#2 solns.

Q (a) $V = V(S)$

$$BS \Rightarrow \quad \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0.$$

method 1: Change $S = e^x$

$$\text{Eqn} \Rightarrow \quad \frac{1}{2}\sigma^2 \frac{d^2V}{dx^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{dV}{dx} - rV = 0.$$

ODE with const coefs $\Rightarrow$ aux eqn

$$\frac{1}{2}\sigma^2\lambda^2 + \left(r - \frac{1}{2}\sigma^2\right)\lambda - r = 0$$

$$\lambda = \frac{\left(\frac{1}{2}\sigma^2 - r\right) \pm \sqrt{\left(\frac{1}{2}\sigma^2 + r\right)^2}}{\sigma^2}$$

$$\lambda_1 = 1, \quad \lambda_2 = -2r/\sigma^2$$

$$\Rightarrow \quad V(S) = C_1 S + C_2{}^{-2r/\sigma^2}$$

[method 2: Assume soln of form $V = S^\alpha$]

(b) $V = A(t)B(S)$

$$BS \Rightarrow \quad \frac{A'}{A} = -\frac{\left(\frac{1}{2}\sigma^2 S^2 \ddot{B} + rS\dot{B} - rB\right)}{B}$$

Both sides $= K$ (const).

$$A' = kA \quad \Rightarrow \quad A(t) = a e^{kt}$$

$$\frac{1}{2}\sigma^2 S^2 \ddot{B} + rS\dot{B} - (r-k)B = 0 \quad \Rightarrow$$ solve in same as Q (a)

#3 Solns (3900).

Q: if $V = e^{\alpha x + \beta \tau} u(x,\tau)$ — $\frac{\partial V}{\partial x} = \alpha e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} \frac{\partial u}{\partial x}$

$\frac{\partial V}{\partial \tau} = \beta e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} \frac{\partial u}{\partial t}$.

etc $\Rightarrow$ sub into BS.

$$u(x,0) = \max\left\{ e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0 \right\}$$

Q: Let $V = u_1 - u_2 \Rightarrow \frac{\partial V}{\partial \tau} = \frac{\partial u_1}{\partial \tau} - \frac{\partial u_2}{\partial \tau}$.

sub into RHS (since u_1, u_2 satisfy BSE) $\Rightarrow$ V satisfies BSE.

$V(x,0) = 0$. Integral positive term position.

- $$\frac{\partial E}{\partial \tau} = \frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} V^2 dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial \tau}(V^2)\, dx = 2\int_{-\infty}^{\infty} V \frac{\partial V}{\partial \tau}\, dx$$

$$= 2\int_{-\infty}^{\infty} V \frac{\partial^2 V}{\partial x^2}\, dx = 2\cancel{\left[V \frac{\partial V}{\partial x} \right]_{-\infty}^{\infty}} - 2\int_{-\infty}^{\infty} \frac{\partial V}{\partial x}\frac{\partial V}{\partial x}\, dx$$

$$= -2\int_{-\infty}^{\infty} \underbrace{\left(\frac{\partial V}{\partial x}\right)^2}_{\text{pos.}} dx \le 0.$$

Q: Forward eqn: $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \Rightarrow u(x,\tau) = \sum_{n=1}^{\infty} a_n \sin(nx) e^{-n^2\tau}$

[a_n's constants]. Note as n increases, $e^{-n^2\tau}$ tend to zero for any τ.

Backward eqn: $\frac{\partial u}{\partial \tau} = -\frac{\partial^2 u}{\partial x^2}$

$\Rightarrow u(x,\tau) = \sum_{n=1}^{\infty} c_n \sin(nx) e^{n^2\tau}$. For any τ,

as n increases $e^{n^2\tau}$ increases without bound.

$$\Delta = \frac{\partial C}{\partial S} = N(d_1) + S N'(d_1) \frac{\frac{1}{S}}{\sigma(T-t)^{1/2}}$$

$$- E e^{-r(T-t)} N'(d_2) \frac{\frac{1}{S}}{\sigma(T-t)^{1/2}}$$

$$= N(d_1) + \frac{1}{\sqrt{2\pi}\,\sigma S (T-t)^{1/2}} \left\{ S e^{-\frac{1}{2} d_1^2} - E e^{-r(T-t)} e^{-\frac{1}{2} d_2^2} \right\}$$

