3. RIEMANNIAN GEOMETRY AND TENSOR ANALYSIS.

3.1. Euclidean n-dimensional spaces.

A Euclidean n-space, E_n, is such that given n variables $x^1, \dots, x^n$, any set of particular values of these variable is regarded as a point in a manifold of n dimensions, and if x^i, $x^i + dx^i$ $(i = 1, \dots, n)$ are co-ordinates of two neighbouring points and these co-ordinates are rectangular cartesian, then the distance between the two points is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \dots + (dx^n)^2. \tag{3.1}$$

For example, an E_3 is such that

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{3.2}$$

If curvilinear co-ordinates are used the expressions (3.1), (3.2) are less simple. In spherical polar co-ordinates (3.2) becomes

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{3.3}$$

If in E_3 we define general curvilinear co-ordinates by

$$u^1 = u^1(x, y, z), \quad u^2 = u^2(x, y, z), \quad u^3 = u^3(x, y, z),$$

which are one-to-one transformations with non-vanishing Jacobian, then (u^1, u^2, u^3) can be used as co-ordinates in E_3. The inverse transformation is

$$x = x(u^1, u^2, u^3), \quad y = y(u^1, u^2, u^3), \quad z = z(u^1, u^2, u^3).$$

If two neighbouring points have rectangular cartesian co-ordinates (x, y, z) and $(x+dx, y+dy, z+dz)$ respectively, and have curvilinear co-ordinates (u^1, u^2, u^3) and $(u^1+du^1, u^2+du^2, u^3+du^3)$, then we have

$$dx = \frac{\partial x}{\partial u^1} du^1 + \frac{\partial x}{\partial u^2} du^2 + \frac{\partial x}{\partial u^3} du^3, \text{ etc.}$$

so that, from (3.2)

$$ds^2 = A\,du^{1\,2} + B\,du^{2\,2} + C\,du^{3\,2} + 2D\,du^2 du^3 + 2E\,du^3 du^1 + 2F\,du^1 du^2, \tag{3.4}$$

where, in general, A, B, C, D, E, F are functions of (u^1, u^2, u^3). Hence, in terms of curvilinear co-ordinates, an E_n is such that

$$ds^2 = g_{ij}\, du^i du^j, \tag{3.5}$$

where the coefficients g_{ij} are, in general, functions of the curvilinear co-ordinates u^i.

3.2 Generalized n-dimensional spaces.

Since the expression (3.5) refers to an E_n, it is possible to transform the curvilinear co-ordinates u^i to cartesian coordinates x^i so that

$$ds^2 = dx^i dx^i = (dx^1)^2 + \cdots + (dx^n)^2,$$

and, in order for this to be so, the g_{ij} must satisfy certain conditions. We may extend the theory of Euclidean spaces by considering spaces in which the 'distance' ds is given by (3.5) but in which the g_{ij} do not satisfy the Euclidean conditions. Such a space is called a Riemannian space of n-dimensions and is denoted by V_n. An E_n is a particular type of V_n. The expression (3.5) is called the metric of the V_n.

Consider the surface of a sphere of radius a. The co-ordinates of a point on the surface are θ and ϕ and this surface is a two-dimensional space. The distance between two neighbouring points on the surface is obtained by putting $r = a$ in the expression (3.3), i.e.

$$ds^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{3.6}$$

In this case $g_{11} = a^2$, $g_{12} = g_{21} = 0$, $g_{22} = a^2\sin^2\theta$. It is not possible to find co-ordinates (x, y) such that (3.6) takes the form $ds^2 = dx^2 + dy^2$, so this surface is a V_2, but not an E_2.

If a surface is developable, such as a cylinder or a cone, then it is Euclidean.

We can generalize still further by considering spaces in which ds^2 is not expressible as a quadratic differential form and also by considering completely general spaces in which ds, i.e. the distance between two neighbouring points, is not defined. Such a space is called an L_n and we now deal with tensors in these spaces.

3.3 Tensor algebra.

Consider an L_n with co-ordinates x^i $(i = 1, \ldots, n)$, and consider another co-ordinate system $x^{i'}$, where

$$x^{i'} = x^{i'}(x^i) \tag{3.7}$$

If P has co-ordinates $x^i, x^{i'}$ in the two systems and a neighbouring point Q has co-ordinates $x^i + dx^i$, $x^{i'} + dx^{i'}$, then

$$dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i , \qquad (3.8)$$

where it is assumed that the co-ordinate transformation (3.7) and its inverse

$$x^i = x^i(x^{i'}) \qquad (3.9)$$

are differentiable and have non-zero Jacobian. Note that co-ordinate transformations may be considered as a relabeling of a point in an L_n or as a mapping of a point in one L_n onto a point in another L_n.

If the transformations are linear, the quantities $\frac{\partial x^{i'}}{\partial x^i}$ and $\frac{\partial x^i}{\partial x^{i'}}$ are constants. However, in general, these quantities are functions of the x^i and $x^{i'}$, respectively, and the values of expressions involving these quantities vary from point to point.

Definitions.

(i) A set of n components A^i transforming according to the expression

$$A^{i'} = \frac{\partial x^{i'}}{\partial x^i} A^i \qquad (3.10)$$

is a contravariant vector, or a contravariant tensor of rank one, or a $(1,0)$ tensor.

