DS: Lecture Notes

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1 INTRODUCTION

We shall review systems of *ordinary differential equations* (DEs) of the form

$$x' = f(x) \tag{1}$$

where $x' = \frac{dx}{dt}$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Since the right-hand-side of (1) does not depend on t explicitly, the DE is called *autonomous*. If f is a linear function, i.e.,

$$f(x) = Ax \tag{2}$$

where A is an $n \times n$ matrix of real numbers, the DE is *linear*. In general f will be non-linear. The vector $x \in \mathbb{R}^n$ is called the *state vector* of the system, and $\mathbb{R}^n$ is called the *state space*.

The function f can be interpreted as a *vector field* on the state space $\mathbb{R}^n$, since it associates with each $x \in \mathbb{R}^n$ an element $f(x)$ on $\mathbb{R}^n$, which can be interpreted as a vector

$$f(x) = (f_1(x), \cdots, f_n(x)) \tag{3}$$

situated at x.

Definition. *A* solution *of the (DE) (1) is a function $\psi : \mathbb{R} \to \mathbb{R}^n$ which satisfies*

$$\psi'(t) = f(\psi(t)) \tag{4}$$

for all $t \in \mathbb{R}$ (the domain of ψ may be a finite interval (α, β)).

The image of the solution curve ψ in $\mathbb{R}^n$ is called an *orbit* of the (DE). Equation (4) implies that the vector field f at x is tangent to the orbit through x. The state of the physical system that is being analyzed is represented by a point $x \in \mathbb{R}^n$. The evolution of the system in time is described by the motion of this point along an orbit of the DE in $\mathbb{R}^n$, with t as time. In this interpretation, the DE implies that *the vector field f is the velocity of the moving point in state space* (this should not be confused with the physical velocity of a physical particle).

One cannot hope to find exact solutions of a non-linear DE (1) for $n \geq 2$ (except in very special cases). One thus has to use either qualitative methods, perturbative methods, or numerical methods, in order to deduce the behavior of the physical system. We shall be interested in qualitative methods (in conjunction with 'numerical experimentation'). The aim of *qualitative analysis* is to understand the qualitative behavior of typical solutions of the DE, for example *the long-term behavior* as $t \to \infty$ of typical solutions. One is also interested in exceptional solutions such as *equilibrium solutions* or *periodic solutions*, since such solutions can significantly influence the long-term behavior of typical solutions. One is also interested in questions of *stability* and the possible existence of *bifurcations*.

The starting point in the qualitative analysis of an autonomous DE (1) in $\mathbb{R}^n$ is to locate the zeros of the vector field, i.e., to find all $a \in \mathbb{R}^n$ such that

$$f(a) = 0 \tag{5}$$

If $f(a) = 0$, then $\psi(t) = a$, for all $t \in \mathbb{R}$, and it is a solution of the DE, since

$$\psi'(t) = f(\psi(t)) \tag{6}$$

is satisfied trivially for all $t \in \mathbb{R}$. A constant solution $\psi(t) = a$ describes an *equilibrium state* of the physical system, and hence the point $a \in \mathbb{R}^n$ is called an *equilibrium point* of the DE. Here is the official definition.

Definition. *Given a DE $x' = f(x)$ in $\mathbb{R}^n$, any point $a \in \mathbb{R}^n$ which satisfies $f(a) = 0$, is called an* equilibrium point *of the DE.*

We are interested in the stability of equilibrium states. In order to address this question it is necessary to study the behaviour of the orbits of the DE close to the equilibrium points. The idea is to consider the linear approximation of the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ at an equilibrium point. We thus assume that the function f is of class $C^1(\mathbb{R}^n)$ (i.e., that the partial derivatives of f exist and are continuous functions on $\mathbb{R}^n$.)

Definition. *The* derivative matrix *of $f : \mathbb{R}^n \to \mathbb{R}^n$ is the $n \times n$ matrix $\mathrm{D}f(x)$ defined by*

$$\mathrm{D}f(x) = \left(\frac{\partial f_i}{\partial x_j}\right), \quad i, j = 1, \cdots, n, \tag{7}$$

where the f_i are the component functions of f.

The *linear approximation* of f is written in terms of the derivative matrix:

$$f(x) = f(a) + \mathrm{D}f(a)(x - a) + R_1(x, a), \tag{8}$$

where $\mathrm{D}f(a)(x-a)$ denotes the $n \times n$ derivative matrix evaluated at a, acting on the vector $(x-a)$, and $R_1(x,a)$ is the *error term*. An important result from advanced calculus is that if f is of class C^1, then the magnitude of the error $\|R_1(x,a)\|$ tends to zero faster than the magnitude of the displacement $\|x-a\|$. Here $\|\cdots\|$ denotes the Euclidean norm on $\mathbb{R}^n$ (i.e., $\|x\| = \sqrt{{x_1}^2 + \cdots + {x_n}^2}$). This means that in general, $R_1(x,a)$ will be small compared to $\mathrm{D}f(a)(x-a)$, for x sufficiently close to a.

If $a \in \mathbb{R}^n$ is an equilibrium point of the DE $x' = f(x)$, we can use (8) to write the DE in the form

$$(NL): \qquad x' = \mathrm{D}f(a)(x-a) + R_1(x,a) \tag{9}$$

assuming that f is of class C^1. We let $u = x - a$, and with the non-linear DE (NL) we associate the linear DE

$$(L): \qquad u' = \mathrm{D}f(a)u \tag{10}$$

which is called *the linearization of (NL) at the equilibrium point* $a \in \mathbb{R}^n$. The question is when do solutions of (L) approximate the solutions of (NL) near $x = a$? i.e., under what conditions can we neglect the error term $R_1(x,a)$? In general the approximation is valid, but in special situations, the approximation can fail. We thus begin with a systematic study of *linear* DEs.

2 LINEAR AUTONOMOUS DIFFERENTIAL EQUATIONS

The initial value problem in one dimension (i.e., A is a 1×1 matrix or equivalently a constant):

$$x' = Ax, \qquad x(0) = a \in \mathbb{R} \tag{11}$$

has the unique solution

$$x(t) = \mathrm{e}^{tA}a, \qquad \text{all } t \in \mathbb{R}. \tag{12}$$

By analogy, we define the *matrix series*:

$$\mathrm{e}^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k \tag{13}$$

called the *exponential* e^A of A, where A is an $n \times n$ real matrix, I is the $n \times n$ identity matrix and $A^2 = A\,A$ etc. (matrix product). A matrix series is said to *converge* if the n^2 infinite series corresponding to the n^2 entries converge in $\mathbb{R}$. The exponential matrix e^A converges for all $n \times n$ matrices A [cf. Hirsch and Smale, page 83, [1]].

We recall that

$$\mathrm{e}^{s+t} = \mathrm{e}^s\,\mathrm{e}^t, \qquad \text{for all } s,t \in \mathbb{R} \tag{14}$$

(which is proved by using the Taylor series as the definition of e^s, and application of the Binomial Theorem and the Cauchy product for absolutely convergent series). The result (14) does not go over to $n \times n$ matrices due to the general non-commutativity of such matrices.

Proposition 1. *If A and B are $n \times n$ real matrices, and $A\,B = B\,A$, then*

$$e^{A+B} = e^A\, e^B \tag{15}$$

Corollary. *If A is an $n \times n$ real matrix, then e^A is invertible and $\left(e^A\right)^{-1} = e^{-A}$.*

In order to be able to calculate e^A for any matrix, it is necessary to simplify A by performing a similarity transformation:

$$B = P^{-1}\,A\,P \tag{16}$$

where P is a non-singular matrix (this corresponds to a change of basis).

Proposition 2. *If $B = P^{-1}\,A\,P$ then $e^B = P^{-1}\,e^A\,P$*

Proof. Simplify $(P^{-1}\,A\,P)^k$. □

Proposition 3 (Jordan Canonical Form). *For any 2×2 real matrix A, there exists a non-singular matrix P such that*

$$J = P^{-1}\,A\,P \tag{17}$$

and J is one of the following matrices:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \tag{18}$$

Proof. We give an algorithm for constructing P in three mutually exclusive cases, which include all possible 2×2 real matrices; as follows

CASE I: A has two linearly independent eigenvectors f_1, f_2 with eigenvalues λ_1, λ_2. Choose $P = (f_1, f_2)$, i.e., f_1, f_2 are the columns of P. Then $A\,P = (Af_1, Af_2) = (\lambda_1 f_1, \lambda_2 f_2) = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Hence $P^{-1}\,A\,P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

CASE II: A has only one eigenvector, with eigenvalue λ. This implies that $(A - \lambda I)^2 = 0$. Choose f_2 such that $(A - \lambda I)f_2 \neq 0$, and let $f_1 = (A - \lambda I)f_2$. Then $\{f_1, f_2\}$ is a basis of $\mathbb{R}^2$, and $(A - \lambda I)f_1 = 0$. Choose $P = (f_1, f_2)$. Then $A\,P = (Af_1, Af_2) = (\lambda f_1, \lambda f_2 + f_1) = P \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

CASE III: A admits no real eigenvector, but admits a complex eigenvector: $A(f_1 + if_2) = (\alpha + i\beta)(f_1 + if_2)$, i.e., $Af_1 = \alpha f_1 - \beta f_2$, $Af_2 = \beta f_1 + \alpha f_2$. Choose $P = (f_1, f_2)$. Then $A\,P = (Af_1, Af_2) = (\alpha f_1 - \beta f_2, \beta f_1 + \alpha f_2) = P \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. □

We now have a complete algorithm for calculating e^A for any 2×2 real matrix A:

(a) find the Jordan canonical form $J = P^{-1} A P$,

(b) calculate e^J,

(c) then $e^A = P e^J P^{-1}$.

The Flow of a Linear DE

Theorem (Fundamental Theorem for Linear Autonomous DEs). *Let A be an $n \times n$ real matrix. Then the initial value problem*

$$x' = Ax, \qquad x(0) = a \in \mathbb{R}^n \tag{19}$$

has the unique solution

$$x(t) = e^{tA} a, \qquad \textit{for all } t \in \mathbb{R}. \tag{20}$$

Proof. 1) Existence: Let $x(t) = e^{tA} a$ then

$$\frac{dx}{dt} = \frac{d\left(e^{tA} a\right)}{dt} = A e^{tA} a = Ax \tag{21}$$

$$x(0) = e^0 a = Ia = a \tag{22}$$

shows that $x(t)$ satisfies the initial value problem (19).
2) Uniqueness: Let $x(t)$ be any solution of (19). It follows that

$$\frac{d}{dt}\left[e^{-tA} x(t)\right] = 0 \tag{23}$$

Thus $e^{-tA} x(t) = C$, a constant. The initial condition implies that $C = a$ and hence $x(t) = e^{tA} a$. □

The unique solution of the DE (19) is given by (20) for all t. Thus, for each $t \in \mathbb{R}$, the matrix e^{tA} maps

$$a \to e^{tA} a \tag{24}$$

(where a is the state at time $t = 0$ and e^{tA} is the state at time t). The set $\{e^{tA}\}_{t \in \mathbb{R}}$ is a 1-parameter family of linear maps of $\mathbb{R}^n$ into $\mathbb{R}^n$, and is called the *linear flow* of the DE.

We write

$$g^t = e^{tA} \tag{25}$$

to denote the flow. *The flow describes the evolution in time of the physical system for all possible initial states.* As the physical system evolves in time, one can think of the state vector x as a moving point in state space, its motion being determined by the flow $g^t = e^{tA}$. The linear flow satisfies two important properties, which also hold for non-linear flows (to follow).

Proposition 4. *The linear flow* $g^t = e^{tA}$ *satisfies*

$$\begin{array}{llr} F1: & g^0 = I & \textit{(identity map)} \\ F2: & g^{t_1+t_2} = g^{t_1} \circ g^{t_2} & \textit{(composition)} \end{array} \tag{26}$$

Proof. Easy consequence of Proposition 1. □

Comment: Properties $F1$and $F2$ imply that the flow $\{g^t\}_{t\in\mathbb{R}}$ forms a *group* under composition of maps.

The flow g^t of the DE (19) partitions the state space $\mathbb{R}^n$ into subsets called *orbits*, defined by

$$\gamma(a) = \left\{ g^t a | t \in \mathbb{R} \right\}. \tag{27}$$

The set $\gamma(a)$ is called the *orbit of the DE through a*. It is the image in $\mathbb{R}^n$ of the solution curve $x(t) = e^{tA}a$. It follows that for $a, b \in \mathbb{R}^n$, either $\gamma(a) = \gamma(b)$ or $\gamma(a) \cap \gamma(b) = \emptyset$, since otherwise the uniqueness of solutions would be violated.

Example. *Consider* $x' = Ax$, $x \in \mathbb{R}^2$, *and* $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, *the linear flow is* $e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. *The action of the flow on* $\mathbb{R}^2$, $a \to e^{tA}a$ *corresponds to a* clockwise rotation about the origin. *Thus if* $a \neq 0$, *the orbit* $\gamma(a)$ *is a circle centered at the origin passing through* a. *The origin is a* fixed point *of the flow, since* $e^{tA}0 = 0$, *for all* $t \in \mathbb{R}$. *The orbit* $\gamma(0) = \{0\}$ *is called a* point orbit. *All other orbits are called* periodic orbits *since* $e^{2\pi A}a = a$, *i.e., the flow maps onto itself after a time* $t = 2\pi$ *has elapsed.*

Example. *Consider* $x' = Ax$, $x \in \mathbb{R}^2$, *and* $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, *then the linear flow is* $e^{tA} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. *The action of the flow on* $\mathbb{R}^2$, $a \to e^{tA}a$, *expands* a_1 *exponentially and contracts* a_2 *exponentially, leaving the product* a_1a_2 *constant. Thus if* a *is not on one of the axes, the orbit* $\gamma(a)$ *is a hyperbola. If* $a \neq 0$ *lies on one of the axes, then the orbit* $\gamma(a)$ *is a half-axis. The origin is again a point orbit, and all other orbits are* non-periodic, i.e., $e^{tA}a \neq a$ *for all* $t \neq 0$, *and* $a \neq 0$. *Note that the* x_1*-axis is the union of three orbits*

$$\{(x_1, 0) | x_1 > 0\} \cup \{0\} \cup \{(x_1, 0) | x_1 < 0\}.$$

Classification of orbits of a DE.

1. *If* $g^t a = a$ *for all* $t \in \mathbb{R}$, *then* $\gamma(a) = \{a\}$ *and it is called a* point orbit. *Point orbits correspond to equilibrium points.*

2. *If there exists a* $T > 0$ *such that* $g^T a = a$, *then* $\gamma(a)$ *is called a* periodic orbit. *Periodic orbits describe a system that evolves periodically in time.*

3. *If $g^t a \neq a$ for all $t \neq 0$, then $\gamma(a)$ is called* a non-periodic orbit.

Comment:

1. Non-periodic orbits can be of great complexity even for linear DEs if $n > 3$ (for non-linear DEs if $n > 2$).

2. A *solution curve* of a DE is a parameterized curve and hence *contains information about the flow of time t*. The *orbits* are paths in state space (or subsets of state space). Orbits which are not point orbits are *directed paths* with the direction defined by increasing time. The orbits thus do no provide detailed information about the flow of time.

For an autonomous DE, the slope of the solution curves depend only on x and hence the tangent vectors to the solution curves define a vector field $f(x)$ in x–space. Infinitely many solution curves may correspond to a single orbit. On the other hand, a non-autonomous DE does not define a flow or a family of orbits.

Canonical Linear Flows in $\mathbb{R}^2$ under Linear Equivalence.

To what extent can a linear DE in $\mathbb{R}^n$ be simplified by making a linear change of coordinates and a linear change of the time variable? Given a linear DE $x' = Ax$ in $\mathbb{R}^n$, introduce new coordinates by $y = Px$, where P is a non-singular matrix, and a new time variable $\tau = kt$, where k is a positive constant. It follows that $y' = By$, where $B = \frac{1}{k} P\,A\,P^{-1}$.