$$= N(d_1) + [\quad] \left\{ S e^{\left[-\frac{1}{2}\left\{\cancel{\left(\frac{\ln(S/E)}{\sigma\sqrt{\ }}\right)^2} + \frac{2\ln(S/E)(r+\frac{1}{2}\sigma^2)\cancel{(T-t)}}{\sigma^2\cancel{(T-t)}} + \frac{(r+\frac{1}{2}\sigma^2)^2(T-t)^2}{\sigma^2(T-t)}\right\}\right]} - E e^{-r(T-t)} e^{-\frac{1}{2}\left[\cancel{\frac{\ln(S/E)^2}{\sigma\sqrt{\ }}} + \frac{2\ln(S/E)(r-\frac{1}{2}\sigma^2)}{\sigma^2} + \left[\frac{(r-\frac{1}{2}\sigma^2)^2}{\sigma^2}(T-t)\right]\right]} \right.$$

$$= N(d_1) + [\ ] e^{-\frac{1}{2}\frac{\ln^2(S/E)}{\sigma^2(T-t)}} \left\{ S e^{-\frac{1}{2}\left[2\ln(S/E)\left(\cancel{r}/\sigma^2 + \frac{1}{2}\right) + \frac{(r+\frac{1}{2}\sigma^2)^2(}{\sigma^2}\right.} - E e^{-r(T-t)} e^{-\frac{1}{2}\left[2\ln(S/E)\left(\cancel{r}/\sigma^2 - \frac{1}{2}\right) + \frac{(r-\frac{1}{2}\sigma^2)^2}{\sigma^2}(T\right.} \right.$$

$$= N(d_1) + [\ ] e^{-\frac{1}{2}[\ ]} e^{-\frac{1}{2}(2\ln(S/E) r/\sigma^2 - \frac{1}{2}\left(\frac{r^2 + \frac{1}{4}\sigma^4}{\sigma^4}\right)(T-t)} \Bigg\{$$

$$S e^{-\frac{1}{2} 2\ln(S/E)\frac{1}{2} - \frac{1}{2}\cdot 2r\frac{\frac{1}{2}\sigma^2}{\sigma^2}(T-t)} \Bigg\}$$

$$- E e^{-r(T-t)} e^{-\frac{1}{2} 2\ln(S/E) - \frac{1}{2} + \frac{1}{2}\, 2r\frac{\frac{1}{2}\sigma^2}{\sigma^2}(T} \Bigg\}$$

$$= N(d_1) + [\quad] \left\{ S e^{-\frac{1}{2}\sigma \ln(S/E) - \frac{1}{2} r(T-t)} - E e^{-r(T-t)} e^{\frac{1}{2}\sigma \ln(S/E) + \frac{1}{2} r(T-t)} \right\}$$

$$= N(d_1) + [\quad] e^{-\frac{1}{2} r(T-t)} \left\{ S e^{-\frac{1}{2}\sigma \ln(S/E)} - E e^{\frac{1}{2}\sigma \ln(S/E)} \right\}$$

$$= N(d_1) + [\quad] \left\{ S e^{(\ln S/E)^{-\frac{1}{2}\sigma}} - E e^{(\ln(S/E))^{\frac{1}{2}\sigma}} \right\}$$

$$= N(d_1) + [\quad] \left\{ S \left(\frac{S}{E}\right)^{-\frac{1}{2}\sigma} - E \left(\frac{S}{E}\right)^{\frac{1}{2}\sigma} \right\}$$

$$= N(d_1) + [\quad] \cancel{\left\{ \frac{S\sqrt{E}}{\sqrt{S}} - E \frac{\sqrt{S}}{\sqrt{E}} \right\}}$$

ZERO

$$= N(d_1)$$

$$C = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$\frac{\partial C}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - rEe^{-r(T-t)}N(d_2)$$

$$- Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}$$

$$= S\frac{1}{\sqrt{2\sqrt{\pi}}}e^{-\frac{1}{2}d_1^2}\left\{ \cancel{\sigma\sqrt{T-t}\,(-)(r+\tfrac{1}{2}\sigma^2)} - \frac{\log(S/E)}{\sigma}\frac{-1}{2}(T-t)^{-3/2} - \frac{(r^2+\frac{1}{2}\sigma^2)}{\sigma}\frac{1}{2}(T-t)^{-1/2}\right\}$$

$$- Ee^{-r(T-t)}\left\{ rN(d_2) + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2} \quad \right\}.$$

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M3900. **Solutions to Assignment 4:**

Q13 (P88): **physical explanation:** the equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + f(x,\tau)$$

is a heat equation with heat source $f(x,\tau)$. If $f(x,\tau) > 0$, there is an additional non-zero heat source, so the temperature $u(x,\tau)$ (without heat source) would increase. Since $u(x,\tau) = 0$ and $u \to 0$ as $x \to \pm\infty$, $u(x,\tau) \geq 0$.