(ii) A set of n components A_i transforming according to the expression

$$A_{i'} = \frac{\partial x^i}{\partial x^{i'}} A_i \qquad (3.11)$$

is a covariant vector, or a covariant tensor of rank one, or a $(0,1)$ tensor.

(iii) A set of n^r components $T^{i_1 i_2 \dots i_r}$ transforming according to the law

$$T^{i_1' i_2' \dots i_r'} = \frac{\partial x^{i_1'}}{\partial x^{i_1}} \cdots \frac{\partial x^{i_r'}}{\partial x^{i_r}} T^{i_1 i_2 \dots i_r} \qquad (3.12)$$

is a contravariant tensor of rank r, or a $(r,0)$ tensor.

(iv) A set of n^r components $T_{i_1 i_2 \dots i_r}$ transforming according to the law

$$T_{i_1' i_2' \dots i_r'} = \frac{\partial x^{i_1}}{\partial x^{i_1'}} \cdots \frac{\partial x^{i_r}}{\partial x^{i_r'}} T_{i_1 \dots i_r} \qquad (3.13)$$

is a covariant tensor of rank r, or a $(0,r)$ tensor.

(v) A set of n^r components $T^{i_1 \cdots i_a}_{j_1 \cdots j_b}$ $(a+b=r)$ transforming according to the law

$$T^{i_1' \cdots i_a'}_{j_1' \cdots j_b'} = \frac{\partial x^{i_1'}}{\partial x^{i_1}} \cdots \frac{\partial x^{i_a'}}{\partial x^{i_a}} \frac{\partial x^{j_1}}{\partial x^{j_1'}} \cdots \frac{\partial x^{j_b}}{\partial x^{j_b'}} T^{i_1 \cdots i_a}_{j_1 \cdots j_b} \qquad (3.14)$$

is a <u>mixed tensor of rank r</u>, or a (a,b) tensor.

(vi) A quantity ϕ which remains unchanged during the co-ordinate transformation, i.e.

$$\phi' = \phi \qquad (3.15)$$

is called a <u>scalar</u> or <u>invariant</u> in L_n.

<u>Examples of tensors.</u>

(a) From (3.8) the coordinate differentials dx^i satisfy the condition (3.10) and so are the components of a contravariant vector. Note that the co-ordinates x^i are not the components of a vector, since the equation $x^{i'} = \frac{\partial x^{i'}}{\partial x^i} x^i$ holds only when the transformations are linear and homogeneous. In this case the x^i are the components of a restricted contravariant vector.

(b) If ϕ is a scalar, then the quantities $\frac{\partial \phi}{\partial x^i}$ satisfy the equations

$$\frac{\partial \phi}{\partial x^{i'}} = \frac{\partial \phi}{\partial x^i} \frac{\partial x^i}{\partial x^{i'}},$$

so that $\frac{\partial \phi}{\partial x^i}$ transforms like a covariant vector. It is called the <u>gradient</u> of ϕ and is written $\phi_{,i}$ thus showing its covariant character.

(c) The Kronecker delta, δ^i_j, is defined by

$$\delta^i_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \qquad (3.16)$$

Now $\delta^i_j = \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{i'}}{\partial x^j} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} \delta^{i'}_{j'}$, so δ^i_j is a mixed tensor of rank two, i.e. a $(1,1)$ tensor, as indicated by the suffixes.

If the components of a tensor A^{ijk} satisfy $A^{ijk} = A^{jik}$, then A^{ijk} is said to be <u>symmetric</u> in the suffixes i and j. If $A^{ijk} = -A^{ikj}$, the tensor is said to be <u>antisymmetric</u> (or <u>skewsymmetric</u>) in the suffixes j and k. It is easily shown that the symmetry properties of a tensor are preserved under transformations as long as the pair of suffixes are either both subscripts or both superscripts. In general, the symmetry properties are not retained if one index is a subscript and one is a superscript. The Kronecker delta is an exception.

A second-rank tensor T^{ij} may be written in the form

$$T^{ij} = \tfrac{1}{2}\left[T^{ij} + T^{ji}\right] + \tfrac{1}{2}\left[T^{ij} - T^{ji}\right], \tag{3.17}$$

which shows that any such tensor may be regarded as the sum of a part which is symmetric and a part which is antisymetric in a given pair of suffixes. We introduce the notation

$$\begin{aligned} T_{k(ij)l\ldots} &= \tfrac{1}{2}\left[T_{kijl\ldots} + T_{kjil\ldots}\right], \\ T_{k[ij]l\ldots} &= \tfrac{1}{2}\left[T_{kijl\ldots} - T_{kjil\ldots}\right], \end{aligned} \tag{3.18}$$

for the symmetric and antisymmetric parts of a tensor $T_{kijl\ldots}$ with respect to a pair of suffixes. The symmetric and antisymmetric parts with respect to non-adjacent suffixes are denoted by

$$T_{(k|i|j)l\ldots} = \tfrac{1}{2}\left[T_{kijl} + T_{jikl}\right], \text{ etc.}$$

3.4 <u>Operations with tensors.</u>

(i) <u>Addition and subtraction.</u>

If $A^{i\ldots j}_{k\ldots l}$ and $B^{i\ldots j}_{k\ldots l}$ are two tensors of the same type, i.e. both (m,n) tensor, then their sum or difference

$$A^{i\ldots j}_{k\ldots l} \pm B^{i\ldots j}_{k\ldots l} = C^{i\ldots j}_{k\ldots l}$$

is also a tensor of the same type. For example, consider

$$\begin{aligned} C^{i'j'} &= A^{i'j'} \pm B^{i'j'} \\ &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}\left(A^{ij} \pm B^{ij}\right) \\ &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}C^{ij}. \end{aligned}$$

Note that the tensors must be added or subtracted at the same point, otherwise $\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}$ will not have the same value for A^{ij} as for B^{ij}, and so is not a common factor.