Definition. *The linear DEs $x' = Ax$ and $x' = Bx$ are said to be* linearly equivalent *if there exists a non-singular matrix P and a positive constant k such that*

$$A = kP^{-1}\,B\,P. \tag{28}$$

Proposition 5. *The linear DEs*

$$x' = Ax, \qquad x' = Bx, \tag{29}$$

are linearly equivalent if and only if there exists an invertible matrix P and a positive constant k such that

$$P\,e^{tA} = e^{ktB}\,P, \qquad \text{for all } t \in \mathbb{R}. \tag{30}$$

Proof. Exponentiation of (29) and differentiation of (30). □

The condition (30), which characterizes linear equivalence, ensures that the linear map P maps each orbit of the flow e^{tA} onto an orbit of the flow e^{tB}.

Definition. *Two linear flows* e^{tA} *and* e^{tB} *on* $\mathbb{R}^n$ *are said to be* linearly equivalent *if there exists a non-singular matrix* P *and a positive constant* k *such that*

$$P\, e^{tA} = e^{ktB}\, P, \qquad \textit{for all } t \in \mathbb{R}. \tag{31}$$

Let us consider three cases corresponding to the three Jordan canonical forms for any 2×2 real matrix A (see Proposition 3.)

CASE I: TWO EIGENDIRECTIONS. By Proposition 3, there exists a matrix P such that $J = P\, A\, P^{-1}$, where

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

It follows that the given DE is linearly equivalent to $y' = Jy$. The flow is

$$e^{tJ} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}.$$

and the eigenvectors are $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The solutions are $y(t) = e^{tJ} b$, $b \in \mathbb{R}^2$, i.e., $y_1 = e^{\lambda_1 t} b_1$ and $y_2 = e^{\lambda_2 t} b_2$. On eliminating t, we obtain

$$\left(\frac{y_1}{b_1}\right)^{\frac{1}{\lambda_1}} = \left(\frac{y_2}{b_2}\right)^{\frac{1}{\lambda_2}}, \quad \text{if } b_1 b_2 \neq 0 \tag{32}$$

$$y_1 = 0, \qquad \text{if } b_1 = 0 \tag{33}$$

$$y_2 = 0, \qquad \text{if } b_2 = 0 \tag{34}$$

These equations define the orbits of the DE $y' = Jy$.

Ia. $\lambda_1 = \lambda_2 < 0$: Attracting Focus (See Fig. 1.)

Ib. $\lambda_1 < \lambda_2 < 0$: Attracting Node (See Fig. 2.)

Ic. $\lambda_1 < \lambda_2 = 0$: Attracting Line (See Fig. 3.)

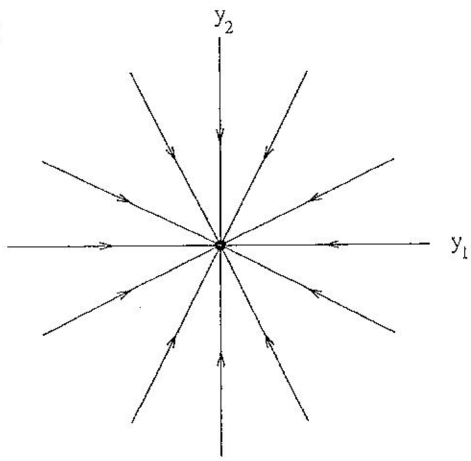
Id. $\lambda_1 < 0 < \lambda_2$: Saddle (See Fig. 4.)

Ie. $\lambda_1 = 0 < \lambda_2$: Repelling Line (time reverse of Fig 3.)

If. $0 < \lambda_1 < \lambda_2$: Repelling Node (time reverse of Fig 2.)

Ig. $0 < \lambda_1 = \lambda_2$: Repelling Focus (time reverse of Fig 1.)

Ia



Ib

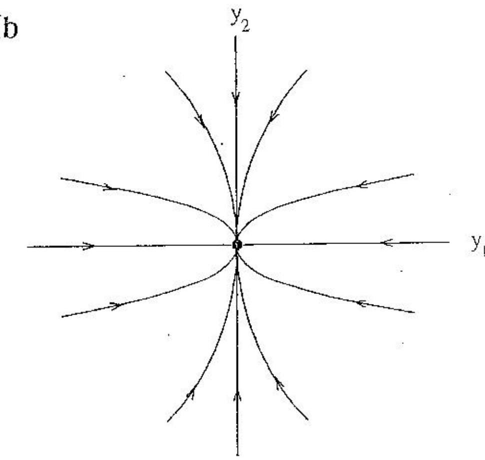
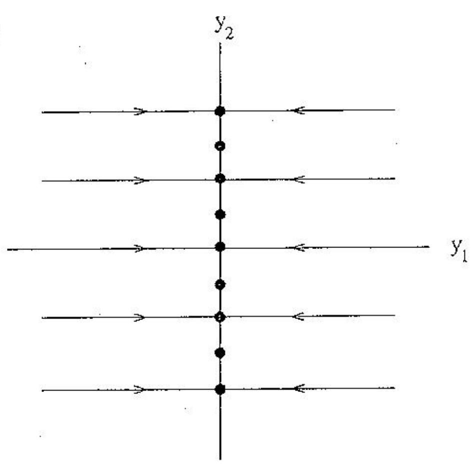


Figure 1: **Ia.** Attracting Focus

Figure 2: **Ib.** Attracting Node

Ic



Id

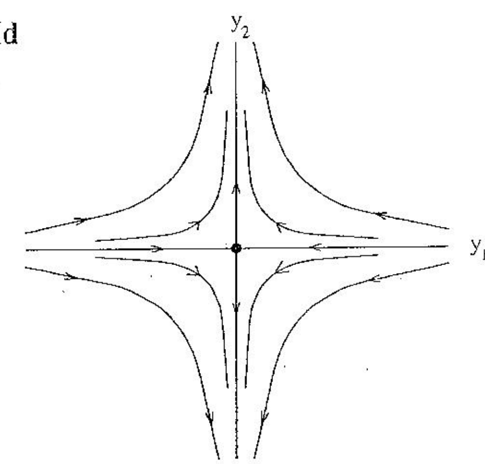


Figure 3: **Ic.** Attracting Line

Figure 4: **Id.** Saddle

CASE II: ONE EIGENDIRECTION. By Proposition 3, there exists a matrix P such that $J = P\,A\,P^{-1}$, where

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

It follows that the given DE is linearly equivalent to $y' = Jy$. The flow is

$$e^{tJ} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

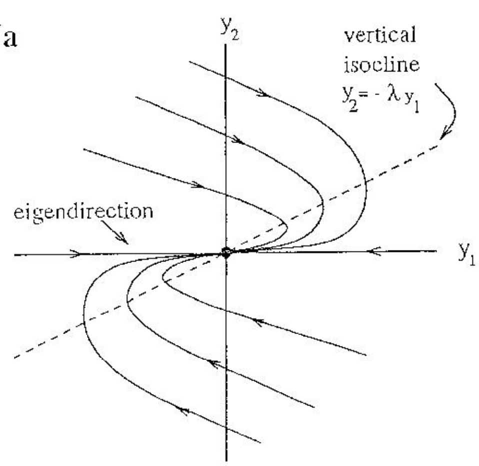
and the single eigenvector is $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We note that if $\lambda \neq 0$, the orbits are given by

$$\begin{aligned} y_1 &= y_2 \left[\frac{b_1}{b_2} + \frac{1}{\lambda} \log \frac{y_2}{y_1} \right], \quad \text{if } b_2 \neq 0 \\ y_2 &= 0, \quad \text{if } b_2 = 0 \end{aligned} \tag{35}$$

IIa. $\lambda < 0$: Attracting Jordan Node (See Fig. 5.)

IIb. $\lambda = 0$: Neutral Line (See Fig. 6.)

IIc. $\lambda > 0$: Repelling Jordan Node (time reverse of Fig. 5.)



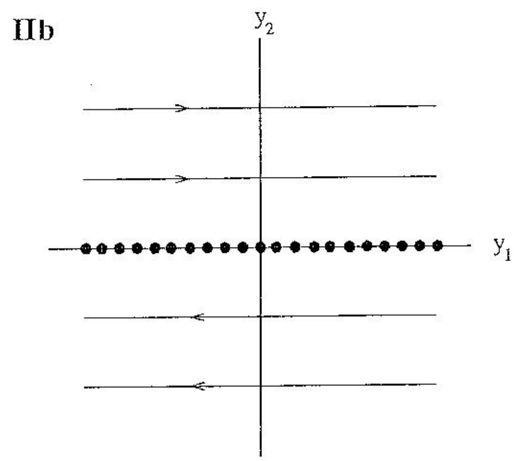


Figure 5: **IIa.** Attracting Jordan Node

Figure 6: **IIb.** Neutral Line

CASE III: NO EIGENDIRECTIONS. By Proposition 3, there exists a matrix P such that $J = PAP^{-1}$, where

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

It follows that the given DE is linearly equivalent to $y' = Jy$. The simplest way to find the orbits is to introduce polar coordinates (r, θ): $y_1 = r\cos\theta$, and $y_2 = r\sin\theta$. The DE becomes $r' = \alpha r$ and $\theta' = -\beta$. It follows that $\frac{dr}{d\theta} = -\frac{\alpha}{\beta} r$ which can be integrated to yield $r = r_0 e^{-\frac{\alpha}{\beta}(\theta - \theta_0)}$. Without loss of generality, we can assume $\beta > 0$, since the DE is invariant under the changes $(\beta, y_1) \to (-\beta, -y_1)$. Thus $\lim_{t\to\infty} \theta = -\infty$ (counterclockwise rotation as t increases).

IIIa. $\alpha < 0$: Attracting Spiral (See Fig. 7.)

IIIb. $\alpha = 0$: Centre (See Fig. 8.)

IIIc. $\alpha > 0$: Repelling Spiral (time reverse of Fig. 7.)

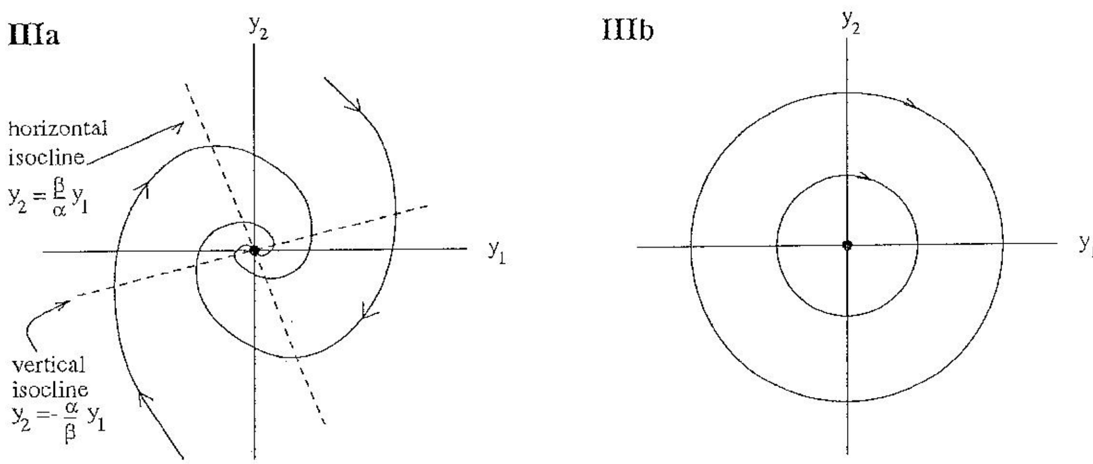


Figure 7: **IIIa.** Attracting Spiral

Figure 8: **IIIb.** Centre

In terms of the Jordan canonical form of two matrices A and B, the corresponding DEs are linearly equivalent if and only if

- A and B have the same number of eigendirections.
- The eigenvalues of A are a multiple (k) of the eigenvalues of B.

This implies that if the DEs have different canonical forms (i.e., belong to different classes Ia–g, IIa–c, IIIa–c) then the DEs are not linearly equivalent. On the other hand, if the DEs have the same canonical form, they will be linearly equivalent if and only if the eigenvalues of A are a multiple of the eigenvalues of B.

Example.

Given DE

$$x' = Ax, \qquad A = \begin{pmatrix} -4 & -3 \\ 2 & 1 \end{pmatrix}$$

(See Fig. 9a)

Eigendirections:

$$u_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \; u_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Change of variables:

Canonical DE

$$y' = Jy, \qquad J = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

(See Fig. 9b)

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$y = Px, \qquad P = \begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}, \qquad A = P^{-1} J P$$

The flows e^{tA} and e^{tJ} are related by $P\,e^{tA} = e^{tJ}\,P$. This implies that the map of the $x_1\,x_2$–plane into the $y_1\,y_2$–plane that is defined by the matrix P, maps orbits of the flow e^{tA} to e^{tJ}.

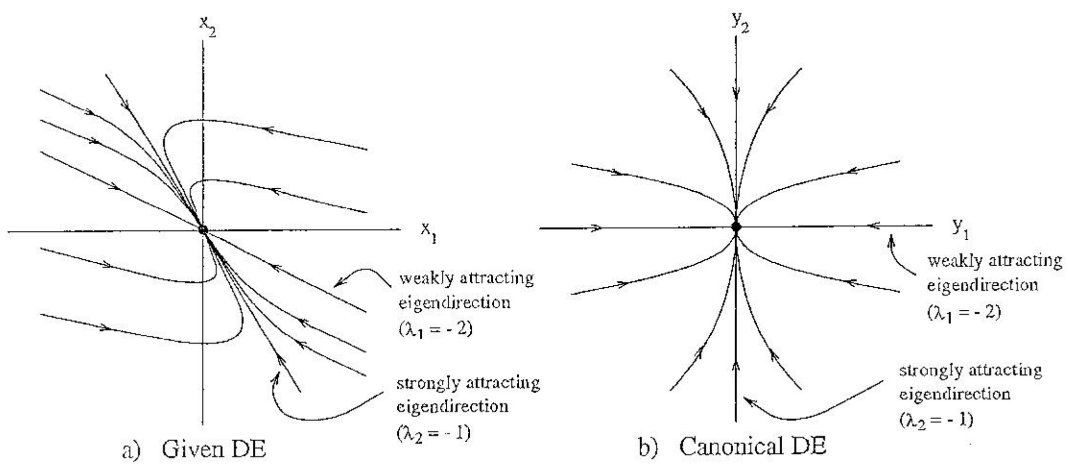


Figure 9: Phase portraits for the Given DE and for the Canonical DE.

Topological Equivalence

We have seen that under linear equivalence, 2–D flows can be simplified to the extent that they can be parameterized by one real-valued parameter and several discrete parameters (e.g., number of independent eigenvectors). Linear equivalence thus acts as a filter, which retains only certain essential features of the flow i.e., the behavior of the orbits near the equilibrium point $(0,0)$. On the other hand, if one is primarily interested in long-term behavior, one can use a finer filter, which eliminates more features, and hence leads to a much simpler (but coarser classification). This is the notion of *Topological Equivalence* of linear flows.

For example, cases Ia, Ib, IIa, and IIIa have the common characteristic that all orbits approach the origin (an equilibrium point) as $t \to \infty$. We would like these flows to be "equivalent" in some sense. We shall show that in fact for all flows of these types, *the orbits of one flow can be mapped onto the orbits of the simplest flow* Ia, using a (non-linear) map $h : \mathbb{R}^2 \to \mathbb{R}^2$: which is a *homeomorphism on* $\mathbb{R}^2$.