For the call $C_1(S,t;E,T,\sigma_1)$, $C_2(S,t;E,T,\sigma_2)$, the following equations hold (since the Call option satisfies BSE):

$$\frac{\partial C_1}{\partial t} + \frac{1}{2}(\sigma_1 - \sigma_2 + \sigma_2)^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + rS\frac{\partial C_1}{\partial S} - rC_1 = 0 \qquad (1)$$

$$\frac{\partial C_2}{\partial t} + \frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 C_2}{\partial S^2} + rS\frac{\partial C_2}{\partial S} - rC_2 = 0 \qquad (2)$$

Eqns. (1)-(2) give

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC + \underline{\frac{1}{2}\left((\sigma_1 - \sigma_2)^2 + 2\sigma_2(\sigma_1 - \sigma_2)\right) S^2 \frac{\partial^2 C_1}{\partial S^2}} = 0 \qquad (3)$$

where $C \equiv C_1 - C_2$. Thus C is a solution of the BSE where the term underlined is effectively the function f above. We need to show this is positive. The term in parentheses is equivalent to $\frac{1}{2}\left({\sigma_1}^2 - {\sigma_2}^2\right)$, which is positive when $\sigma_1 > \sigma_2$, and $\frac{\partial^2 C_1}{\partial S^2} \geq 0$ (why?). This is sufficient. However, strictly speaking, we must transform to the heat eqn. and show that 'transformed f' is positive to use the first part of the question (see below).

In order to simplify (3) to the heat equation, we change the variables as follows:

$$S \equiv Ee^x, \quad t \equiv T - \frac{2\tau}{\sigma_2^2}, \quad C \equiv Ee^{\alpha x + \beta\tau} u(x,\tau), \quad \alpha = -\frac{1}{2}(\frac{2r}{\sigma_2^2} - 1), \quad \beta \equiv -\frac{1}{4}(\frac{2r}{\sigma_2^2} + 1)^2$$

In new variables, (3) becomes

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \underline{\frac{1}{2}\left((\sigma_1 - \sigma_2)^2 + 2\sigma_2(\sigma_1 - \sigma_2)\right) E^2 e^{2x} \frac{\partial^2 C_1}{\partial S^2}} = \frac{\partial^2 u}{\partial x^2} + \underline{\frac{1}{2}\left((\sigma_1 - \sigma_2)^2 + 2\sigma_2(\sigma_1 - \sigma_2)\right) Ee^x \frac{e^{-\frac{1}{2}d_1^2}}{\sigma_1\sqrt{2\pi\tau}}} = \frac{\partial^2 u}{\partial x^2} + \underline{f(x,\tau)}$$

from the def. of $f(x,\tau)$, we know $\underline{f(x,\tau) \geq 0}$. So

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + f(x,\tau)$$

Next we need to varify $u(x,0) = 0$, and $u \to 0$ as $x \to \pm\infty$, $u(x,\tau) \geq 0$.

$$\underline{u(x,0)} \equiv E^{-1}e^{-\alpha x}C(S,T) = E^{-1}e^{-\alpha x}\{C_1(S,T) - C_2(S,T)\} = E^{-1}e^{-\alpha x}\{max(S-E,0) - max(S-E,0)\}\underline{=0}$$

$$\underline{\lim_{x\to+\infty} u(x,0)} = E^{-1}\lim_{x\to+\infty} e^{-\alpha x - \beta\tau} \lim_{S\to\infty} C(S,T) = E^{-1}\lim_{x\to+\infty} e^{-\alpha x - \beta\tau}\{S - S\}\underline{=0}$$

and

$$\underline{\lim_{x\to-\infty} u(x,0)} = E^{-1}\lim_{x\to-\infty} e^{-\alpha x - \beta\tau} \lim_{S\to 0} C(S,T) = E^{-1}\lim_{x\to-\infty} e^{-\alpha x - \beta\tau}\{0 - 0\}\underline{=0}$$

Therefore from first part, we obtain $u(x,\tau \geq 0)$. By $C \equiv Ee^{\alpha x + \beta\tau}u(x,\tau)$, we get $C_1 - C_2 \equiv C \geq 0$ whenever $\sigma_1 > \sigma_2$.

For put option $P(S,t;E,T,\sigma)$, $P_1 - P_2 \geq 0$ whenever $\sigma_1 > \sigma_2$. Because $P(S,t;E,T,\sigma)$ satisfy the BSE, $\frac{\partial^2 P}{\partial S^2} > 0$ and both $P(\sigma_1)$ and $P(\sigma_2)$ have the same E, T, and hence we can show that the inequality holds.