(ii) <u>Outer multiplication.</u>

Two tensors, say $A^{ij}{}_k$ and $B_{lm}{}^{np}$, can be multiplied together to give a new tensor $C^{ijnp}_{klm} = A^{ij}{}_k B_{lm}{}^{np}$, whose rank is the sum of the ranks of the two tensors and which has the same number of subscripts and superscripts. It is easily shown that C^{ijnp}_{klm} transforms as such a tensor since

$$\begin{aligned} C^{i'j'n'p'}_{k'l'm'} = A^{i'j'}{}_{k'} B_{l'm'}{}^{n'p'} &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^k}{\partial x^{k'}}A^{ij}{}_k \frac{\partial x^l}{\partial x^{l'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{n'}}{\partial x^n}\frac{\partial x^{p'}}{\partial x^p}B_{lm}{}^{np} \\ &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^l}{\partial x^{l'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{n'}}{\partial x^n}\frac{\partial x^{p'}}{\partial x^p}A^{ij}{}_k B_{lm}{}^{np} \end{aligned}$$

$$\text{i.e. } C^{i'j'n'p'}_{k'l'm'} = \left(\qquad\qquad \text{"} \qquad\qquad \right) C^{ijnp}_{klm}$$

(iii) Inner multiplication.

If the above tensors are multiplied together but with summation over a pair of suffixes, the result is a tensor of lower rank. For example, $A^{ij}{}_k B_{im}{}^{np} = C^{jnp}_{km}$ is a tensor of rank 5.

The inner product of two vectors, A^i and B_j, is a scalar because

$$A^{i'} B_{i'} = \frac{\partial x^{i'}}{\partial x^i} A^i \frac{\partial x^j}{\partial x^{i'}} B_j = \delta_i{}^j A^i B_j = A^i B_i = A^j B_j,$$

so that $A^i B_i = \phi$, the scalar product of the two vectors.

(iv) Contraction.

A mixed tensor such as T^{ij}_k may be contracted by setting a subscript and a superscript as the same letter, e.g. T^{ij}_i. This gives a quantity which as n components, instead of n^3 components, and transforms as a vector, i.e.

$$\begin{aligned} T^{j'} \equiv T^{i'j'}_{i'} &= \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^k}{\partial x^{i'}} T^{ij}_k \\ &= \delta_i{}^k \frac{\partial x^{j'}}{\partial x^j} T^{ij}_k \\ &= \frac{\partial x^{j'}}{\partial x^j} T^{ij}_i = \frac{\partial x^{j'}}{\partial x^j} T^j \end{aligned}$$

It follows that a contracted tensor is of rank two less than that of the original tensor, i.e. a (m,n) tensor contracts to a $(m-1, n-1)$ tensor. Note that the tensor resulting from inner multiplication is the contraction of the corresponding outer multiple.

(v) Quotient Theorem.

This allows us to test whether or not a set of quantities form the components of a tensor.

Theorem. A set of symbols, whose product (inner or outer) with an arbitrary tensor is a tensor, forms a tensor.

Let A^i_j be the set of symbols and B^k an arbitrary tensor, and suppose that

$$A^i{}_j B^j = C^i,$$

where C^i is a tensor. We have to show that $A^i{}_j$ is a tensor.

Transforming to a new frame, we have

$$A^{i'}{}_{j'} B^{j'} = C^{i'}.$$

Since B^j, C^i are tensors, we have

$$B^{j'} = \frac{\partial x^{j'}}{\partial x^j} B^j, \quad C^{i'} = \frac{\partial x^{i'}}{\partial x^i} C^i,$$

and thus

$$A^{i'}_{\ j'}\frac{\partial x^{j'}}{\partial x^j}B^j = \frac{\partial x^{i'}}{\partial x^i}C^i$$
$$= \frac{\partial x^{i'}}{\partial x^i}A^i_{\ j}B^j$$
$$\therefore \left(\frac{\partial x^{j'}}{\partial x^j}A^{i'}_{\ j'} - \frac{\partial x^{i'}}{\partial x^i}A^i_{\ j}\right)B^j = 0$$

Since B^j is arbitrary we can choose its components to be $(1,0,0,\ldots)$, and then $(0,1,0,\ldots)$, and so on. This shows that the quantity in the bracket is identically zero, i.e.

$$\frac{\partial x^{j'}}{\partial x^j}A^{i'}_{\ j'} = \frac{\partial x^{i'}}{\partial x^i}A^i_{\ j}.$$

Inner multiplication by $\frac{\partial x^j}{\partial x^{k'}}$ gives

$$A^{i'}_{\ k'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{k'}}A^i_{\ j},$$

so that $A^i_{\ j}$ is a tensor of the type indicated by its suffixes.

(vi) <u>Conjugate tensors.</u>

Consider a symmetric covariant tensor g_{ij} with determinant $g \equiv |g_{ij}| \neq 0$. Let G^{ij} be the cofactor of the element g_{ij} in the determinant. We shall now show that $g^{ij} = G^{ij}/g$ is a symmetric covariant tensor.