Definition. *A map* $h : \mathbb{R}^n \to \mathbb{R}^n$ *is a* homeomorphism on $\mathbb{R}^n$ *if and only if*

1. *h is one-to-one and onto,*
2. *h is continuous,*
3. *h^{-1} is continuous.*

Definition. *Two linear flows* e^{tA} *and* e^{tB} *on* $\mathbb{R}^n$ *are said to be* topologically equivalent *if there exists a homeomorphism* h *on* $\mathbb{R}^n$ *and a positive constant* k *such that*

$$h\left(\mathrm{e}^{tA}x\right) = \mathrm{e}^{ktB}h(x), \qquad \text{for all } x \in \mathbb{R}^n \text{ and for all } t \in \mathbb{R}. \tag{36}$$

Example. *The linear flows* e^{tA}, $A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ *and* e^{tB}, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *are topologically equivalent. The homeomorphism* $h : \mathbb{R}^2 \to \mathbb{R}^2$ *is given by* $y = h(x) = \begin{pmatrix} h_1(x_1) \\ h_2(x_2) \end{pmatrix}$, *where* $h_1 : \mathbb{R} \to \mathbb{R}$ *and* $h_2 : \mathbb{R} \to \mathbb{R}$ *are defined by*

$$h_1(x_1) = \begin{cases} \sqrt{x_1}, & \text{if } x_1 \geq 0 \\ -\sqrt{-x_1}, & \text{if } x_1 < 0 \end{cases}, \qquad h_2(x_2) = x_2.$$

Definition. *A* hyperbolic *linear flow in* $\mathbb{R}^2$ *is one in which the real parts of the eigenvalues are all non-zero (i.e.,* $\Re e(\lambda_i) \neq 0, \quad i = 1, 2$*).*

Proposition 6. *Any* hyperbolic *linear flow in* $\mathbb{R}^2$ *is topologically equivalent to the linear flow* e^{tA}, *where* A *is one of the following matrices:*

1. $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, *the standard sink.*
2. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *the standard source.*

3. $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, *the standard saddle.*

Proof. Establish appropriate homeomorphisms. □

As regards to the *non-hyperbolic* linear flows in $\mathbb{R}^2$ one can infer by inspection of the portraits that none of the 5 canonical flows [i.e., the centre, the attracting and repelling line, and neutral 2–space ($A = 0$)] are topologically equivalent (their asymptotic behavior as $t \to \infty$ differs). Thus, two non-hyperbolic linear flows in $\mathbb{R}^2$ are topologically equivalent if and only if they are linearly equivalent.

Proposition 7. *Any non-hyperbolic linear flow in $\mathbb{R}^2$ is linearly (and hence topologically) equivalent to the flow e^{tA}, where A is one of the following matrices:*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{37}$$

These five flows are topologically inequivalent.

Linear Stability and Linear Sinks in $\mathbb{R}^n$

Suppose that a physical system in an equilibrium state is disturbed. Does it *remain close to* (stable) or *approach* (asymptotically stable) the equilibrium state as time passes ($t \to \infty$)?

Definition.

1. *The equilibrium point 0 of a linear DE $x' = Ax$ in $\mathbb{R}^n$ is* stable *if for all neighborhoods U of 0, there exists a neighborhood V of 0 such that $g^t V \subseteq U$ for all $t \geq 0$, where $g^t = e^{tA}$ is the flow of the DE.*

2. *The equilibrium point 0 of a linear DE $x' = Ax$ in $\mathbb{R}^n$ is* asymptotically stable *if it is stable and if, in addition, for all $x \in V$, $\lim_{t\to\infty} \|g^t x\| = 0$.*

Proposition 8. *Let $A \in M_n(\mathbb{R})$. Then each entry of the matrix e^{tA} is a unique linear combination of the functions*

$$t^k e^{\alpha t} \cos(\beta t), \qquad t^k e^{\alpha t} \sin(\beta t), \tag{38}$$

where $\alpha + i\beta$ runs through all eigenvalues of A with $\beta \geq 0$ ($\beta = 0$ gives the real eigenvalues) and k takes on all values $0, 1, 2, \cdots, n-1$ less than the multiplicity of the corresponding eigenvalue.

Proof. [cf. Hirsch and Smale, page 135, [1]] □

Proposition 9 (Characterization of a Sink). *Let $A \in M_n(\mathbb{R})$. Then*

$$\lim_{t\to\infty} e^{tA}a = 0 \qquad \text{for all } a \in \mathbb{R}^n \tag{39}$$

if and only if $\Re e(\lambda) < 0$ for all eigenvalues of A.

Proof. $\Rightarrow$ Suppose that $\Re e(\lambda) < 0$, then (39) follows from Proposition 8. $\Leftarrow$ Suppose (39) and that $\lambda = \alpha + i\beta$ is an eigenvalue of A with $\alpha \geq 0$, then we obtain a contradiction using Proposition 8. □

Equation (39) means that if $\Re e(\lambda) < 0$ then *all* solutions $x(t)$ of the DE $x' = Ax$ approach the equilibrium 0 in the long term, i.e.,

$$\lim_{t\to\infty} x(t) = 0 \in \mathbb{R}^n \tag{40}$$

Thus if $A \in M_n(\mathbb{R})$ is such that $\Re e(\lambda) < 0$ for all eigenvalues, then we say that the equilibrium point 0 of the DE $x' = Ax$ is a *sink* in $\mathbb{R}^n$. If we replace A by $-A$ and t by $-t$, we obtain the time reverse of Proposition 9. Thus if $A \in M_n(\mathbb{R})$ is such that $\Re e(\lambda) > 0$ for all eigenvalues, then we say that the equilibrium point 0 of the DE $x' = Ax$ is a *source* in $\mathbb{R}^n$.

Proposition 10 (Exponential Attraction to a Sink). *Let $A \in M_n(\mathbb{R})$. If there exists a constant k such that all eigenvalues of A satisfy $\Re e(\lambda) < -k < 0$ then there exists a positive constant M such that*

$$\left\| e^{tA}x \right\| \leq M e^{-kt}\|x\| \qquad \text{for all } x \in \mathbb{R}^n, \quad \text{for all } t \geq 0 \tag{41}$$

Proof. From Proposition 8 and the fact that for any $\epsilon > 0$ and $n > 0$, there exists a constant C such that $t^n < Ce^{\epsilon t}$ for all $t \geq 0$. □

Corollary. *If the equilibrium point $0 \in \mathbb{R}^n$ is a sink of the DE $x' = Ax$, then 0 is an asymptotically stable equilibrium point.*

Although Proposition 10 guarantees that any initial state is attracted at an exponential rate in time to a linear sink (the equilibrium point 0), it does not imply that the distance from 0, i.e., $\left\| e^{tA}x \right\|$ decreases monotonically with t. In other words, as the orbits approach 0, they do not necessarily cut the *spheres* $\|x\| = R$ in the inward direction. However, as one might expect, one can find a family of concentric *ellipsoids*, such that as the orbits approach 0, they intersect the ellipsoids in the inward direction.

Proposition 11. *If the equilibrium point 0 of the DE $x' = Ax$ in $\mathbb{R}^n$ is a sink, then there exists a positive definite quadratic form*

$$V(x) = x^T Q x \tag{42}$$

which is monotone decreasing along all orbits, except for the equilibrium point 0. (Note: *Q is a $n \times n$ symmetric matrix such that $V(x) = x^T Q x > 0$ for all $x \neq 0 \in \mathbb{R}^n$. The level sets $V(x) = C > 0$ are ellipsoids in $\mathbb{R}^n$*).

Proof. Differentiate (42) and use Liapunov's Lemma. □

Let us state Liapunov's Lemma without proof.

Liapunov's Lemma. *Let $A \in M_n(\mathbb{R})$. If all eigenvalues of A satisfy $\Re e(\lambda) < 0$, then there exists a symmetric positive definite matrix Q such that*

$$A^T Q + Q A = -I \tag{43}$$

Comments:

1. The matrix Q in Liapunov's lemma can be found explicitly by solving the linear system of equations (43) for Q.
2. The function V in Proposition 11 is an example of a *Liapunov function* for the equilibrium point 0. Such functions will play an important role when we discuss non-linear stability later.
3. Proposition 11 can also be used to prove that any linear sink in $\mathbb{R}^n$ is topologically equivalent to the standard sink.

3 NON-LINEAR DIFFERENTIAL EQUATIONS

For non-linear DEs, one does not expect to be able to write down the flow explicitly. Indeed the aim of the subject of dynamical systems is to describe the qualitative properties of a non-linear flow *without knowing the flow explicitly.*

Example. We shall illustrate a quick way to draw the orbits for a 1-D DE. *Consider the 1-D DE $x' = f(x) = x(1-x)$, it is the sign of $f(x)$ that determines the direction of each orbit which corresponds to increasing t. (See Fig. 10)*

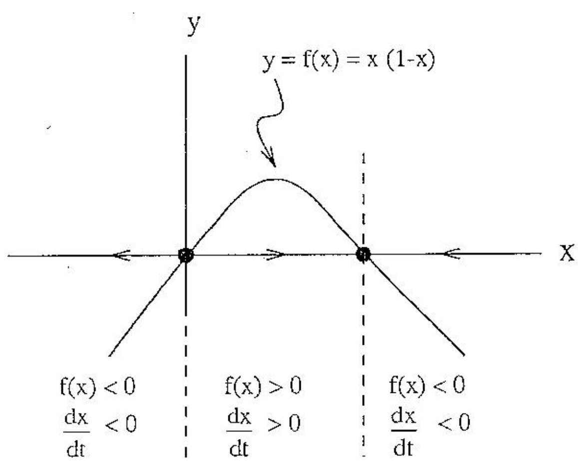


Figure 10: Quick way to draw orbits for a 1-D DE.

Example. *Consider the non-linear DE in* $\mathbb{R}^2$, $x' = f(x)$ *where*

$$f(x) = \begin{pmatrix} x_1(1-x_1) \\ -2x_2 \end{pmatrix} \qquad \text{and } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or equivalently

$$x_1' = x_1(1-x_1), \qquad x_2' = -2x_2.$$

The 2-D flow $\{g^t\}$ *is defined by*

$$g^t a = \begin{pmatrix} \frac{e^t a_1}{e^t a_1 + 1} \\ e^{-2t} a_2 \end{pmatrix}, \qquad \text{where } a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{44}$$

$\{g^t\}$ *is a one-parameter family of non-linear maps of* $\mathbb{R}^2 \to \mathbb{R}^2$. *Since the set of values of* t *for which* $\{g^t a\}$ *is defined depends on* a, $\{g^t\}$ *is called the* local flow. *We note that the flow satisfies the same group properties as in the linear case. Because the DEs are un-coupled, one can also obtain an explicit expression for the orbits in* $\mathbb{R}^2$. *By eliminating* t *we obtain*

$$\frac{dx_1}{dx_2} = -\frac{x_1(1-x_1)}{2x_2}, \qquad \text{for } x_2 \neq 0,$$

and hence

$$x_1^2 x_2 = k(1-x_1)^2, \qquad k = \text{ constant}. \tag{45}$$

There are two equilibrium points, i.e., $f(x) = 0$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

These are necessarily fixed points of the flow:

$$g^t \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad g^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

as may be verified using (44). By studying the flow (44), and equation (45) for the orbits, one can obtain a qualitative sketch of the orbits (see Fig. 11). Note that the behavior of the orbits near the equilibrium points can be inferred by approximating equation (45):

$$\text{if } x_1 \approx 0, \text{ then } x_1^2 x_2 \approx k$$

$$\text{if } x_1 \approx 1, \text{ then } x_2 \approx k(1-x_1)^2$$

Note that if the initial state a satisfies $a_1 > 0$, then

$$\lim_{t\to\infty} g^t a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

i.e., the *long term behaviour of the system is to approach the equilibrium state* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

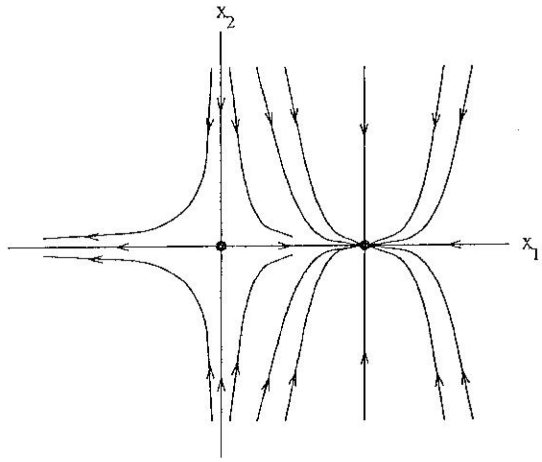


Figure 11: Phase portrait for the system $x_1' = x_1(1 - x_1), \quad x_2' = -2x_2$.

Hamiltonian DEs in 2-D.

We now discuss a class of DEs in $\mathbb{R}^2$ whose orbits can be studied directly, even though the flow cannot be found explicitly. Consider a particle moving in 1-D under the influence of a force which depends only on position. If q denotes the position of the particle, and m its mass, then Newton's Second Law states that

$$m\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = F(q). \tag{46}$$

Since $F = F(q)$, by assumption, there exists a function $V(q)$ defined up to an additive constant, called the *potential function*, such that

$$F(q) = -V'(q). \tag{47}$$

In general, we introduce the *linear momentum*

$$p = m\frac{\mathrm{d}q}{\mathrm{d}t}. \tag{48}$$

Then the equation of motion (46), with (47), leads to the 2-D DE

$$q' = \frac{1}{m}p, \qquad p' = -V'(q). \tag{49}$$

The state vector of the system (mass moving subject to the force) is $(q, p) \in \mathbb{R}^2$. The energy of the system is the sum of kinetic and potential energy:

$$\frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + V(q). \tag{50}$$

When expressed in terms of q and p, the energy is referred to as the *Hamiltonian of the system*, denoted by H:

$$H(q,p) = \frac{1}{2m}p^2 + V(q). \tag{51}$$

In terms of H, the DE (49) assumes the form

$$q' = \frac{\partial H}{\partial p}, \qquad p' = -\frac{\partial H}{\partial q}. \tag{52}$$

These equations are called *Hamilton's equations* for the mechanical system. An important consequence of (52) is that H is constant along any solution curve:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\partial H}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0. \tag{53}$$

This expresses *conservation of energy*, and is expected in view of the physical situation.

Let us make some comments on the orbits of a Hamiltonian DE (52), with H of the form (51),

- Since H is constant along any solution curve, the orbits in $q\,p$-space are contained in the level sets of the Hamiltonian function, given by $H(q,p) = C$.
- The *equilibrium points* of the DE are given by $p = 0$ and $V'(q) = 0$, i.e., they are determined by the critical points of $V(q)$.
- The isocline corresponding to $q' = 0$ (vertical tangent lines) is given by $p = 0$, and the isocline(s) corresponding to $p' = 0$ (horizontal tangent lines) are given by $V'(q) = 0$.
- For orbits with $H(q,p) = C$, the q-values are determined by the inequality $V(q) \leq C$, as follows from equation (51).

Example: A magneto-elastic beam. *[cf. Guckenheimer and Holmes, page 83 [2]] The Hamiltonian is*

$$H(q,p) = \frac{1}{2m}p^2 - \frac{1}{2}\beta^2 q^2 + \frac{1}{4}q^4,$$

and the Hamiltonian DE (the equations of motion) is

$$q' = \frac{1}{m}p, \qquad p' = \beta^2 q - q^3.$$

where q denotes the displacement of the beam from line of symmetry and the force gives rise to the potential $V(q) = -\frac{1}{2}\beta^2 q^2 + \frac{1}{4}q^4$. The phase portrait is given by Fig. 12.

Terminology: Any 2-D DE of the form (52) is called a *Hamiltonian DE*, irrespective of whether the Hamiltonian $H(q,p)$ is of the form (51). If H is of the form (51), the resulting

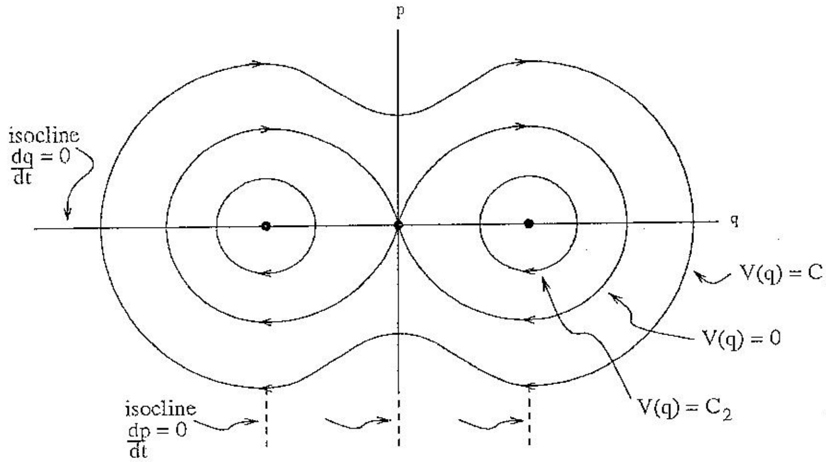


Figure 12: Phase portrait for the magneto-elastic beam.

