduce polar coordinates

$$\begin{cases} x_1 - \bar{x}_1 = r\cos\theta \\ x_2 - \bar{x}_2 = r\sin\theta \end{cases} \qquad \text{—— (2)}$$



Solutions of the DE can be specified by giving

$$r(t) \text{ and } \theta(t)$$

Since $\bar{x}$ is asymptotically stable,

$$\lim_{t\to+\infty} r(t) = 0, \qquad \text{—— (2)}$$

if $r(0)$ is sufficiently close to zero.

We say that the equilibrium point $\bar{x}$ is a <u>non-linear spiral</u> if

$$\lim_{t \to +\infty} \theta(t) = \pm\infty$$

for any solution $(r(t), \theta(t))$ for which (2) holds.

<u>Proposition IV.4.1</u>: Consider the DE

$$\frac{dx}{dt} = f(x) \qquad \text{— (NL)}$$

in $\mathbb{R}^2$, where f is of class C^1. Consider the linearization

$$\frac{du}{dt} = Df(\bar{x})\,u \qquad \text{— (L)}$$

at the equilibrium point $\bar{x}$. If 0 is an attracting spiral point of (L), then $\bar{x}$ is an attracting spiral point of (NL).

<u>Proof</u>: Perform a linear change of coordinates so that $\bar{x} = (0,0)$, and so that $Df(\bar{x})$ is in canonical form:

$$Df(\bar{x}) = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix} \quad \text{with} \quad \alpha > 0.$$

By using the linear approximation, the DE can be written in the form:

$$\frac{dx_1}{dt} = -\alpha x_1 + \beta x_2 + g_1(x_1, x_2) \qquad \text{— (3)}$$

$$\frac{dx_2}{dt} = -\beta x_1 - \alpha x_2 + g_2(x_1, x_2)$$

[see equation (1) on p IV.2.2]. Since f is of class C^1, the non-linear terms satisfy

$$\lim_{(x_1,x_2)\to(0,0)} \frac{g_i(x_1,x_2)}{\sqrt{x_1^2 + x_2^2}} = 0, \quad i = 1,2, \qquad \text{— (4)}$$

[see equation (2) on p IV.2.2]

Introduce polar coordinates according to (2). By using the Chain Rule it follows from equations (3) that

$$x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} = r^2 \frac{d\theta}{dt}$$

and hence that

$$\frac{d\theta}{dt} = -\beta + \frac{x_1 g_2 - x_2 g_1}{r^2} \tag{5}$$

Since O is an attracting spiral point of (L), the Hartman-Grobman theorem (or Theorem IV.1.2) implies that

$$\lim_{t\to\infty} r(t) = 0, \tag{6}$$

if $r(0)$ is sufficiently close to zero. It follows from equation (4) that

$$\lim_{r\to 0} \frac{x_1 g_2 - x_2 g_1}{r^2} = 0$$

and hence from (6) that

$$\lim_{t\to +\infty} \frac{x_1 g_2 - x_2 g_1}{r^2} = 0$$

if $r(0)$ is sufficiently close to zero. Thus equation can be written

$$\frac{d\theta}{dt} = -\beta + o(1) \quad , \quad \text{as } t \to \infty,$$

in terms of the order symbol o. Thus

$$\theta(t) = -\beta t + o(t) = t\left[-\beta + \frac{o(t)}{t}\right]$$

and hence

$$\lim_{t\to\infty} \theta(t) = \pm\infty$$

depending on the sign of β, for any solution $(r(t), \theta(t))$ with $r(0)$ sufficiently close to zero. This establishes that $\bar{x}$ is a non-linear attracting spiral. □

Reference: Coddington & Levinson, page 376, Theorem 2.2

2° Non-linear Nodes

An asymptotically stable equilibrium point is said to be an attracting non-linear node if all orbits, with the exception of two orbits, are tangent to a single line as they approach the equilibrium point.

Referring to Proposition IV.4.1, we have

Proposition IV.4.2: If O is an attracting node of (L) then $\bar{x}$ is an attracting node of (NL).

Proof: The proof involves a more delicate analysis than Prop. IV.3.1. See Coddington & Levinson, p 384, Theorem 5.1.

Remark: A similar result holds for Jordan nodes. See Coddington & Levinson, page 387.

3° Non-linear Focus. (Optional)

An asymptotically stable equilibrium point $\bar{x}$ is said to be an *attracting non-linear focus* if all orbits sufficiently close to $\bar{x}$ approach $\bar{x}$ in a definite direction as $t \to \infty$, and, given any direction, there exists an orbit which tends to $\bar{x}$ in this direction.

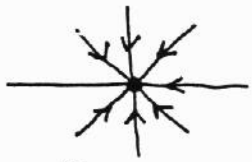
If 0 is a focus of (L), it does not follow in general that $\bar{x}$ is a non-linear focus of (NL)

Example: Consider the DE

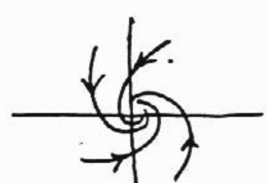
$$\frac{dx_1}{dt} = -x_1 - \frac{x_2}{\log(x_1^2+x_2^2)}, \quad \frac{dx_2}{dt} = -x_2 + \frac{x_1}{\log(x_1^2+x_2^2)},$$

where the vector field is defined to be $(0,0)$ when $(x_1,x_2) = (0,0)$. The vector field is then C^1 on $\mathbb{R}^2$.

The equilibrium point $(0,0)$ is a focus of the linearization, but is a spiral point of the non-linear DE.



linear



nonlinear

To verify the nonlinear behaviour, introduce polar coordinates, obtaining

$$\frac{dr}{dt} = -r, \quad \frac{d\theta}{dt} = \frac{1}{\log r}$$

This disagreement between linear and nonlinear occurs because the vector field is not sufficiently smooth — it is not of class C^2.

Proposition IV.4.3:
Suppose that the vector field f is of class C^2. If O is an attracting focus of (L) then $\bar{x}$ is an attracting focus of NL.

Proof: Coddington + Levinson, page 377.

Exercise: Why does the above example not violate the Hartman-Grobman theorem?

4° The Case of a Centre (Optional)

A stable equilibrium point $\bar{x}$ is said to be a non-linear centre if in some neighbourhood of $\bar{x}$, the orbits are periodic orbits which enclose $\bar{x}$.

Recall that the Hartman-Grobman Thm. does not apply if $(0,0)$ is a centre of the linearization. ie one cannot conclude that $\bar{x}$ is a centre of the non-linear DE. But one can still draw a useful conclusion.

Proposition IV.4.4: If O is a centre of (L), then $\bar{x}$ is either a centre, an attracting spiral, or a repelling spiral of (NL).
Proof: Coddington + Levinson, p382, Theorem 4.1.