From the theory of determinants we have

$$g_{ij}G^{kj} = g\delta_i^{\ k}, \quad g_{ij}G^{ik} = g\delta_j^{\ k} \tag{3.19}$$

so that, dividing by g, we find

$$g_{ij}g^{kj} = \delta_i^{\ k}, \quad g_{ij}g^{ik} = \delta_j^{\ k} \tag{3.20}$$

Choose an arbitrary contravariant vector A^i, then $B_i = g_{ij}A^j$ is also an arbitrary vector, since $g \neq 0$ implies that the components of A^i can always be calculated from this last equation. Now we have

$$g^{ij}B_i = g^{ij}g_{ik}A^k = \delta^j_{\ k}A^k = A^j,$$

using equation (3.20). Hence, by the quotient theorem, g^{ij} is a contravariant tensor, and it follows that it is symmetric like g_{ij}. The tensors g_{ij}, g^{ij} are said to be conjugate to one another.

(vii). <u>Tensor equations.</u>

It follows from the transformation laws that if all components of a given tensor vanish in one co-ordinate system, then all components vanish in all co-ordinate systems. Hence, if a physical law is expressed in tensor form by saying that one tensor equals another, then, since the difference of the tensors is zero, it follows that the tensors are equal in any other co-ordinate system so that the validity of the law is independent of the co-ordinate system employed. Also if a tensor equation is established in a special co-ordinate system, then it is valid in general.

3.5. Tensor densities.

Given a transformation of co-ordinates

$$x^{i'} = x^{i'}(x^{i}),$$

we denote by J the Jacobian determinant of the transformation, i.e. $J = \left|\frac{\partial x^{i}}{\partial x^{i'}}\right|$.

We define a relative tensor, $\mathfrak{R}^{i\ldots j}_{k\ldots l}$, as a quantity which transforms according to the rule

$$\mathfrak{R}^{i'\ldots j'}_{k'\ldots l'} = J^{W}\frac{\partial x^{i'}}{\partial x^{i}}\cdots\frac{\partial x^{j'}}{\partial x^{j}}\frac{\partial x^{k}}{\partial x^{k'}}\cdots\frac{\partial x^{l}}{\partial x^{l'}}\mathfrak{R}^{i\ldots j}_{k\ldots l}, \tag{3.21}$$

where W is a positive or negative integer. $\mathfrak{R}$ is said to be a relative tensor of weight W having covariant and contravariant characteristics as indicated by the suffixes.

If $W = 0$, the relative tensor is an ordinary tensor. If $W = 1$, the relative tensor is called a tensor density.

The following results are easily proved:

(i) The addition or subtraction of two relative tensor of the same type and weight results in new relative tensors of the same type and weight.

(ii) The outer product of two relative tensors is a relative tensor whose weight is the sum of the weights of its factors.

(iii) A relative tensor may be contracted, reducing its rank by two, but leaving its weight unaltered.

Note that area and volume elements are tensor densities of zero rank, i.e. scalar densities. For example, in E_3, using cartesian co-ordinates, the volume element dV is

$$dV = dx\,dy\,dz = dx^{1}dx^{2}dx^{3}$$

and using spherical polars it is

$$dV' = r^{2}\sin\theta\,dr\,d\theta\,d\phi = (x^{1'})^{2}\sin x^{2'}\,dx^{1'}dx^{2'}dx^{3'}.$$

In this transformation from $x^{i'}$ to x^i, $J = \left|\frac{\partial x^i}{\partial x^{i'}}\right| = r^2 \sin^2\theta = (x^{1'})^2 \sin u^{2'}$, so the volume elements obey

$$dx^1 dx^2 dx^3 = J\, dx^{1'} dx^{2'} dx^{3'}$$

and so the volume elements are scalar densities.

Consider a contravariant tensor density of rank n, $\varepsilon^{ij\ldots k}$, which is antisymmetric with respect to every suffix. Its components are determined as follows:

$$\begin{aligned}\varepsilon^{ij\ldots k} &= 0, \text{ if any two suffixes are the same,}\\ &= \varepsilon^{12\ldots n}, \text{ if } i, j, \ldots, k \text{ is an even permutation of } 1, 2, \ldots, n.\\ &= -\varepsilon^{12\ldots n}, \text{ if } i, j, \ldots, k \text{ is an odd permutation of } 1, 2, \ldots, n.\end{aligned}$$

If in some co-ordinate system we take $\varepsilon^{12\ldots n} = +1$, then in the primed system

$$\begin{aligned}\varepsilon^{1'2'\ldots n'} &= J \frac{\partial x^{1'}}{\partial x^i}\frac{\partial x^{2'}}{\partial x^j} \cdots \frac{\partial x^{n'}}{\partial x^k}\varepsilon^{ij\ldots k}\\ &= JK\end{aligned}$$

where K is the determinant with $i'j$-element $\frac{\partial x^{i'}}{\partial x^j}$, i.e. $K = \left|\frac{\partial x^{i'}}{\partial x^j}\right|$. Now $JK = \left|\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^j}\right| = \left|\frac{\partial x^i}{\partial x^j}\right| = |\delta^i_j| = 1$, so $\varepsilon^{1'2'\ldots n'} = 1$, and hence it follows that $\varepsilon^{ij\ldots k}$ has the same components $0, \pm 1$ in all frames.