DE describes a mechanical system, and will be called a *classical Hamiltonian DE*. The concept of a Hamiltonian DE can be generalized to higher dimensions:

$$q_i' = \frac{\partial H}{\partial p_i}, \qquad p_i' = -\frac{\partial H}{\partial q_i}, \tag{54}$$

where the state vector is $x = (q_1, \cdots, q_n, p_1, \cdots, p_n) \in \mathbb{R}^{2n}$. The motion of a spherical pendulum, or motion of a particle in a plane under a central potential (e.g., planetary motion) leads to a Hamiltonian DE with $n = 2$ i.e., state space is $\mathbb{R}^4$.

The Flow of a Non-Linear DE

We begin by stating the standard existence-uniqueness theorem for the initial value problem (IVP) for a DE in $\mathbb{R}^n$.

Theorem (Existence-Uniqueness). *Consider the IVP*

$$x' = f(x), \qquad x(0) = a \in \mathbb{R}^n. \tag{55}$$

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class $C^1(\mathbb{R}^n)$, then for all $a \in \mathbb{R}^n$, there exists an interval $(-\delta, \delta)$ and a unique function $\psi_a : (-\delta, \delta) \to \mathbb{R}^n$ such that

$$\psi_a'(t) = f(\psi_a(t)), \qquad \psi_a(0) = a. \tag{56}$$

Proof. The idea is to rewrite the IVP as an integral equation, and use Picard iterates. [cf. Hirsch and Smale, page 162, [1]] □

Comment: If the hypothesis (f be of class C^1) is weakened, then uniqueness may fail, (e.g., the DE $x' = x^{\frac{2}{3}}$ in $\mathbb{R}$, has two solutions which satisfy the initial condition $x(0) = 0$, namely $x(t) = 0$ and $x(t) = \frac{1}{27}t^3$. Note that $f(x) = x^{\frac{2}{3}}$ is continuous but not C^1.

The existence-uniqueness theorem is a local result — it guarantees existence of a solution in some interval $(-\delta, \delta)$ centered at $t = 0$. Since we are interested in the long-term behaviour of solutions, we would like the solutions to be defined for all $t \geq 0$. We can extend the interval of definition of the solution $\psi_a(t)$ by successively reapplying the theorem, and in this way obtain a *maximal interval of definition* of the solution $\psi_a(t)$. We shall denote this maximal interval by (α, β).

Definition. *We say that the solution $\psi_a(t)$ has* finite escape time *β_a if*

$$\lim_{t \to \beta^-{}_a} \|\psi_a(t)\| = +\infty \tag{57}$$

Theorem (Maximality). *Let $\psi_a(t)$ be the unique solution of the DE $x' = f(x)$, where $f \in C^1(\mathbb{R}^n)$, which satisfies, $\psi_a(0) = a$, and let (α_a, β_a) denote the maximal interval on which $\psi_a(t)$ is defined. If β_a is finite, then*

$$\lim_{t \to \beta^-{}_a} \|\psi_a(t)\| = +\infty \tag{58}$$

Proof. [cf. Hirsch and Smale, pages 171–172, [1]] □

Corollary. *Consider the DE $x' = f(x)$, $f \in C^1(\mathbb{R}^n)$. If a solution $\psi_a(t)$ is bounded for $t \geq 0$, then the solution is defined for all $t \geq 0$.*

Comment: One can always modify a given DE $x' = f(x)$, $x \in \mathbb{R}^n$, and $f \in C^1(\mathbb{R}^n)$, so that the orbits are unchanged, but such that all solutions are defined for all $t \in \mathbb{R}$. The idea is to re-scale the vector field f (the velocity of the state point x):

$$f(x) \to \lambda(x) f(x), \tag{59}$$

where $\lambda(x) : \mathbb{R}^n \to \mathbb{R}$ is a C^1-function (a scalar) which is *positive* on $\mathbb{R}^n$ (in order to preserve the direction of time). This rescaling does not change the direction of the vector field, hence the orbits are unchanged. However, one can choose λ so that $\|\lambda f\|$ is bounded e.g.,

$$\lambda(x) = \frac{1}{1 + \|f(x)\|}. \tag{60}$$

Proposition 12. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1(\mathbb{R}^n)$, and $\lambda : \mathbb{R}^n \to \mathbb{R}$ is $C^1(\mathbb{R}^n)$ and positive, then $x' = f(x)$ and $x' = \lambda(x) f(x)$ have the same orbits, and λ can be chosen so that all solutions of the second DE are defined for all $t \in \mathbb{R}$.*

Proof. [cf. Nemytskii and Stepanov, Theorem 3.22, page 19, and Theorem 1.31, page 9, [3]] □

Definition. *Consider a DE $x' = f(x)$, where f is of class $C^1(\mathbb{R}^n)$, whose solutions are defined for all $t \in \mathbb{R}$. Let $\psi_a(t)$ be the unique maximal solution which satisfies $\psi_a(0) = a$. The* flow of the DE *is defined to be the one-parameter family of maps $\{g^t\}_{t\in\mathbb{R}}$ such that $g^t : \mathbb{R}^n \to \mathbb{R}^n$ and $g^t a = \psi_a(t)$ for all $a \in \mathbb{R}^n$.*

The flow $\{g^t\}$ is defined in terms of the solution function $\psi_a(t)$ of the DE by

$$g^t a = \psi_a(t). \tag{61}$$

It is important to understand the difference between $\psi_a(t)$ and $g^t a$ conceptually:

- *For a fixed $a \in \mathbb{R}^n$*, $\psi_a : \mathbb{R}^n \to \mathbb{R}^n$ gives the state of the system $\psi_a(t)$ *for all $t \in \mathbb{R}$*, with $\psi_a(0) = a$ initially.
- *For a fixed $t \in \mathbb{R}$*, $g^t : \mathbb{R}^n \to \mathbb{R}^n$ gives the state of the system $g^t a$ at time t *for all initial states a.*

The solution function $\psi_a(t)$ satisfies $\psi_a'(t) = f(\psi_a(t))$, $\psi_a(0) = a$. Hence $\psi_a'(0) = f(a)$. By definition of the flow, it follows that

$$\left.\frac{d}{dt}(g^t a)\right|_{t=0} = f(a), \tag{62}$$

which is simply a statement of the fact that the vector field f is tangent to the orbits of the DE.

Proposition 13. *Let $\{g^t\}$ be the flow of a DE $x' = f(x)$, then*

$$\begin{array}{lll} F1: & g^0 = I & \textit{(identity map)} \\ F2: & g^{t_1+t_2} = g^{t_1} \circ g^{t_2} & \textit{(composition)} \end{array} \tag{63}$$

Proof. Use the translational property of solutions of an autonomous DE. □

Theorem (Smoothness of a Flow). *If $f \in C^1(\mathbb{R}^n)$, then the flow $\{g^t\}$ of the DE $x' = f(x)$ consists of C^1 maps.*

Proof. [cf. Hirsch and Smale, pages 298–300, [1]] □

Comment: The significance of this result is that the solutions of the DE depend smoothly on the initial state.

Definition. *The* orbit through a, *denoted $\gamma(a)$ is defined to be*

$$\gamma(a) = \left\{x \in \mathbb{R}^n | x = g^t a, \text{ for all } t \in \mathbb{R}\right\} \tag{64}$$

As in the linear case, orbits are classified as *point orbits*, *periodic orbits*, and *non-periodic orbits*. Sometimes it is convenient to work with the *positive orbit through a* denoted $\gamma^+(a)$ and defined by

$$\gamma^+(a) = \left\{x \in \mathbb{R}^n | x = g^t a, \text{ for all } t \geq 0\right\} \tag{65}$$

Long-Term Behaviour and Limit Sets

Consider a physical system with initial state vector $x \in \mathbb{R}^n$, whose evolution is described by a DE $x' = f(x)$, which determines a flow $\{g^t\}_{t\in\mathbb{R}}$. A fundamental question is: What is the *long-term behaviour* of the system as $t \to \infty$, starting at an initial state a when $t = 0$? In other words, what happens to the positive orbit through a defined by (65) as $t \to +\infty$?

The simplest behaviour is that the system, starting at state a, *approaches an equilibrium state* as $t \to \infty$, i.e., $\lim_{t\to\infty} g^t a = p$. In this case, we say that the *ω-limit set of the initial point a* is the equilibrium point p, and write

$$\omega(a) = \{p\} \tag{66}$$

The next simplest behaviour is that the system, starting at state a, approaches periodic evolution, i.e., the orbit approaches a periodic orbit γ. In this situation, $\lim_{t\to\infty} g^t a$ does not exist, since the orbit does not approach a unique point. However, for any point $p \in \gamma$, we can choose a sequence of times $\{t_n\}$, with $\lim_{n\to\infty} t_n = \infty$, such that $\lim_{n\to\infty} g^{t_n} a = p$. In this case we say that the ω-limit set of the initial point a is the periodic orbit γ, and write

$$\omega(a) = \gamma. \tag{67}$$

These examples motivate the definition to follow.

Definition. *Consider the DE $x' = f(x)$ in $\mathbb{R}^n$, and the associated flow $\{g^t\}_{t\in\mathbb{R}}$. Given an initial point $a \in \mathbb{R}^n$, a point $p \in \mathbb{R}^n$ is said to be an* ω-limit point of a *if there exists a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that $\lim_{n\to\infty} g^{t_n} a = p$. The set of all ω-points of a is called* the ω-limit set of a, *denoted by $\omega(a)$.*

Example. *Consider the system*

$$x_1' = x_1(1 - x_1), \qquad x_2' = -x_2.$$

The equilibrium points are $(0,0)$ and $(1,0)$. For $a = (a_1, a_2)$ we have

$$\begin{aligned} a_1 > 0, &\quad \omega(a) = (1,0); \\ a_1 = 0, &\quad \omega(a) = (0,0); \\ a_1 < 0, &\quad \omega(a) = \emptyset. \end{aligned}$$

To help in identifying the ω-limit set of an initial state a, we consider the following question: What subsets of $\mathbb{R}^n$ can be ω-limit sets for a flow $\{g^t\}$? This is a difficult question, and is unsolved if $n > 2$. But there is a simple necessary condition which is indispensable in identifying $\omega(a)$.

Proposition 14. *An ω-limit set $\omega(a)$ of a flow $\{g^t\}$ is a whole orbit of the flow, or is the union of more than one whole orbit.*

Proof. We simply prove that if $y \in \omega(a)$, then the orbit through y given by

$$\gamma(y) = \left\{g^t y \mid t \in \mathbb{R}\right\} \tag{68}$$

is contained in $\omega(a)$. □

It is also important to know that an ω-limit set is non-trivial (i.e., not the empty set).

Proposition 15. *If the positive orbit through a,*

$$\gamma^+(a) = \left\{ g^t a | t \geq 0 \right\} \tag{69}$$

is bounded, then $\omega(a) \neq \emptyset$.

Proof. By the Bolzano-Weierstrass theorem, the bounded set $\{g^n a | n \in N\}$ has at least one limit point. □

Trapping Sets and the Global Liapunov Theorem

In this section we discuss a method for locating the ω-limit sets of a certain class of DEs, namely those which admit a so-called *Liapunov function.*

Definition. *Given a DE* $x' = f(x)$ *in* $\mathbb{R}^n$, *a set* $S \subseteq \mathbb{R}^n$ *which is the union of whole orbits of the DE, is called an* invariant set for the DE.

For example if we have a Hamiltonian DE in $\mathbb{R}^2$, then the level sets $H(x_1, x_2) = k$ are invariant sets, since H is constant along any orbit. More generally, we have the concept of a *first integral.*

Definition. *A function* $H : \mathbb{R}^n \to \mathbb{R}$ *of class* C^1, *that is not constant on any open subset of* $\mathbb{R}^n$, *is called a* first integral *of the DE* $x' = f(x)$ *if* H *is constant on every orbit:*

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) = 0 \qquad \textit{for all } t. \tag{70}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) = \nabla H(x(t)) \cdot f(x(t)), \tag{71}$$

using the chain rule and the DE, it follows that $H(x)$ is a first integral of the DE $x' = f(x)$ if and only if

$$\nabla H(x) \cdot f(x) = 0 \qquad \text{for all } x \in \mathbb{R}^n, \tag{72}$$

and $H(x)$ is not identically constant on any open subset of $\mathbb{R}^n$. If one has a first integral (e.g., a Hamiltonian function) then the orbits of the DE are contained in the one-parameter family of level sets $H(x) = k$.

It sometimes happens that one has a function $F : \mathbb{R}^n \to \mathbb{R}$ such that *only a particular level set of* F *is an invariant set.* For example look at the DE, $x_1' = x_1(1 - x_1)$, $x_2 = -2x_2$, and consider the functions $F(x_1, x_2) = x_1$ and $G(x_1, x_2) = x_2$. The level sets $F = 0$ and $F = 1$ are invariant sets since $x_1 = 0$ and $x_1 = 1$ satisfy the DE; but F *is not a first integral.* Similarly the level set $G = 0$ is an invariant set. *These invariant sets play a major role in determining the portrait of the orbits.*

Proposition 15. *Given a DE $x' = f(x)$, in $\mathbb{R}^n$, and a function $G : \mathbb{R}^n \to \mathbb{R}$ of class C^1. If $\nabla G(x) \cdot f(x) = 0$ for all x such that $G(x) = k$, then the level set $G(x) = k$ is an invariant set of the DE.*

Proof. The vector field f is tangent to the level set $G(x) = k$, and hence for any initial state a with $G(a) = k$, the orbit $\gamma(a)$ lies in the level set. □

In order to determine an ω-limit set, it is helpful to know that an orbit enters a bounded set S and *never leaves it*. Such a set is called a trapping set.

Definition. *Given a DE $x' = f(x)$ in $\mathbb{R}^n$, with flow g^t, a subset $S \subset \mathbb{R}^n$ is said to be a* trapping set *of the DE if it satisfies*

1. *S is a closed and bounded set,*
2. *$a \in S$ implies $g^t a \in S$ for all $t \geq 0$.*

The usefulness of trapping sets lies in this result; *if S is a trapping set of a DE $x' = f(x)$, then for all $a \in S$, the ω-limit set $\omega(a)$ is non-empty and is contained in S.*

Example. *Consider the DE*

$$x_1' = \gamma_1(1 - x_1 - \alpha x_2)x_1, \qquad x_2' = \gamma_2(1 - \beta x_1 - x_2)x_2,$$

with $x_1 \geq 0$, $x_2 \geq 0$, $\gamma_1, \gamma_2, \alpha$ and β are positive constants with $\alpha < 1$ and $\beta < 1$. By inspection we see that the x_1-axis ($x_2 = 0$) and the x_2-axis ($x_1 = 0$) are invariant sets. By inspection, for sufficiently large x_1 and/or x_2, then $x_1' < 0$ and $x_2' < 0$. Thus the set

$$S_k = \{(x_1, x_2) | x_1 + x_2 \leq k, x_1 \geq 0, x_2 \geq 0\}$$

is a trapping set for the DE.

The Global Liapunov Theorem

Consider a DE $x' = f(x)$ in $\mathbb{R}^n$ and let $V : \mathbb{R}^n \to \mathbb{R}$ be $C^1(\mathbb{R}^n)$. We can calculate the rate of change of V along a solution of the DE:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) &= \frac{\partial V}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \cdots + \frac{\partial V}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t} \qquad \text{by the Chain Rule} \\ &= \nabla V(x(t)) \cdot f(x(t)) \equiv \dot{V}(x) \end{aligned} \tag{73}$$

using $x_i' = f_i$ and the definition of scalar product in $\mathbb{R}^n$. Suppose that $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$. Then for any orbit $\gamma(a)$ in a trapping set S, $V(x)$ will keep decreasing along $\gamma(a)$ until the orbit approaches its ω-limit set $\omega(a)$. One thus expects that $\omega(a)$ will consist of points for which $\dot{V}(x) = 0$. In this way, one obtains a strong restriction on the possible ω-limit sets.

Theorem (Global Liapunov Theorem). *Consider the DE $x' = f(x)$ in $\mathbb{R}^n$, and let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. If $S \subset \mathbb{R}^n$ is a trapping set, and $\dot{V}(x) \le 0$ for all $x \in S$, then for all $a \in S$, $\omega(a) \subseteq \left\{x \in S | \dot{V}(x) = 0\right\}$.*

Proof. [cf. Hale, Theorem 1.3, page 296, [4]] □

Comments:

1. A function $V : \mathbb{R}^n \to \mathbb{R}$ which satisfies the above theorem for $x \in S \subset \mathbb{R}^n$ is called a *Liapunov function on S.*
2. One can often use the level sets of the function V to define the trapping set S in the theorem.
3. In applying the theorem, we note that we simply have to find *whole orbits* that are contained in the set $\left\{x \in S | \dot{V}(x) = 0\right\}$ to obtain the ω-limit set $\omega(a)$.