Q6(p88): The key points are to calculate $\frac{\partial v}{\partial \tau}$ and $\frac{\partial^2 v}{\partial x^2}$. We have

$$\frac{\partial v}{\partial \tau} = \frac{\partial e^{-k\tau}V(\xi,\tau)}{\partial \tau} = -ke^{-k\tau}V(\xi,\tau)+e^{-k\tau}\frac{\partial V(\xi,\tau)}{\partial \xi}\frac{\partial \xi}{\partial \tau}+e^{-k\tau}\frac{\partial V}{\partial \tau} = -ke^{-k\tau}V(\xi,\tau)+e^{-k\tau}\frac{\partial V(\xi,\tau)}{\partial \xi}(k-1)+e^{-k\tau}\frac{\partial V}{\partial \tau}$$

and

$$\frac{\partial v}{\partial x} = \frac{\partial e^{-k\tau}V(\xi,\tau)}{\partial x} = e^{-k\tau}\frac{\partial V(\xi,\tau)}{\partial \xi}\frac{\partial \xi}{\partial x} = e^{-k\tau}\frac{\partial V(\xi,\tau)}{\partial \xi}; \qquad \frac{\partial^2 v}{\partial x^2} = e^{-k\tau}\frac{\partial^2 V(\xi,\tau)}{\partial \xi^2}$$

Substituting this into Eqn (5.10) we obtain the heat eqn. $\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2}$. The solution is then given as an integral in which the limit of integration depends on $\xi\tau^{-1/2}$ (why does this lead to problem?).

Q20 In the case of an asset having zero volatility, we have that μ is equal to r. So there is no contradiction between the two statements. To see that $\mu = r$ in the current case, note the **stochastic differential equation** (p20)

$$\frac{dS}{S} = \sigma\, dX + \mu\, dt$$

In the current case $\sigma = 0$, so the stochastic differential equation becomes

$$\frac{dS}{S} = \mu\, dt \qquad \Rightarrow \qquad S = S_0\, e^{\mu\, t}$$

So keeping the asset S is equivalent to saving money in the bank. The '*interest*' (risk-free profit) is μ which is equal to the interest r (by arbitrage). Thus, when $\sigma = 0$, $\mu = r$, so that μ is completely specified by r.

Q19(p89) The explicit expression for the European call is

$$\underline{C(S,t;E,T)} = S\,N(d_1) - E\,e^{-r(T-t)}N(d_2) = E\left(\frac{S}{E}N(d_1(\frac{S}{E},t,T)) - e^{-r(T-t)}N(d_2(\frac{S}{E},t,T))\right) \equiv \underline{E\,F(\frac{S}{E},t,T)}$$

where

$$d_{1,2} = \frac{\log(S/E) + (r \pm \frac{\sigma^2}{2}(T-t))}{\sigma\sqrt{T-t}}$$

So $C(S,t;E,T)$ is a function of S, t, E and T.

As a function of S and t, $C(S,t;E,T)$ satisfies the BSE (3.9) (on p43). We substitute

$$\frac{\partial}{\partial S}C(S,t;E,T) = F'(\frac{S}{E}); \qquad \frac{\partial^2}{\partial S^2}C(S,t;E,T) = \frac{F''(S/E)}{E}$$

into the BSE, and obtain

$$\frac{\partial}{\partial t}C(S,t;E,T) + \frac{\sigma^2 S^2}{2\,E}F''(S/E) + r\,SF'(S/E) - r\,E\,F(S/E) = 0 \tag{4}$$

where $C(S,t;E,T) = E\,F(S/E)$, $F' = \frac{\partial}{\partial \xi}F(\xi,t,T)$ and $\xi = S/E$.

Next we consider $C(S,t;E,T)$ as a function of E and t. By calculation, we get

$$\frac{\partial}{\partial t}C(S,t;E) + \frac{\sigma^2\,E^2}{2}\frac{\partial^2}{\partial E^2}C(S,t;E) - r\,E\frac{\partial}{\partial E}C(S,t;E) = \frac{\partial}{\partial t}C(S,t;E) + \frac{\sigma^2 S^2}{2\,E}F''(S/E) + r\,SF'(S/E) - r\,E\,F(S/E)$$

By (4) we know that RHS of above is equal to zero. Therefore we arrive at

$$\underline{\frac{\partial}{\partial t}C(S_0,t;E) + \frac{\sigma^2\,E^2}{2}\frac{\partial^2}{\partial E^2}C(S_0,t;E) - r\,E\frac{\partial}{\partial E}C(S_0,t;E) = 0} \tag{5}$$

Note that in the above

$$\frac{\partial C}{\partial E} = \frac{C}{E} - \frac{S}{E}F'(S/E); \qquad \frac{\partial^2 C}{\partial E^2} = \frac{S^2}{E^2}\frac{\partial^2 C}{\partial S^2}; \qquad etc. \tag{6}$$