The tensor density $\varepsilon^{ij\ldots k}$ is known as the <u>Levi-Civita tensor density</u>, or the <u>permutation symbol</u>.

The determinants of second-rank tensors are relative invariants. For example, if A_{ij} is a covariant tensor, then

$$\begin{aligned} A_{i'j'} &= \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} A_{ij}\\ \text{i.e. } |A_{i'j'}| &= \left|\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} A_{ij}\right|\\ &= \left|\frac{\partial x^i}{\partial x^{i'}}\right|\left|\frac{\partial x^j}{\partial x^{j'}}\right|\left|A_{ij}\right|\\ \text{i.e. } |A_{i'j'}| &= J^2 |A_{ij}|, \end{aligned} \qquad (3.22)$$

so that $|A_{ij}|$ is a relative invariant of weight 2.

Similarly, the determinant $|A^{ij}|$ of a contravariant tensor is a relative invariant of weight -2, while the determinant $|A^i{}_j|$ of a mixed tensor is an invariant.

3.6 <u>Parallel displacement and covariant differentiation</u>.

In order to have further operations with tensors, the general space L_n must have some structure. In particular, a satisfactory definition of differentiation of tensors is not possible in the basic L_n.

Now the partial derivative of a tensor (other than a scalar) is not a tensor. For example, if A_i is a vector, then

$$A_{i'} = \frac{\partial x^i}{\partial x^{i'}} A_i ,$$

so that

$$\begin{aligned} \frac{\partial A_{i'}}{\partial x_{j'}} \equiv \; & A_{i',j'} \\ = \; & \left(\frac{\partial x^i}{\partial x^{i'}} A_i \right)_{,j'} \\ = \; & \frac{\partial x^i}{\partial x^{i'}} A_{i,j'} + \frac{\partial^2 x^i}{\partial x^{i'} \partial x_{j'}} A_i \\ \text{i.e.} \quad A_{i',j'} = \; & \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x_{j'}} A_{i,j} + \frac{\partial^2 x^i}{\partial x^{i'} \partial x_{j'}} A_i , \end{aligned} \tag{3.23}$$

thus showing that $A_{i,j}$ is not a tensor, unless we a restricted to linear transformations.

Let P and Q be neighbouring points with co-ordinates x^i and $x^i + dx^i$ and let A_i and $A_i + dA_i$ [i.e. $A_i(x^i + dx^i) = A_i(x^i) + dA_i$] be the vectors of a covariant vector field associated with P and Q. Since these tensors are associated with different points, their transformation laws will be different, so their difference, dA_i, will not be a vector. However,

$$dA_i = \frac{\partial A_i}{\partial x_j} dx^j , \tag{3.24}$$

and dx^j is a vector, so $\frac{\partial A_i}{\partial x_j}$ cannot be a tensor, as before. In order to obtain a derivative which is a tensor we must find a procedure which involves comparison of two vectors defined at the same point.

In an E_n, a vector A_i may undergo parallel displacement from a point P to another point Q so that it suffers no change in magnitude and direction. If rectangular cartesian co-ordinates are employed then the displaced vector will have the same components as the original vector, but if curvilinear co-ordinates are used, the components of the displaced vector and the original vector will not have the same components, since the axes at P and Q will not, in general, be in the same directions. In the more general L_n, we say that parallel displacement (in some sense) takes place giving us a displaced vector with components $A_i + \delta A_i$

which can be compared with the field vector $A_i + dA_i$ at the same point Q. Since these are both vectors defined at the same point, their difference $dA_i - \delta A_i$ is a vector. Generalizing (3.24) we write

$$dA_i - \delta A_i = A_{i;j}\,dx^j, \tag{3.25}$$

where, by the quotient rule, $A_{i;j}$ is a second rank tensor known as the <u>covariant derivative</u> of A_i.

We have defined the tensor derivative of a vector in terms of parallel displacement, but we have yet to define parallel displacement in L_n. The definition that we adopt must correspond to the usual definition in E_n since E_n is a special case of L_n. In E_n, when cartesian co-ordinates are used, the components of a vector A_i remain unchanged and also the scalar product $A_i B^i$ of two vectors must remain unchanged (since it is a scalar) so that the angle between the vectors is constant. In L_n we assume that when A_i undergoes an infinitesimal parallel displacement, its scalar product with an arbitrary vector B^i remains invariant.

To define parallel displacement in L_n, first consider a vector A_i in E_n. Let x^i be curvilinear co-ordinates and let $x^{i'}$ be rectangular cartesian co-ordinates in E_n. Then

$$A_i = \frac{\partial x^{i'}}{\partial x^i} A_{i'} \quad \text{and} \quad A_{i'} = \frac{\partial x^i}{\partial x^{i'}} A_i. \tag{3.26}$$

If A_i is parallelly displaced from P to Q, the cartesian components $A_{i'}$ do not change, i.e. $\delta A_{i'} = 0$. From (3.26) we have

$$\begin{aligned}\delta A_i &= \delta\left(\frac{\partial x^{i'}}{\partial x^i} A_{i'}\right) = \delta\left(\frac{\partial x^{i'}}{\partial x^i}\right) A_{i'} \\ &= \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}\,dx^j A_{i'}\end{aligned} \tag{3.27}$$

Substituting for $A_{i'}$, we find

$$\delta A_i = \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}\frac{\partial x^k}{\partial x^{i'}} A_k\,dx^j,$$

$$\text{i.e. } \delta A_i = \Gamma^k_{ij} A_k\,dx^j, \tag{3.28}$$

where

$$\Gamma^k_{ij} = \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}\frac{\partial x^k}{\partial x^{i'}} \tag{3.29}$$

Conforming to this, we define δA_i in L_n by (3.28), where the n^3 quantities Γ^k_{ij} are determined arbitrarily at every point of L_n, subject to being continuous functions of x^i and possessing continuous partial derivatives of the necessary order. The quantities Γ^k_{ij} are the components of an <u>affinity</u> which specifies an <u>affine connection</u> between points of L_n.