Example. *Consider the DE*

$$x_1' = x_2, \qquad x_2' = -\alpha x_2 - x_1^3, \quad \alpha > 0.$$

We note that when $\alpha = 0$, the DE is Hamiltonian, with $H = \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4$. So we let $V(x) = \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4$. It follows that $\dot{V}(x) = -\alpha x_2^2 \le 0$ on $\mathbb{R}^2$. The level sets $V(x) = k$ are simple closed curves and $\nabla V(x)$ points outwards. Thus $S_k = \left\{x \in \mathbb{R}^2 | V(x) \le 0\right\}$ is a trapping set. Thus for all $a \in S_k$, $\omega(a) \subseteq \left\{x \in S_k | \dot{V}(x) = 0\right\} = \{x \in S_k | x_2 = 0\}$. However, when $x_2 = 0$, the DE implies that $x_2' \ne 0$ unless $x_1 = 0$. Thus the equilibrium point $(0,0)$ *is the only whole orbit with $x_2 = 0$, and hence $\omega(a) = \{(0,0)\}$, for all $a \in S_k$. Finally, note that for all $a \in \mathbb{R}^2$, $a \in S_k$* for some k. *Thus $\omega(a) = \{(0,0)\}$, for all $x \in \mathbb{R}^2$.*

4 LIAPUNOV'S STABILITY THEOREM

The goal is to show that the stability of an equilibrium point can be ascertained, subject to a condition, by studying the linearization of the DE. The basic definitions are the same as in the linear case, with the linear flow e^{tA} being replaced by g^t

Definition.

1. *The equilibrium point $\bar{x}$ of a DE $x' = f(x)$ in $\mathbb{R}^n$ is* stable *if for all neighborhoods U of $\bar{x}$, there exists a neighborhood V of $\bar{x}$ such that $g^t V \subseteq U$ for all $t \ge 0$, where g^t is the flow of the DE.*
2. *The equilibrium point $\bar{x}$ of a DE $x' = f(x)$ in $\mathbb{R}^n$ is* asymptotically stable *if it is stable and if, in addition, for all $x \in V$, $\lim_{t\to\infty} \|g^t x - \bar{x}\| = 0$*

Now, consider a non-linear DE $x' = f(x)$ in $\mathbb{R}^n$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a $C^1(\mathbb{R}^n)$. The rate of change of V along a solution of the DE is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \nabla V(x(t)) \cdot f(x(t)) \equiv \dot{V}(x) \tag{74}$$

Thus, if $\dot{V}(x) < 0$ *for all* t *then* $V(x)$ *decreases with time along the corresponding orbit.* From a geometrical point of view, the orbits cut the level sets $V(x) = k$ in the direction away from $\nabla V(x)$. Suppose that $\bar{x}$ is an equilibrium point of the DE. If $V(\bar{x}) = 0$ and $V(x) > 0$ for all $x \in U - \{\bar{x}\}$, where U is a neighborhood of $\bar{x}$, then we expect the level sets of V in U to be concentric curvès (n=2) or concentric spheres (n=3); consequently when $\dot{V} < 0$ for all $x \in U - \{\bar{x}\}$, any orbit in $U - \{\bar{x}\}$ will cut the level sets of V in the inward direction, and we expect that this will continue until the orbit is forced to approach the equilibrium point $\bar{x}$ as $t \to \infty$, showing that the equilibrium point is *asymptotically stable.* If, instead, $\dot{V} \leq 0$ for all $x \in U - \{\bar{x}\}$, then U may contain periodic orbits, and we only obtain the weaker conclusion that $\bar{x}$ is *stable.* Finally, if $\dot{V} > 0$ for all $x \in U - \{\bar{x}\}$, then the orbits are forced away from $\bar{x}$, which is thus an *unstable* equilibrium point.

Theorem (Liapunov Stability Theorem). *Let $\bar{x}$ be an equilibrium point of the DE $x' = f(x)$ in $\mathbb{R}^n$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that $V(\bar{x}) = 0$, $V(x) > 0$ for all $x \in U - \{\bar{x}\}$, where U is a neighborhood of $\bar{x}$.*

1. *If $\dot{V}(x) < 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is asymptotically stable.*
2. *If $\dot{V} \leq 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is stable.*
3. *If $\dot{V}(x) > 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is unstable.*

Proof. This can be proved as a corollary of the Global Liapunov Theorem. □

A function $V : \mathbb{R}^n \to \mathbb{R}$ which satisfies $V(\bar{x}) = 0$, $V(x) > 0$ for all $x \in U - \{\bar{x}\}$, and $\dot{V}(x) \leq 0$ (respectively < 0) for all $x \in U - \{\bar{x}\}$, is called a *Liapunov function* (respectively, a *strict Liapunov function*) for the equilibrium point $\bar{x}$. Hence we obtain the following:

Theorem (Criterion for Asymptotic Stability). *Let $\bar{x}$ be an equilibrium point of the DE $x' = f(x)$ in $\mathbb{R}^n$. If all eigenvalues of the derivative matrix $\mathrm{D}f(\bar{x})$ satisfy $\Re e(\lambda) < 0$, then the equilibrium point $\bar{x}$ is asymptotically stable.*

Proof. Consider the linear approximation of $f(x)$ at $\bar{x}$, $A(x - \bar{x})$, where $A = \mathrm{D}f(\bar{x})$. Since $\Re e(\lambda) < 0$ for all eigenvalues of A, by Liapunov's Lemma there exists a symmetric positive definite matrix Q such that $A^T Q + Q A = -I$. To complete the proof, we note that $V(x) = x^T Q x$ is a strict Liapunov function for $\bar{x}$. □

Linearization and the Hartman-Grobman Theorem

Consider the DE, $x_1' = x_1(1-x_1)$, $\quad x_2' = -2x_2$, studied earlier. We consider the orbits of the linearizations at the equilibrium points $(0,0)$ and $(1,0)$. At $(0,0)$, $A = \mathrm{D}f(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$, a saddle. At $(1,0)$, $A = \mathrm{D}f(1,0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, an attracting node. For each equilibrium point, there is a homeomorphism h which maps the orbits of the linearized flow in a neighborhood of O onto the orbits of the non-linear flow in a neighborhood of the equilibrium point. In other words, the linearizations give a reliable description of the non-linear orbits *near the equilibrium points.*

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Suppose that $\bar{x}$ is a critical point of g, i.e., $\nabla g(\bar{x}) = 0$. Let

$$Hg(\bar{x}) = \begin{pmatrix} g_{11}(\bar{x}) & g_{12}(\bar{x}) \\ g_{21}(\bar{x}) & g_{22}(\bar{x}) \end{pmatrix} \tag{75}$$

be the Hessian matrix of g at $\bar{x}$. The second derivative test determines whether $\bar{x}$ is a local maximum (respectively local minimum, saddle point) of g, subject to a certain restriction, namely $\det[Hg(\bar{x})] \neq 0$, where the second derivative test may fail. In a similar manner, the linearization of a non-linear DE can fail to give reliable information about the orbits, if a certain restriction does not hold.

Example. *Consider the non-linear DE* $x_1' = -x_1$, $\quad x_2' = x_2^3$, *which describes a non-linear saddle with orbits* $x_1 = ke^{\frac{1}{2}x_2^2}$. *The linearization at* $(0,0)$ *is* $u' = Au$, $\quad A = \mathrm{D}f(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, *which describes an attracting line.*

The linear and non-linear flows are not topologically equivalent in a neighborhood of the equilibrium point, and hence the linearization fails. The source of the failure is that the matrix $\mathrm{D}f(0,0)$ *has a zero eigenvalue.*

Example. *Consider the DE* $x_1' = -x_2 - x_1(x_1^2 + x_2^2)$, $\quad x_2' = x_1 - x_2(x_1^2 + x_2^2)$, *a non-linear spiral. The linearization at* $(0,0)$ *is* $u' = Au$, $\quad A = \mathrm{D}f(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, *representing a centre.*

Again, the linear and non-linear flows are not topologically equivalent in a neighborhood of the equilibrium points, hence the linearization fails. The source of the failure is that the matrix $\mathrm{D}f(0,0)$ *has eigenvalues with zero real parts.*

Theorem (Hartman-Grobman). *Let* $\bar{x}$ *be an equilibrium point of the DE* $x' = f(x)$ *in* $\mathbb{R}^n$, *where* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is of class* C^1. *If all the eigenvalues of the matrix* $\mathrm{D}f(\bar{x})$ *satisfy* $\Re e(\lambda) \neq 0$, *then there is a homeomorphism* $h : U \to \bar{U}$ *of a neighborhood* U *of* O *onto a neighborhood* $\bar{U}$ *of* $\bar{x}$ *which maps orbits of the linear flow* $e^{t\mathrm{D}f(\bar{x})}$ *onto orbits of the non-linear flow* g^t *of the DE, preserving the parameter* t.

Proof. [cf. Hartman, pages 244-250, [5]] □

The Hartman-Grobman Theorem can be stated more concisely using the concept of topological equivalence, which can be generalized to non-linear flows.

Definition. *Two flows* g^t *and* $\tilde{g}^t$ *on* $\mathbb{R}^n$ *are said to be* topologically equivalent *if there is a homeomorphism* $h : \mathbb{R}^n \to \mathbb{R}^n$ *which maps orbits of* g^t *onto orbits of* $\tilde{g}^t$, *and preserves the direction of the parameter* t.

Then we can state: If $\bar{x}$ is a *hyperbolic equilibrium point*, then the flow of the DE $x' = f(x)$ and the flow of its linearization $u' = Df(\bar{x})u$, are *locally* topologically equivalent.
Comment: An equilibrium point $\bar{x}$ of a non-linear DE is said to be *hyperbolic* if *all* eigenvalues of the matrix $Df(\bar{x})$ satisfy $\Re e(\lambda) \neq 0$.

Saddle Points and the Stable Manifold Theorem

Definition. *An equilibrium point* $\bar{x}$ *of a DE* $x' = f(x)$ *in* $\mathbb{R}^n$ *is a* saddle point *if the real parts of the eigenvalues of the matrix* $Df(\bar{x})$ *are all non-zero, and not all of one sign. [i.e., a saddle point is a hyperbolic (all* $\Re e(\lambda) \neq 0$*) equilibrium point which is neither a sink (all* $\Re e(\lambda) < 0$*) nor a source (all* $\Re e(\lambda) > 0$*).]*

The Hartman-Grobman theorem gives a qualitative local description of a (non-linear) saddle. In particular in $\mathbb{R}^2$ we have (see Fig. 13).

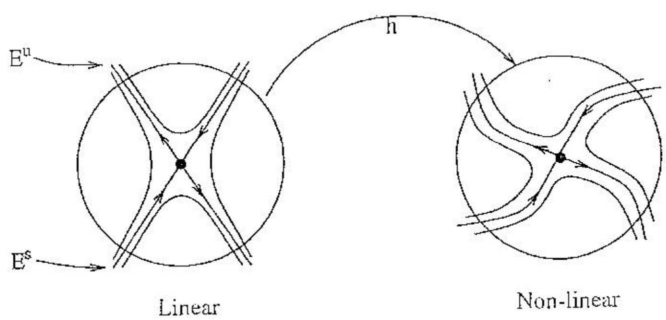


Figure 13: The linear and non-linear flows in a neighborhood of a saddle point.

The 1-D subspace E^s that is spanned by the eigenvector which corresponds to the eigenvalue $\lambda_1 < 0$ is called the *stable subspace* of the equilibrium point, and the 1-D subspace E^u that is spanned by the eigenvector which corresponds to the eigenvalue $\lambda_2 > 0$ is called the *unstable subspace.*

Definition. *Let* $\bar{x}$ *be a saddle point of the DE* $x' = f(x)$ *in* $\mathbb{R}^n$, *and let* U *be a neighborhood of* $\bar{x}$. *The* local stable manifold of $\bar{x}$ in U *is defined by*

$$W^s(\bar{x}, U) = \left\{ x \in U | g^t x \xrightarrow[\infty]{t} \bar{x}, g^t x \in U \text{ for all } t \geq 0 \right\}. \quad (76)$$

We can now state the following theorem (without proof).

Theorem (Stable Manifold Theorem). *Let $\bar{x}$ be a saddle point of $x' = f(x)$ in $\mathbb{R}^n$, where f is of class C^1, and let E^s be the stable subspace of the linearization at $\bar{x}$. Then there exists a neighborhood U of $\bar{x}$ such that the local stable manifold $W^s(\bar{x}, U)$ is a smooth (C^1) curve which is tangent to E^s at $\bar{x}$.*

Comment: One can define, in an analogous way the local unstable manifold of $\bar{x}$ in U denoted $W^u(\bar{x}, U)$, and similarly there is an "Unstable Manifold Theorem."

Local Behaviour Near a Non-linear Sink

Suppose that $\bar{x}$ is an equilibrium point of a non-linear DE $x' = f(x)$ in $\mathbb{R}^2$. Suppose that all eigenvalues of the matrix $Df(\bar{x})$ satisfy $\Re e(\lambda) < 0$, (i.e., $\bar{x}$ is a sink). The Hartman-Grobman theorem asserts that in some neighborhood of $\bar{x}$, the flow of the non-linear DE is topologically equivalent to the flow of the linearization $u' = Df(\bar{x})u$, where $u = x - \bar{x}$. In this section we give a more detailed description of the non-linear orbits near a sink.

Let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be an asymptotically stable equilibrium point of the DE $x' = f(x)$. In order to describe the orbits near $\bar{x}$, we introduce polar coordinates

$$\begin{aligned} x_1 - \bar{x}_1 &= r\cos\theta \\ x_2 - \bar{x}_2 &= r\sin\theta \end{aligned} \tag{77}$$

Since $\bar{x}$ is asymptotically stable,

$$\lim_{t\to+\infty} r(t) = 0 \tag{78}$$

if $r(0)$ is sufficiently close to zero. We say that the equilibrium point $\bar{x}$ is a *non-linear spiral* if

$$\lim_{t\to+\infty} \theta(t) = \pm\infty \tag{79}$$

for any solution $(r(t), \theta(t))$ for which (78) holds.

Proposition 16. *Consider the DE*

$$(NL): \qquad x' = f(x) \tag{80}$$

in $\mathbb{R}^2$, where f is of class C^1. Consider the linearization

$$(L): \qquad u' = Df(\bar{x})u \tag{81}$$

at the equilibrium point $\bar{x}$. If O is an attracting spiral point of (L), then $\bar{x}$ is an attracting spiral point of (NL).

Proof. [cf. Coddington and Levinson, Theorem 2.2, page 376 [6]] □

Proposition 17. *If O is an attracting node of (L) then $\bar{x}$ is an attracting node of (NL).*

Proof. [cf. Coddington and Levinson, Theorem 5.1, page 384, [6]] □

Comment: A similar result holds for Jordan nodes. [cf. Coddington and Levinson, page 387, [6]]

An asymptotically stable equilibrium point $\bar{x}$ is said to be an *attracting non-linear focus* if all orbits sufficiently close to $\bar{x}$ approach $\bar{x}$ in a definite direction as $t \to \infty$, and given any direction there exists an orbit which tends to $\bar{x}$ in this direction.
Comment: If O is a focus of (L), it does not necessarily follow in general that $\bar{x}$ is a non-linear focus of (NL).

Proposition 18. *Suppose that the vector field f is of class C^2. If O is an attracting focus of (L) then $\bar{x}$ is an attracting focus of (NL).*

Proof. [cf. Coddington and Levinson, page 377, [6]] □

A stable equilibrium point $\bar{x}$ is said to be a *non-linear centre* if in some neighborhood of $\bar{x}$, the orbits are periodic orbits which enclose $\bar{x}$. Recall that the Hartman-Grobman theorem does not apply if O is a *centre* of (L), i.e., *one cannot conclude that $\bar{x}$ is a centre of (NL).* But one can still draw a useful conclusion.

Proposition 19. *If O is a centre of (L), then $\bar{x}$ is either a centre, an attracting spiral, or a repelling spiral of (NL).*

Proof. [cf. Coddington and Levinson, Theorem 4.1, page 382, [6]] □

5 PERIODIC ORBITS AND LIMIT SETS IN THE PLANE

We have seen that a linear DE can admit a family of periodic orbits, corresponding to a physical system whose motions are undamped oscillations. Of greater interest is the case where a DE admits an *isolated periodic orbit,* i.e., the orbit has a neighborhood U which contains no other periodic orbits. This is in fact only possible for a non-linear DE. In this situation, the periodic orbit γ may attract neighboring orbits, thereby describing a physical system which has an *oscillatory steady state* which is stable. We say that such a system undergoes *self-sustained oscillations.* The main goal in this section is to discuss isolated periodic orbits of DEs in the plane.

It should be noted that the question of existence of periodic orbits is a difficult one. In 1900, as part of problem 16 of his famous list, David Hilbert posed the question: What is the maximum number of *isolated* periodic orbits of an autonomous DE $x' = f(x)$ in $\mathbb{R}^2$, if the components of the vector field f are *polynomial functions?* This problem is still unsolved even for the case of quadratic polynomials (degree 2). For awhile the upper bound was thought to be 3 but an example with 4 isolated periodic orbits has been found.

Non-Existence of Periodic Orbits

Dulac's criterion for excluding periodic orbits for a DE in $\mathbb{R}^2$ is based on Green's theorem.

Theorem (Green's). *If g_1 and g_2 are of class C^1 on an open set $D \subset \mathbb{R}^2$ and C is a simple closed curve in D, whose interior R is in D then*

$$\oint_C g_1\, dx_1 + g_2\, dx_2 = \iint_R \left(\frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2}\right) dx_1\, dx_2 \tag{82}$$

where C is oriented counter-clockwise.

Recall that the line integral is evaluated as an ordinary integral by introducing parametric equations for C:

$$\oint_C g_1 dx_1 + g_2 dx_2 = \int_{t=a}^{b} [g_1(x_1(t), x_2(t))x_1'(t) + g_2(x_1(t), x_2(t))x_2'(t)]\, dt \tag{83}$$

where $(x_1(b), x_2(b)) = (x_1(a), x_2(a))$. Since (x_1', x_2') is tangent to C it follows that if the vector field (g_1, g_2) is orthogonal to C at each point of C then

$$\oint_C g_1 dx_1 + g_2 dx_2 = 0 \tag{84}$$

The idea is to apply Green's theorem to a periodic orbit γ of the DE $x' = f(x)$ in $\mathbb{R}^2$, where $f(x) = (f_1(x), f_2(x))$. The essential point is to note that the vector field $(f_2(x), -f_1(x))$ is orthogonal to the periodic orbit γ, since (f_1, f_2) is tangent to γ, and $(f_1, f_2) \cdot (f_2, -f_1) = 0$. We apply Green's theorem to a periodic orbit γ, with $(g_1, g_2) = (f_2, -f_1)$. It follows that

$$\iint_R \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1\, dx_2 = 0 \tag{85}$$

Thus, if the DE is such that $\operatorname{div}(f) = \nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} > 0$ (< 0), for all $x \in D$, where D is a simply connected open set, then a contradiction arises, and we conclude that D contains no periodic orbits. The requirement that D be simply connected is necessary in order to ensure that the interior of γ is contained in D.

The preceding argument may be generalized by noting that for any scalar function B, the vector field $(Bf_2, -Bf_1)$ is orthogonal to the periodic orbit γ. The condition which excludes periodic orbits becomes $\operatorname{div}(Bf) = \nabla \cdot Bf = \frac{\partial Bf_1}{\partial x_1} + \frac{\partial Bf_2}{\partial x_2} > 0$ (< 0).

Proposition 20 (Dulac's Criterion). *If $D \subseteq \mathbb{R}^2$ is a simply connected open set and $\operatorname{div}(Bf) = \frac{\partial}{\partial x_1}(Bf_1) + \frac{\partial}{\partial x_2}(Bf_2) > 0$, (< 0) for all $x \in D$ where B is a C^1 function, then the DE $x' = f(x)$ where $f \in C^1$ has no periodic orbit which is contained in D.*

Proof. Essentially given above. □

Comment: The function $B(x_1, x_2)$ is called a *Dulac function* for the DE in the set D.

Example. *A classical Hamiltonian DE in 2-D,*

$$x_1' = x_2, \qquad x_2' = -V'(x_1)$$

typically admits a family of periodic orbits. Modify the DE by adding linear damping:

$$x_1' = x_2, \qquad x_2' = -\alpha x_2 - V'(x_1) \qquad \alpha < 0.$$

It follows that

$$\operatorname{div}(f) = \frac{\partial}{\partial x_1}(x_2) + \frac{\partial}{\partial x_2}(-\alpha x_2 - V'(x_1)) = -\alpha < 0$$

for all $x \in \mathbb{R}^2$. *Thus the* damped DE admits no periodic orbits, *irrespective of the form of the potential* $V(x_1)$.

The second criterion for excluding periodic orbits, which is valid in $\mathbb{R}^n$, $n \geq 2$, follows from the observation that if *a function* $V(x)$ *is monotone decreasing along an orbit of a DE, then that orbit cannot be periodic.*

Proposition 21. *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a* C^1 *function. If* $\dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0$ *on a subset* $D \subseteq \mathbb{R}^n$, *then any periodic orbit of the DE* $x' = f(x)$ *which lies in* D, *belongs to the subset* $\left\{x | \dot{V}(x) = 0\right\} \cap D$.

Self-Sustained Oscillations–An Example

We consider the DE

$$x'' + (x^2 - 1)x' + x = 0 \quad \text{or} \quad \begin{cases} x_1' = x_2, \\ x_2' = -x_1 - (x_1^2 - 1)x_2. \end{cases}$$

This DE, the famous *Van der Pol DE*, arises in the study of electrical circuits. It admits an isolated periodic orbit which is not easy to locate. So we begin by considering a modification of the DE, for which the periodic orbit can be found explicitly.

Consider the DE, $x_1' = x_2$, $x_2' = -x_1 - (x_1^2 + x_2^2 - 1)x_2$. the origin $(0,0)$ is the only equilibrium point. In order to determine whether orbits approach $(0,0)$ we consider $V(x_1, x_2) = x_1^2 + x_2^2$. Along any solution

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x_1(t), x_2(t)) = -\left[V(x_1(t), x_2(t)) - 1\right] x_2^2. \tag{86}$$

$V(x_1(t), x_2(t)) = 1$ satisfies this DE; and hence *the circle* $x_1^2 + x_2^2 = 1$ *is a periodic orbit.* Further any annulus $1 \leq x_1^2 + x_2^2 \leq R$ is positively invariant, and V is a Liapunov function i.e., $\dot{V} \leq 0$ by (86). It follows from the Global Liapunov theorem that for *any initial state* a *exterior to the circle* $x_1^2 + x_2^2 = 1$, *the* ω*-limit set* $\omega(a)$ *is the periodic orbit* $x_1^2 + x_2^2 = 1$. Similarly, the annulus $0 < x_1^2 + x_2^2 \leq 1$ is positively invariant, and it follows from equation (86) that V is a Liapunov function. Thus, *for any initial state* $a \neq 0$ *inside the circle* $x_1^2 + x_2^2 = 1$, *the* ω*-limit set* $\omega(a)$ *is the periodic orbit* $x_1^2 + x_2^2 = 1$. This type of periodic orbit is called a *limit cycle.*

Definition. *An isolated periodic orbit γ of a DE $x' = f(x)$ in $\mathbb{R}^2$, is called a* stable limit cycle *if there exists a neighborhood U of γ such that $\omega(a) = \gamma$ for all $a \in U$.*

The Poincaré-Bendixson Theorem

Consider the DE $x_1' = x_2, \quad x_2' = -x_1 + (1 - x_1^2 - \frac{1}{4}x_2^2)x_2$. Let $V(x_1, x_2) = x_1^2 + x_2^2$. It follows that $\dot{V}(x_1, x_2) = (1 - x_1^2 - \frac{1}{4}x_2^2)x_2^2$. On the circle $C_1 : x_1^2 + x_2^2 = 1$, it follows that $\dot{V}(x_1, x_2) = \frac{3}{4}x_2^4 \geq 0$. On the circle $C_2 : x_1^2 + x_2^2 = 4$ we have that $\dot{V}(x_1, x_2) = -\frac{3}{4}x_1^2 x_2^2 \leq 0$. It follows that the annulus S bounded by C_1 and C_2 is a trapping set. Thus for any initial state $a \in S$, the ω-limit set $\omega(a)$ must be a whole orbit (or the union of whole orbits), *we conjecture that $\omega(a)$ must be a periodic orbit.* The validity of this conjecture constitutes the Poincaré-Bendixson Theorem.

Before discussing the theorem, we need a few preliminary concepts.

Definition. *A* local section *of the flow of a DE in $\mathbb{R}^2$ is a smooth curve segment Σ such that the vector field f of the DE satisfies $n \cdot f \neq 0$ on Σ, where n is normal to Σ.*

Comment: This implies that no equilibrium points of f lie on Σ, and by continuity, that orbits pass through Σ in one direction only.

Definition. *Let x be a regular (i.e., non-equilibrium) point of the flow i.e., $f(x) \neq 0$. Let Σ be a local section through x. A* flow-box *for x is a neighborhood of x of the form $N = \left\{ g^t\Sigma \middle| \, |t| < \delta \right\}$ for some $\delta > 0$.*

Finally we need the following properties of ω-limit sets:

1. $\omega(a)$ is the union of whole orbits,
2. $\omega(a)$ is a closed set,
3. if $\omega(a)$ is bounded, then $\omega(a)$ is connected (i.e., is not the union of disjoint sets),
4. the following Lemma.

Lemma (Fundamental Lemma on ω-limit sets in $\mathbb{R}^2$). *Let $\omega(a)$ be an ω-limit set of a DE in $\mathbb{R}^2$. If $y \in \omega(a)$, then the orbit through y, $\gamma(y)$, cuts any local section Σ in at most one point.*

Proof. [cf. Hirsch and Smale, Proposition 2, page 246, [1]] also [cf. Hale, Corollary 1.1, page 53, [4]] □

Comment: The Lemma is not valid for a flow in $\mathbb{R}^3$ (a local section is a smooth surface segment, for a flow on a 2-torus).

Theorem (Poincaré-Bendixson). *Let $\omega(a)$ be a non-empty ω-limit set of the DE $x' = f(x)$ in $\mathbb{R}^2$, where $f \in C^1$. If $\omega(a)$ is a bounded subset of $\mathbb{R}^2$ and $\omega(a)$ contains no equilibrium points, then $\omega(a)$ is a periodic orbit.*

Proof. [cf. Hirsch and Smale, Chapter 11, [1]] □

In applications it is often convenient to use the following Corollary of the Poincaré-Bendixson theorem.

Corollary. *Let K be a positively invariant subset of the DE $x' = f(x)$ in $\mathbb{R}^2$, where $f \in C^1$. If K is a closed and bounded set, then K contains either a periodic orbit or an equilibrium point.*

There is one further result which can help to locate isolated periodic orbits.

Proposition 21. *Any periodic orbit of a C^1 DE on $\mathbb{R}^2$ encloses an equilibrium point.*

The Van der Pol DE and Liénard,'s Theorem

In 1922, a Dutch scientist, Balthasar van der Pol, published a paper concerning oscillations in radio circuits containing triode valves (now obsolete). The analysis was based on the DE

$$x'' + \mu(x^2 - 1)x' + x = 0, \qquad \mu > 0 \tag{87}$$

now known as the *van der Pol DE*, and used as a simple model for systems which can undergo self-sustained oscillations. The French scientist Alfred Liénard, was also interested in self-sustained oscillations and in 1928 published an analysis of a DE with a more general dissipative term:

$$x'' + g(x)x' + x = 0, \tag{88}$$

now known as *Liénard's DE*.

If we let $G(x) = \int_0^x g(s)\,ds$ then the Liénard DE is equivalent to

$$x_1' = x_2 - G(x_1), \qquad x_2' = -x_1 \tag{89}$$

where, in the case of the van der Pol DE, $G_p(x) = \mu(\frac{1}{3}x^3 - x)$. The existence of an isolated periodic orbit of (89) depends on the shape of the vertical isocline, which is given by $x_2 = G(x_1)$. The requirement is that this curve should be qualitatively the same as the van der Pol isocline, i.e., $x_2 = G_p(x_1)$ for large x_1 and for x_1 close to 0.

Following Liénard, consider the following conditions:

L1: G is an odd function,

L2: $\lim_{x\to\infty} G(x) = +\infty$, and there exists $\beta > 0$ such that $G(x) > 0$, $G'(x) > 0$ for $x > \beta$.

L3: There exists an α, with $0 < \alpha \leq \beta$, such that $G(x) < 0$ for $0 < x < \alpha$.

The corollary to the Poincaré-Bendixson theorem can then be used to prove the existence of a periodic orbit of the DE (89).

Theorem (Liénard). *If G satisfies conditions L1–L3, then the DE (88) admits a periodic orbit.*

Proof. [cf. Hale, pages 57-59, [4]] □

Comment: If $\alpha = \beta$, in conditions L2 and L3, it can further be shown that there is a unique periodic orbit, which attracts nearby orbits. [cf. Hale, Theorem 1.6, page 60, [4]]

The Fundamental Theorem for ω-limit sets in $\mathbb{R}^2$

The fundamental property of an ω-limit set is that it consists of one whole orbit, or that it is the union of more than one whole orbit. The simplest situations are

- $\omega(a)$ is an equilibrium point, i.e., the system approaches an equilibrium state as $t \to +\infty$,
- $\omega(a)$ is a periodic orbit, i.e., the system approaches an oscillatory steady state as $t \to +\infty$.

In order to motivate the fundamental theorem for ω-limit sets in $\mathbb{R}^2$, we begin by discussing an example in which the ω-limit set is the union of two orbits, giving an example of a *cycle-graph*.

Consider the DE

$$x_1' = x_2, \qquad x_2' = 2x_1 - 3x_1^2 - \mu x_2(x_1^3 - x_1^2 + \frac{1}{2}x_2^2), \tag{90}$$

For $\mu = 0$, this reduces to a Hamiltonian DE with

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + x_1^3 - x_1^2, \tag{91}$$

We wish to sketch the portrait of the orbits when μ is positive but close to 0, so that the DE (90) can be thought of as a perturbation of the Hamiltonian DE. The portrait for the case $\mu = 0$ is given in Fig. 14.

We use H as given by equation (91) as a Liapunov function for the DE (90). It follows that

$$\dot{H}(x_1, x_2) = \nabla H \cdot f = -2\mu x_2^2 H(x_1, x_2). \tag{92}$$

Hence, since $\dot{H} = 0$ when $H = 0$, the level set $H = 0$ consists of orbits of the DE (as is the case when $\mu = 0$). Thus the set $S = \{(x_1, x_2)\,|H \leq 0, x_1 \geq 0\}$ is a closed and bounded positively invariant set. Also, it follows from equation (92) that if $\mu > 0$, then $\dot{H} \geq 0$ in S and hence the orbits in S cross the level sets $H(x_1, x_2) = C < 0$ in the *outward* direction. Further we can apply the Global Liapunov theorem to H on the set S to conclude that for any initial state a in the interior of S (except for the equilibrium point), *the ω-limit set is the union of the homoclinic orbit and the equilibrium point* $(0, 0)$. This justifies the second phase portrait in Fig. 14. This ω-limit set, which is the union of two orbits, is an example of a *cycle-graph*.