We now have

$$dA_i - \delta A_i = A_{i;j}\,dx^j = A_{i,j}\,dx^j - \Gamma^k_{ij} A_k\,dx^j,$$

so the covariant derivative is given by

$$A_{i;j} = A_{i,j} - \Gamma^k_{ij} A_k \tag{3.30}$$

and this is a tensor.

Note that $A_{i;j}$ and $A_{i,j}$ are equal if the components of Γ^k_{ij} vanish over some region of L_n. This will be true only in the reference frame employed; transformation to another frame will, in general, result in non-zero Γ^k_{ij}. Only covariant derivatives can appear in tensor equations which are to be true in all frames.

3.7 <u>Transformation and properties of the affinity.</u>

Although the components of the affinity may be chosen arbitrarily in any other frame are fixed by its transformation law, which we shall now find.

Since $A_{i;j}$ is a tensor we have

$$A_{i';j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} A_{i;j}$$

and also

$$A_{i'} = \frac{\partial x^i}{\partial x^{i'}} A_i$$

Now

$$A_{i';j'} = A_{i',j'} - \Gamma^{k'}_{i'j'} A_{k'}$$

so that

$$\frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} A_{i;j} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} A_{i,j} + \frac{\partial^2 x^i}{\partial x^{i'} \partial x^{j'}} A_i - \Gamma^{k'}_{i'j'} A_k \frac{\partial x^k}{\partial x^{k'}} \tag{3.31}$$

But

$$A_{i;j} = A_{i,j} - \Gamma^k_{ij} A_k,$$

$$\text{i.e.} \quad \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} A_{i;j} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} A_{i,j} - \Gamma^k_{ij} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} A_k. \tag{3.32}$$

Comparing (3.31) and (3.32) we have

$$\Gamma^{k'}_{i'j'} \frac{\partial x^k}{\partial x^{k'}} = \frac{\partial^2 x^k}{\partial x^{i'} \partial x^{j'}} + \Gamma^k_{ij} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}}$$

Multiplying both sides by $\frac{\partial x^{\ell'}}{\partial x^k}$ we obtain

$$\Gamma^{k'}_{i'j'} = \frac{\partial x^{k'}}{\partial x^k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\Gamma^k_{ij} + \frac{\partial x^{k'}}{\partial x^k}\frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}} \qquad (3.33)$$

which is the required transformation law.

From equation (3.33) we see that if the components of Γ^k_{ij} are zero in one frame, they are not necessarily zero in any other frame. In fact, in general, there is no frame in which the components vanish over a region of L_n.

The major properties of an affinity are:

(i) If Γ^k_{ij} is symmetric with respect to i and j in one frame, then it is symmetric in every frame.

(ii) The difference of two affinities, defined over a region of L_n, is a tensor since

$$\Gamma^{k'}_{i'j'} - \bar{\Gamma}^{k'}_{i'j'} = \frac{\partial x^{k'}}{\partial x^k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}(\Gamma^k_{ij} - \bar{\Gamma}^k_{ij}),$$

from (3.33), so that $\Gamma^k_{ij} - \bar{\Gamma}^k_{ij}$ is a tensor of rank three.

(iii) The average $\frac{1}{2}(\Gamma^k_{ij} + \bar{\Gamma}^k_{ij})$ of two affinities is an affinity, from (3.33).

(iv) It follows that we can write

$$\Gamma^k_{ij} = \Gamma^k_{(ij)} + \Gamma^k_{[ij]},$$

so that $\Gamma^k_{[ij]}$ is a tensor and $\Gamma^k_{(ij)}$ is an affinity. If $\Gamma^k_{[ij]}$ is zero in one frame then it is zero in all, giving an alternative proof of (i).

(v) If we are restricted to linear transformations, then Γ^k_{ij} is a tensor, because the last term in (3.33) will then be zero.

(vi) For any affinity we can make $\Gamma^k_{(ij)} = 0$ in at least one co-ordinate system at one point.