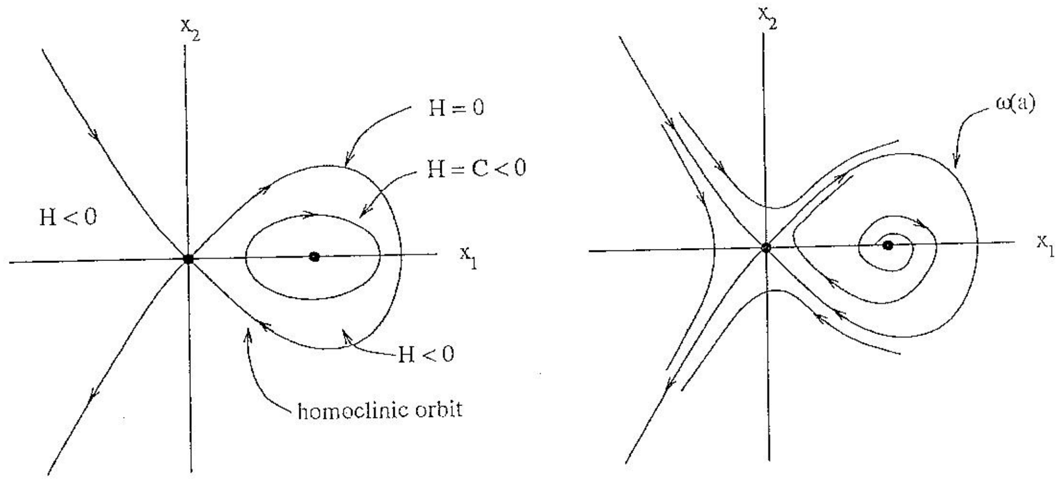


Figure 14: Phase portraits for $\mu = 0$ and $\mu \neq 0$.

Comment: It is of interest to describe the behaviour of a solution which corresponds to the above orbit $\gamma(a)$, as $t \to +\infty$. The functions $x_1(t), x_2(t)$ are not asymptotically periodic. The reason for this is that at an equilibrium point, the velocity of the point $(x_1(t), x_2(t))$ in state space (i.e., the vector field f) is zero, and thus $(x_1(t), x_2(t))$ lingers near $(0,0)$ for successively longer time intervals as $t \to +\infty$.

The Fundamental Theorem

Let us give some more examples of invariant sets (i.e., unions of orbits) which can arise as ω-limit sets in $\mathbb{R}^2$. (See Fig. 15). and some invariant sets which *cannot* arise as ω-limit sets. (See Fig. 16). The four invariant sets in Fig. 15 are examples of cycle graphs.

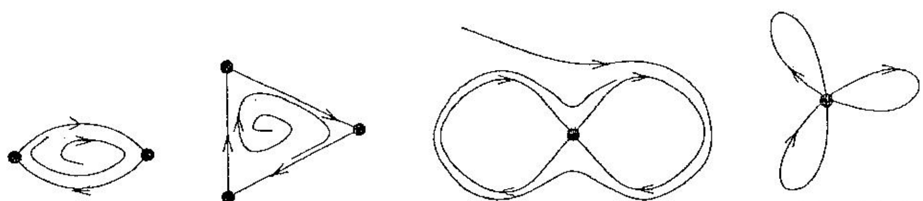


Figure 15: Examples of unions of orbits which can arise as ω-limit sets in $\mathbb{R}^2$.

Definition. *A* cycle graph *S of a DE $x' = f(x)$ in $\mathbb{R}^2$ is a connected union of orbits such that*

1. *for all $x \in S$, $\omega(x) = \{p\}$ and $\alpha(x) = \{q\}$, where p and q are equilibrium points in S.*
2. *for all equilibrium points $p \in S$, there exists points $x, y \in S$ such that $\omega(x) = \{p\}$, $\alpha(y) = \{q\}$, and the number of equilibrium points in S is finite.*

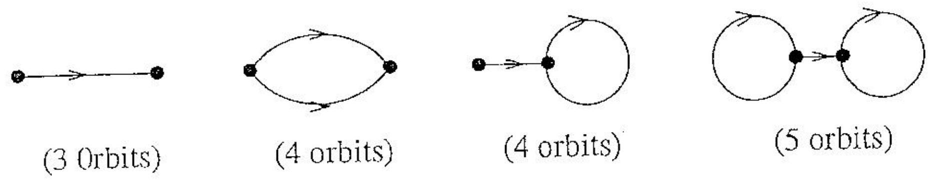


Figure 16: Examples of unions of orbits which *cannot* arise as ω-limit sets in $\mathbb{R}^2$.

3. *the orientations of the orbits define a continuous closed path in S.*

Comment: In 1 and 2, $\alpha(x)$ denotes the α-limit set of the point x, which is the set of past limit points of x, i.e.,

$$\alpha(x) = \left\{ y \middle| y = \lim_{n\to\infty} g^{t_n} x, \text{ and } t_n \overset{n}{\underset{\infty}{\rightarrow}} -\infty \right\}. \tag{93}$$

We can now state:

Theorem. *Consider a DE $x' = f(x)$ in $\mathbb{R}^2$. Let $a \in \mathbb{R}^2$ be an initial point such that $\{g^t a | t \geq 0\}$ lies in a closed bounded subset $K \subset \mathbb{R}^2$. If K contains only a finite number of equilibrium points then one of the following holds:*

1. *$\omega(a)$ is an equilibrium point*

2. *$\omega(a)$ is a periodic orbit*

3. *$\omega(a)$ is a cycle graph.*

Proof. The proof is based on the fundamental lemma of ω-limit sets in $\mathbb{R}^2$. [cf. Hale, Theorem 1.3, page 230, [4]] also [cf. Lefshetz, page 230, [7]]. □

Comment: This theorem does not generalize to DEs in $\mathbb{R}^n$, $n \geq 3$, or to DEs on the 2–torus. Indeed, the problem of describing all possible ω-limit sets in $\mathbb{R}^n$, $n \geq 3$, is presently unsolved.

6 STRUCTURAL STABILITY AND BIFURCATION THEORY

Structural Stability

In the theory of DEs, the word "stability" arises in two contexts, distinguished by the names *Liapunov Stability* and *Structural Stability*. *Liapunov Stability Question:* If a given physical system is perturbed from an equilibrium state, or from an oscillatory steady state, does the system remain close? Mathematically, one is concerned with the behaviour of orbits in a neighborhood of an equilibrium point, or of a periodic orbit.

On the other hand, the second concept arises from the *Structural Stability Question:* Consider the DE $x' = f(x)$ in $\mathbb{R}^n$, with flow g^t. Suppose that the vector field $f(x)$ is perturbed, giving a vector field $\tilde{f}(x)$, and the DE $x' = \tilde{f}(x)$, with flow $\tilde{g}^t$. Is the flow $\tilde{g}^t$ topologically equivalent to the flow g^t? In particular is the long-term behaviour of solutions of the two DEs the same? If the answer is NO, we say that the given vector field is *structurally unstable* (formal definition to follow).

We now give three examples of vector fields which are structurally unstable, thereby motivating three necessary conditions for structural instability.

Example.

Given DE: $x_1' = x_1^2, \quad x_2' = -x^2$ (one singular point)

Perturbed DE: $x_1' = -\mu + x_1^2, \quad x_2' = -x^2$ with $\mu > 0$ (two singular points)

As $\mu \to 0^+$ the flow $\tilde{g}^t$ of the perturbed DE reduces to the flow g^t of the given DE. However for $\mu > 0$, $\tilde{g}^t$ is not topologically equivalent to g^t. This shows that *the given vector field (DE) is not structurally stable.* The instability is due to the fact that *the equilibrium point* $(0,0)$ *of the given DE is not hyperbolic*, i.e., $Df(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

Example.

Given DE: $r' = -r(r^2-1)^2, \quad \theta' = -1$

Perturbed DE: $r' = [\mu - (r^2-1)^2]r, \quad \theta' = -1$ with $\mu > 0$,

in polar coordinates (r, θ). The flow $\tilde{g}^t$ of the perturbed DE is not topologically equivalent to the flow g^t of the given DE (since $\tilde{g}^t$ has two periodic orbits, and g^t has only one). This shows that the given vector field (DE) is not structurally stable. The instability is due to the fact that the *periodic orbit* $r = 1$ *does not attract or repel orbits in some neighborhoods.*

Example.

Given DE: $x_1 = -x_1^2 + 1, \quad x_2' = 2x_1x_2$

Perturbed DE: $x_1 = -x_1^2 + 1, \quad x_2' = 2x_1x_2 - \mu(1 - x_1^2)$, with $\mu > 0$.

The orbit which "joins" the two saddle points in the phase protrait of the given DE is called a *saddle connection.* Since the saddle connection is broken when $\mu > 0$, the flow of the perturbed DE is not topologically equivalent to the flow of the given DE. This shows that the given vector field (DE) is not structurally stable. The instability is due to *the presence of the saddle connection.*

Definition of Structural Stability

The statement "a vector field $\tilde{f}$ is a perturbation of a vector field f" means that the difference $\tilde{f} - f$ is small in some sense. In order to make this precise, one needs to define the *norm of a vector field.* For simplicity, we restrict our considerations to a bounded subset of $\mathbb{R}^n$.

Let $C^1(\mathcal{D})$ denote the set of C^1 vector fields which are defined on the subset $\mathcal{D} \subset \mathbb{R}^n$, where $\mathcal{D} = \left\{ x \in \mathbb{R}^n \middle| \, \|x\| \le R \right\}$, i.e., the solid sphere of radius R. We define a norm on $C^1(\mathcal{D})$, called the C^1*-norm*, by

$$\|f\|_1 = \max_{x \in \mathcal{D}} \|f\| + \max_{x \in \mathcal{D}} \|Df(x)\|,$$

where

$$\|f\| = \max_{1\le j\le n} |f_j(x)|; \qquad \|Df(x)\| = \max_{1\le j,k\le n} \left|\frac{\partial f_j(x)}{\partial x_k}\right|.$$

We now use this norm to define structural stability.

We restrict our consideration to vector fields in $C^1(\mathcal{D})$ which point inwards on the boundary of $\mathcal{D}$, so that $\mathcal{D}$ is a positively invariant set for the DE $x' = f(x)$. It is not essential that $\mathcal{D}$ be a solid sphere. We could use any subset of $\mathbb{R}^n$ that is homeomorphic to $\mathcal{D}$.

Definition. *A vector field $f \in C^1(\mathcal{D})$ which points inwards on the boundary of $\mathcal{D}$ is said to be structurally stable on $\mathcal{D}$ if there exists an $\epsilon > 0$ such that for all vector fields $\tilde{f}(x) \in C^1(\mathcal{D})$, $\left\|f - \tilde{f}\right\| < \epsilon$, then the flow determined by $\tilde{f}$ is topologically equivalent to the flow determined by f.*

Structural Stability in 2–D

The previous three examples suggest that the following conditions are necessary for $f(x) \in C^1(\mathcal{D})$ to be structurally stable on $\mathcal{D}$:

SS1: All equilibrium points are hyperbolic

SS2: All periodic orbits are hyperbolic

SS3: There are no saddle connections.

In the above, equilibrium points etc., refer to the DE determined by the vector field, i.e., $x' = f(x)$. These three conditions are in fact also sufficient in *two dimensions* as stated in the following theorem.

Theorem. *Suppose that the disc $\mathcal{D} = \left\{x \middle| \, \|x\| \le R\right\} \subset \mathbb{R}^2$ is a positively invariant set for the DE $x' = f(x)$. Then the vector field $f(x)$ is structurally stable on $\mathcal{D}$ if and only if conditions SS1, SS2, and SS3 are satisfied.*

Proof. [cf. Andronov and Pontrijagin, pages 247–251, [8]] □

The next question that arises is: Are "most" vector fields structurally stable, or are only a small subset structurally stable? In the 2-D case it was proved by Peixoto in 1962 that "most" vector fields are in fact structurally stable [9], where "most" has the following meaning.

Let $C^1_0(\mathcal{D})$ be the set of C^1 vector fields on $\mathcal{D} = \left\{x \in \mathbb{R}^2 \middle| \, \|x\| \le R\right\}$ which point inwards on the boundary of $\mathcal{D}$. Then we can state

Theorem. *The subset $C^1_0(\mathcal{D})$ which consists of all vector fields which are structurally stable on $\mathcal{D}$ is an open and dense subset of $C^1_0(\mathcal{D})$.*

Proof. [cf. Peixoto, pages 101–120, [9]] □

Comments:

1. "Open" and "Dense" are defined in terms of the C^1-norm in $C^1_0(\mathcal{D})$.

2. We have not defined formally the term *hyperbolic periodic orbit.* A periodic orbit is hyperbolic if it attracts (or repels) all orbits in some neighborhood, at an exponential rate. This can be expressed analytically in terms of eigenvalues of a certain matrix. [cf. Hirsch and Smale, Theorem 2, page 277, [1]]. (They use the term "periodic attractor (repellor)" for hyperbolic periodic orbit.)

Bifurcations of Equilibria

Consider a DE in $\mathbb{R}^n$ of the form $x' = f(x, \mu)$ where μ is a real parameter. Here f maps $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and hence, for each value of μ, defines a vector field on $\mathbb{R}^n$. Bifurcation theory, as applied to DEs, is the study of how the portrait of the orbits change as μ varies. One is interested in finding the values of μ for which a qualitative change in the orbit occurs. One thus considers values of μ for which the vector field $f(x, \mu)$ is not structurally stable.

Definition. *A value μ_0 for which the vector field $f(x, \mu)$ is not structurally stable (on a suitable disc) is called* a bifurcation value *of μ.*

The simplest bifurcations are those for which the lack of structural stability is due to the presence of a *non-hyperbolic equilibrium point.* Let us consider a simple bifurcation in one dimension that occurs when an equilibrium point has a zero eigenvalue.

Example. *Consider the DE $x' = \mu x - x^3$ where $x \in \mathbb{R}$ and μ is a parameter. The equilibrium points are given by $x(\mu - x^2) = 0$ (See Fig. 17).*

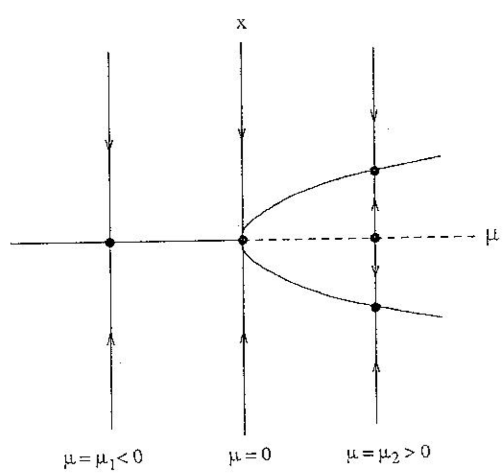


Figure 17: Bifurcation diagram for the DE $x' = \mu x - x^3$.

Comments:

1. The result of this bifurcation is the creation of two new equilibrium points, and a transfer of stability to the two new points. For obvious reasons, this is called a *pitchfork bifurcation*.

2. Other kinds of bifurcation in 1-D include the creation of two equilibrium points from one (called a *saddle-node bifurcation* in 2-D) [e.g., $x' = \mu - x^2 \quad x \in \mathbb{R}$], or the transfer of stability between two equilibrium points (called a *transcritical bifurcation*), [e.g., $x' = \mu x - x^2 \quad x \in \mathbb{R}$].

3. Often, the problem of identifying these bifurcation values in higher dimensional DEs can lead to lengthy algebra.

The Hopf Bifurcation

Consider the DE in polar coordinates $r' = (\mu - r^2)r, \quad \theta' = 1$ in $\mathbb{R}^2$. There is an equilibrium point at $r = 0$. For $\mu > 0$, $r = \sqrt{\mu}$ defines a periodic orbit of the DE. In addition, the linearization at the equilibrium point $r = 0$ is $Df(0, \mu) = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$, which implies that $r = 0$ is an attracting spiral if $\mu < 0$ and a repelling spiral if $\mu > 0$. Also note that if $\mu \leq 0$ then $r' < 0$ for all $r > 0$. The portraits in the cases $\mu < 0$, $\mu = 0$, and $\mu > 0$ are shown in Fig. 18. The information in these portraits can be presented more concisely in a *bifurcation diagram* in the μr-plane. (See Fig. 19.) Note that $\mu = 0$ is the biurcation value. The result of this bifurcation is 1) the creation of a stable equilibrium point, 2) the transfer of stability from an equilibrium point to a periodic orbit. This is the simpliest example of a *Hopf bifurcation*.

One can rotate the previous diagram about the μ-axis (to include the angular variable θ) to obtain a bifurcation diagram which shows the actual orbits. (See Fig. 20.)