<u>Proof</u>. Let the given point be O and let it be the origin of the co-ordinate system so that, at O, $x^i = 0$. Transform to a new co-ordinate system, $x^{i'}$, by the transformation

$$x^i = \delta^i_{i'}x^{i'} + \tfrac{1}{2}a^i_{j'k'}x^{j'}x^{k'} + O[(x^{i'})^3], \qquad (3.34)$$

where the $a^i_{j'k'}$ are constants. Now

$$\frac{\partial x^i}{\partial x^{i'}} = \delta^i_{i'} + a^i_{j'i'}x^{j'} + \dots$$

$$\frac{\partial^2 x^i}{\partial x^{i'}\partial x^{j'}} = a^i_{j'i'} + \dots$$

At O, $x^i = 0$, so

$$\frac{\partial x^i}{\partial x^{i'}} = \delta^i_{i'}, \quad \frac{\partial^2 x^i}{\partial x^{i'}\partial x^{j'}} = a^i_{j'i'}.$$

$$\text{i.e. } \Gamma^{k'}_{(i'j')} = \delta^{k'}_k\delta^i_{i'}\delta^j_{j'}\Gamma^k_{(ij)} + \delta^{k'}_k a^k_{(j'i')},$$

where we note that, from (3.34), $a^{i'}_{j'k'}$ is symmetric. By choosing $a^{k}_{j'i'}$ in the correct way, $\Gamma^{k'}_{(i'j')} = 0$ (at O only), i.e.

$$a^{k}_{(j'i')} = -\delta^{i}_{i'}\delta^{j}_{j'}\Gamma^{k}_{(ij)}.$$

Such a co-ordinate system is called a <u>geodesic co-ordinate system</u> at the point O, which is its <u>pole</u>. In such a system, if Γ^{k}_{ij} is symmetric, the covariant and partial derivatives are identical at O, which is useful since, if a tensor equation is valid in a geodesic system, it is valid in all frames.

3.8. <u>Covariant derivatives of tensors</u>.

For a covariant vector A_i we have

$$A_{i;j} = A_{i,j} - \Gamma^{k}_{ij}A_k. \qquad (3.35)$$

For other types of tensor we find:

(i) <u>Scalar, ϕ</u>. When ϕ undergoes parallel displacement it remains unaltered, i.e. $\delta\phi = 0$ and so

$$\phi_{;i} = \phi_{,i}, \qquad (3.36)$$

which is a vector, as required.

(ii) <u>Contravariant vector B^i</u>. Now A_iB^i is a scalar, so

$$(A_iB^i)_{;j} = (A_iB^i)_{,j}$$

$$\text{i.e. } A_{i;j}B^i + A_iB^i{}_{;j} = A_{i,j}B^i + A_iB^i{}_{,j}$$

$$\text{i.e. } (A_{i;j} - A_{i,j})B^i = -(B^i{}_{;j} - B^i{}_{,j})A_i$$

$$-\Gamma^{k}_{ij}A_kB^i = -(B^i{}_{;j} - B^i{}_{,j})A_i$$

$$\therefore B^i{}_{;j}A^j = (B^i{}_{,j} + \Gamma^{i}_{kj}B^k)A_i$$

Choose A_i to be an arbitrary vector, so

$$B^i{}_{;j} = B^i{}_{,j} + \Gamma^{i}_{kj}B^k, \qquad (3.37)$$

which is the covariant derivative of B^i.

(iii) <u>Tensor D^i_{jk}</u>. This transforms like $A^iB_jC_k$ and we find that

$$D^i_{jk;l} = D^i_{jk,l} + \Gamma^{i}_{ml}D^m_{jk} - \Gamma^{m}_{jl}D^i_{mk} - \Gamma^{m}_{kl}D^i_{jm}. \qquad (3.38)$$

The extension to tensors of all ranks is obvious.

The usual rules apply to covariant differentiation, i.e.

(a) If $T^i{}_{jk} = A^i{}_{jk} + B^i{}_{jk}$, then $T^i{}_{jk;l} = A^i{}_{jk;l} + B^i{}_{jk;l}$. (3.39)

(b) If $T^i = A^i{}_{jk}B^{jk}$, then $T^i{}_{;l} = A^i{}_{jk;l}B^{jk} + A^i{}_{jk}B^{jk}{}_{;l}$ (3.40)

Note that the Kronecker delta is a constant under covariant differentiation, since

$$\delta^i_{j;k} = \delta^i_{j,k} + \Gamma^i_{\ell k}\delta^\ell_j - \Gamma^\ell_{jk}\delta^i_\ell ,$$

but $\delta^i_{j,k} = 0$, so

$$\delta^i_{j;k} = \Gamma^i_{jk} - \Gamma^i_{jk} = 0.$$

3.9 <u>Example.</u>

Consider an E_2 with plane polar co-ordinates (r,θ). Take cartesian co-ordinates (x,y) as the unprimed system and (r,θ) as the primed system, i.e. $(x,y) \equiv (x^1, x^2)$; $(r,\theta) \equiv (x^{1'}, x^{2'})$.

$$x = r\cos\theta \ , \quad y = r\sin\theta \ ; \quad r = \sqrt{x^2+y^2} \ , \quad \tan\theta = \frac{y}{x}.$$

For the (x,y) system, $\Gamma^k_{ij} = 0$.

$$\frac{\partial x^{1'}}{\partial x^1} = \cos\theta, \quad \frac{\partial x^{1'}}{\partial x^2} = \sin\theta, \quad \frac{\partial x^{2'}}{\partial x^1} = -\frac{1}{r}\sin\theta, \quad \frac{\partial x^{2'}}{\partial x^2} = \frac{1}{r}\cos\theta$$

$$\frac{\partial^2 x^i}{\partial x^{1'}\partial x^{1'}} = 0, \quad \frac{\partial^2 x^1}{\partial x^{1'}\partial x^{2'}} = -\sin\theta, \quad \frac{\partial^2 x^2}{\partial x^{1'}\partial x^{2'}} = \cos\theta, \quad \frac{\partial^2 x^i}{\partial x^{2'}\partial x^{2'}} = -x^i.$$