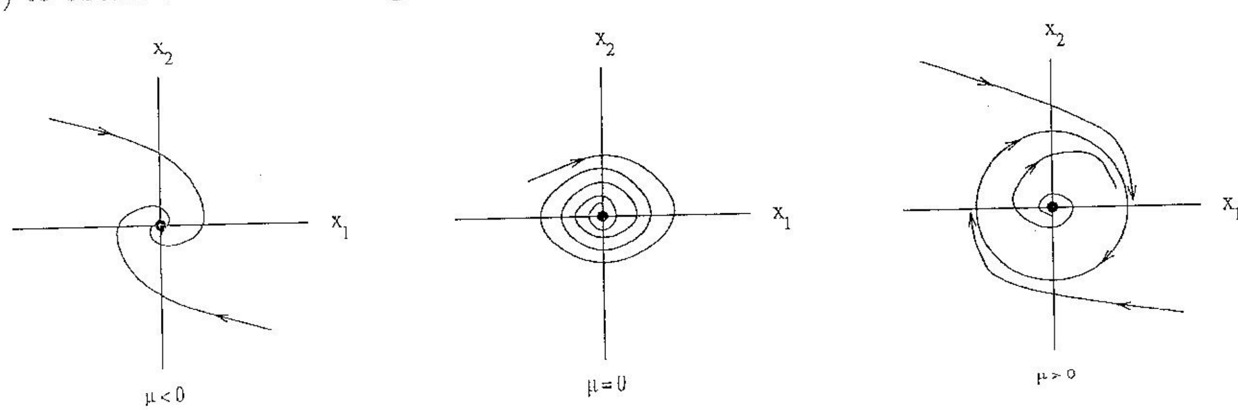


Figure 18: Phase portraits for the DE $r' = (\mu - r^2)r$ for differing values of μ.

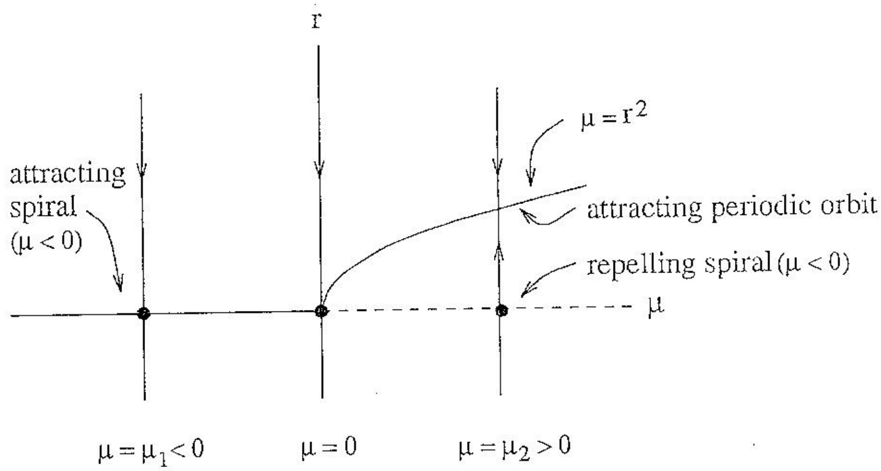


Figure 19: Bifurcation diagram for the DE $r' = (\mu - r^2)r$.

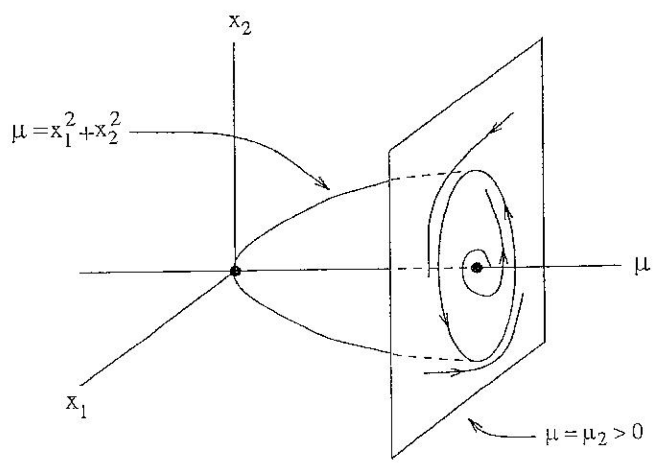


Figure 20: 3-D Bifurcation diagram for the Hopf bifurcation.

Theorem (Hopf). *Consider the DE* $x' = f(x, \mu)$ *in* $\mathbb{R}^2$, *where* $f \in C^3$. *Suppose* $f(0, \mu) = 0$ *for all* $\mu \in I \subset \mathbb{R}$, *and that* $\mathrm{D}f(0, \mu)$ *has eigenvalues* $\alpha(\mu) + i\beta(\mu)$. *If*

H1: *there exists* $\mu_0 \in I$ *such that* $\alpha(\mu_0) = 0$, $\beta(\mu_0) \neq 0$, $\alpha'(\mu_0) \neq 0$

H2: *the equilibrium point* $x = 0$ *is not a nonlinear center when* $\mu = \mu_0$

then

C: *there exists a* $\delta > 0$ *such that for each* $\mu \in (\mu_0, \mu_0 + \delta)$ *or* $\mu \in (\mu_0 - \delta, \mu_0)$, *the DE has a unique periodic orbit (when restricted to a sufficiently small neighborhood of* $x = 0$*).*

Proof. [cf. Hopf, vol. 94 , pages 1-22 and vol. 95, pages 3-22, [10]] □

Comments:

1. The hypothesis H1 guarantees that the equilibrium point $x = 0$ is *non-hyperbolic* when $\mu = \mu_0$, and changes stability at $\mu = \mu_0$.
2. The hypothesis H2 excludes the degenerate case, in which all the periodic orbits occur at $\mu = \mu_0$, as in the linear case (for example $x_1' = \mu x_1 + x_2, \quad x_2' = -x_1 + \mu x_2$.)

The Hopf theorem can be generalized to higher dimensions. The essential requirement is that the derivative matrix has *one pair* of pure imaginary eigenvalues and no other eigenvalues with zero real part. [cf. Guckenheimer and Holmes, page 151 [2]]

7 HIGHER DIMENSIONS

Invariant Tori and Quasiperiodic Orbits

Consider the motion of an undamped symmetric 2-mass oscillator, whose motion can be described by the DE

$$x_1' = \omega_1 x_2, \quad x_2' = -\omega_1 x_1, \quad x_3' = \omega_2 x_4, \quad x_4' = -\omega_2 x_3. \tag{94}$$

Our goal is to describe the ω-limit set $\omega(a)$ for a given initial state $a \in \mathbb{R}^4$. We let

$$x_1 = r_1 \sin\theta_1, \quad x_2 = r_1 \cos\theta_1, \quad x_3 = r_2 \sin\theta_2, \quad x_4 = r_1 \cos\theta_2, \tag{95}$$

where θ_1 and θ_2 assume values between 0 and 2π, and the values 0 and 2π are identified since they describe the same points in $\mathbb{R}^4$. The DE becomes

$$r_1' = 0, \quad \theta' = \omega_1, \quad r_2' = 0 \quad \theta_2' = \omega_2, \tag{96}$$

The solutions are

$$r_1 = C_1, \quad \theta = \alpha_1 + \omega_1 t, \quad r_2 = C_2 \quad \theta_2 = \alpha_2 + \omega_2 t, \tag{97}$$

where the constants $C_1, C_2, \alpha_1, \alpha_2$ are determined by the initial state. The orbits of the DE thus lie in the 2-surfaces $r_1 = C_1$, $r_2 = C_2$. Since these surfaces are parameterized by two variables θ_1 and θ_2 which have values modulo 2π (0 and 2π are identified), *each 2-surface with $C_1 > 0$ and $C_2 > 0$ is a 2-torus.* We note that these tori are invariant sets of the DE since they are unions of orbits. The nature of the orbits on each 2-torus depends on the values of the two constants ω_1 and ω_2, which are the two natural frequencies of oscillation of the physical system. In order to illustrate this suppose that $\omega_1 = \frac{1}{4}, \quad \omega_2 = 1$. (See Fig. 21.)

More generally, suppose that $\frac{\omega_1}{\omega_2} = \frac{n}{m}$, where m, n are positive integers without common factors. If $T = \frac{2\pi m}{\omega_1} = \frac{2\pi n}{\omega_2}$, then the solutions (97) satisfy

$$\theta_1(t + T) = \theta_1(t) + 2\pi m = \theta_1(t) \pmod{2\pi}$$

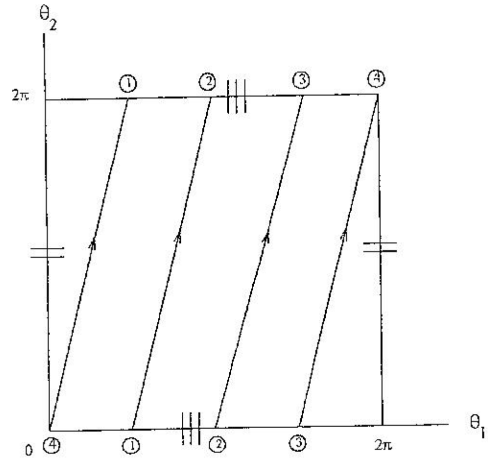


Figure 21: Plane representation of a 2-torus (= and ||| identified). The diagram shows an orbit corresponding to $\alpha_1 = \alpha_2 = 0$ in the solution. The points with the same number are identified, which shows that the orbit is periodic.

$$\theta_2(t+T) = \theta_1(t) + 2\pi n = \theta_2(t) \ (\text{mod} 2\pi)$$

Thus the solutions are periodic of period T. The corresponding orbit on one of the invariant tori is thus periodic, and eventually closes up as it winds around the torus.

On the other hand, if $\frac{\omega_1}{\omega_2}$ is *irrational*, then the orbits are not periodic, and hence do not close up as they wind around the invariant tori. What is not immediately obvious is that as the orbit winds around the torus, it passes arbitrarily close to each point of the torus. We say that the orbit is *everywhere dense on the torus.* [cf. Arnold, [11]]

We summarize the results as follows.

Proposition 22. *Consider the DE on the 2-torus defined by* $\theta_1 = \omega_1, \quad \theta_2 = \omega_2$.

1. *if* $\frac{\omega_1}{\omega_2}$ *is rational, then the orbits are periodic.*

2. *if* $\frac{\omega_1}{\omega_2}$ *is irrational, then the orbits are everywhere dense on the 2-torus.*

We can now draw the following conclusion concerning the ω-limit sets of the original DE (94). Consider an initial state $a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ and let $C_1 = a_1^2 + a_2^2, \quad C_2 = a_3^2 + a_4^2$.

1. If $\frac{\omega_1}{\omega_2}$ is rational, and $C_1, C_2 > 0$, then $\omega(a)$ is the periodic orbit $\gamma(a)$ which lies on the 2-torus $r_1 = C_1 \ r_2 = C_2$.

2. If $\frac{\omega_1}{\omega_2}$ is irrational, and $C_1, C_2 > 0$, then $\omega(a)$ is the 2-torus $r_1 = C_1 \ r_2 = C_2$.

Note that in case 2, $\omega(a)$ is the union of an uncountable infinity of whole orbits, including the orbit through a.

Quasiperiodic orbits

We now discuss the type of functions which describe the solutions of a DE for which the ω-limit set is a 2-torus.

Definition. *Suppose that $\Psi : \mathbb{R}^2 \to \mathbb{R}^n$ is periodic of period 2π in each argument. Suppose that $\omega_1, \omega_2 \in \mathbb{R}$ are rationally independent, i.e., $n_1\omega_1 + n_2\omega_2 \neq 0$ for all non-zero integers n_1, n_2. Then the function $f : \mathbb{R} \to \mathbb{R}^n$ defined by $f(t) = \Psi(\omega_1 t, \omega_2 t)$ is said to be 2-quasiperiodic.*

Example. *Consider the DE $x' = Ax$ in $\mathbb{R}^4$ where*

$$A = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix}.$$

Note the DE is linear and we can write the the solution $x(t) = e^{tA}$ where

$$e^{tA} = \begin{pmatrix} B(\omega_1 t) & 0 \\ 0 & B(\omega_2 t) \end{pmatrix}.$$

$$B(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Thus we can write $x(t) = \Psi(\omega_1 t, \omega_2 t)a$ with $\Psi : \mathbb{R}^2 \to \mathbb{R}^4$

$$\Psi(u_1, u_2) = \begin{pmatrix} B(u_1) & 0 \\ 0 & B(u_2) \end{pmatrix}$$

and is periodic of period 2π in each argument. Thus if ω_1, ω_2 are rationally independent, and the initial state a satisfies $a_1^2 + a_2^2 \neq 0$, $a_3^2 + a_4^2 \neq 0$ then the solution $x(t)$ is a 2-quasiperiodic function. The orbit through a, $\gamma(a)$ is dense on the two torus defined by $x_1^2 + x_2^2 = a_1^2 + a_2^2$, $x_3^2 + x_4^2 = a_3^2 + a_4^2$, and is called a 2-quasiperiodic orbit.

Attracting Sets and Long-Term Behaviour

Intuitively, an attracting set is a generalization of an asymptotically stable equilibrium point or periodic orbit.

Definition. *Given a DE $x' = f(x)$ in $\mathbb{R}^n$, a closed invariant set $A \subset \mathbb{R}^n$ is said to be an attracting set if there exists a neighborhood U of A such that*

1. *$g^t U \subseteq U$ for all $t \geq 0$*

2. *$\omega(a) \subseteq A$ for all $a \in U$,*

where g^t is the flow of the DE, and $\omega(a)$ is the ω-limit set of the point a.

Definition. *The* basin of attraction *of an attracting set A is the subset of $\mathbb{R}^n$ defined by* $\rho(A) = \{x \in \mathbb{R}^n | \omega(x) \subseteq A\}$.

If a DE has an attracting set A, then for all initial states a in the basin of attraction $\rho(A)$ the physical system approaches a "steady state" of some sort. The nature of the steady state is determined by the orbits which form the attractor. Some possibilities are summarized in the table to follow.

Attracting Sets	Long-term steady-state behaviour
Equilibrium Point	Equilibrium State
Periodic Orbit	Periodic
Invariant 2-Torus with dense orbits	2-Quasiperiodic
Invariant k-Torus with dense orbits	k-Quasiperiodic
"Strange Attractor"	"Chaotic" (none of the above)

Comment: Chaotic behaviour and Strange attractors are subjects of current research, and as yet there is no agreement on the definitions of the concepts. A Strange attractor is not a piecewise smooth surface, and can have a structure like that of a Cantor set. Chaotic behaviour occurs when neighboring orbits diverge (separate) from each other at an exponential rate, while remaining bounded, a phenomenon that is referred to as "sensitive dependence on initial conditions". [cf. Milnor, pages 177–195 [12]; Auslander, Bhatia, and Siefert, pages 55–56 [13]]

The previous example which admits invariant 2-tori does not admit an attracting set — the invariant 2-tori do not attract neighboring orbits. Likewise, a linear DE in $\mathbb{R}^2$ with a centre does not admit an attracting set. But just as one can use non-linearity to create an attracting periodic orbit, one can also create an attracting 2-torus.

Example. *Consider the DE in* $\mathbb{R}^4$,

$$\begin{aligned} x_1' &= \omega x_2 + x_1(\mu - r^2) \\ x_2' &= -\omega x_1 + x_2(\mu - r^2) \\ x_3' &= \nu x_4 + x_3(\lambda - R^2) \\ x_4' &= \nu x_3 + x_4(\lambda - R^2) \end{aligned}$$

with $r^2 = x_1^2 + x_2^2$ *and* $R^2 = x_3^2 + x_4^2$, *and* $\omega, \nu, \mu, \lambda$ *constants.*

In terms of "polar coordinates"

$$x_1 = r\sin\theta, \quad x_2 = r\cos\theta, \quad x_3 = R\sin\psi, \quad x_4 = R\cos\psi,$$

the DE becomes

$$\begin{aligned} r' &= (\mu - r^2)r, & \theta' &= \omega, \\ R' &= (\lambda - R^2)R, & \psi' &= \nu. \end{aligned}$$

It follows that if $\mu > 0$ and $\lambda > 0$ then the equations $r = \sqrt{\mu}$ and $R = \sqrt{\lambda}$ define an attracting set which is an invariant 2-torus. If in addition, ω and ν are rationally independent, then the orbits on the 2-torus are dense on the 2-torus. Thus the long-term behaviour of the system would be *quasiperiodic* (with two frequencies).

The next step in understanding the long-term behaviour is to study examples which exhibit chaotic behaviour.

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