Since $\Gamma^k_{ij} = 0$, $\Gamma^{k'}_{i'j'} = \frac{\partial x^{k'}}{\partial x^k}\frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}}$ and is symmetric. Then

$$\Gamma^{1'}_{1'1'} = 0 \ , \quad \Gamma^{1'}_{1'2'} = \cos\theta(-\sin\theta) + \sin\theta(\cos\theta) = 0,$$

$$\Gamma^{1'}_{2'2'} = \cos\theta(-r\cos\theta) + \sin\theta(-r\sin\theta) = -r$$

$$\Gamma^{2'}_{1'2'} = -\frac{1}{r}\sin\theta(-\sin\theta) + \frac{1}{r}\cos\theta(\cos\theta) = \frac{1}{r}$$

$$\Gamma^{2'}_{2'2'} = -\frac{1}{r}\sin\theta(-r\cos\theta) + \frac{1}{r}\cos\theta(-r\sin\theta) = 0.$$

i.e. $\Gamma^{1'}_{2'2'} = -r$ and $\Gamma^{2'}_{1'2'} = \Gamma^{2'}_{2'1'} = \frac{1}{r}$ are the only non-zero Γ's.

3.10. <u>Curl of a covariant vector</u>

Consider a covariant vector A_i. Then $A_{i;j} = A_{i,j} - \Gamma^k_{ij}A_k$. Twice the antisymmetric part of this expression is

$$2A_{[i;j]} = 2A_{[i,j]} - 2\Gamma^k_{[ij]}A_k \qquad (3.41)$$

The first term is the covariant curl, the second term is the ordinary curl $(A_{i,j} - A_{j,i})$ and the third term is a tensor. Hence, the ordinary curl is a tensor. If Γ^k_{ij} is symmetric, the covariant curl and ordinary curl are identical.

3.11. <u>Covariant differentiation of relative tensors.</u>

By using the concept of parallel displacement and requiring that the covariant derivatives of $\varepsilon^{ij\dots k}$ and $\varepsilon_{ij\dots k}$ be identically zero (since they have the same components in all frames and at all points), it can be shown that the covariant derivative of a relative tensor of weight W is

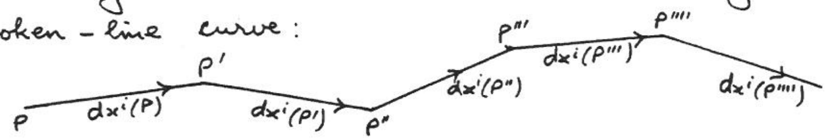
$$\mathfrak{R}^{\ell}{}_{i;j} = \mathfrak{R}^{\ell}{}_{i,j} + \Gamma^{\ell}_{kj}\mathfrak{R}^{k}{}_{i} - \Gamma^{k}_{ij}\mathfrak{R}^{\ell}{}_{k} - W\Gamma^{k}_{kj}\mathfrak{R}^{\ell}{}_{i}, \qquad (3.42)$$

and, in particular, for a relative invariant $\mathfrak{R}$,

$$\mathfrak{R}_{;i} = \mathfrak{R}_{,i} - W\Gamma^{k}_{ki}\mathfrak{R}. \qquad (3.43)$$

3.12. Geodesics.

Starting with some infinitesimal displacement vector, dx^i, at a point P, we may parallel-transport this vector along its own direction to the point P'. This gives a new infinitesimal vector at P' that we can displace along its own direction to P''. Continuing this process we obtain a broken-line curve:



Hence, one can parallel-transport the vector dx^i from any point to some other point. As the size of the displacement tends to zero, the broken line becomes a continuous curve. This curve starts from P with a well-defined direction and continues to another point at a finite distance. Such a curve is called a geodesic.

Now introduce a parameter to designate points along the curve. Let s be chosen to transform as a scalar and to have an invariant value at P and P'. Then $\frac{dx^i}{ds}$ is a vector. The condition for a geodesic is that the components of the vector $\left(\frac{dx^i}{ds}\right)_P$ displaced to a point P' are identical with those of the vector $\left(\frac{dx^i}{ds}\right)_{P'}$, i.e.

$$\left(\frac{dx^i}{ds}\right)_P - \Gamma^i_{jk}\frac{dx^j}{ds}\Delta x^k = \left(\frac{dx^i}{ds}\right)_{P'}, \qquad (3.44)$$

where Δx^k corresponds to dx^k used in equation (3.28), for example. Dividing this equation by Δs, the displacement from P to P', and taking the limit as $\Delta s \to 0$:

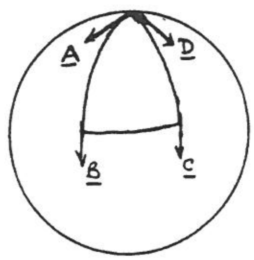
$$\lim_{\Delta s \to 0}\left\{\frac{(dx^i/ds)_{P'} - (dx^i/ds)_P}{\Delta s} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{\Delta x^k}{\Delta s}\right\} = 0$$

$$\text{i.e.} \quad \frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \qquad (3.45)$$

This is the equation for a geodesic in terms of a scalar parameter s for a curve. Note that, in E_n using cartesian co-ordinates, (3.45) becomes $\frac{d^2x^i}{ds^2} = 0$, which is the equation for straight lines. Hence, in E_n geodesics are straight lines.

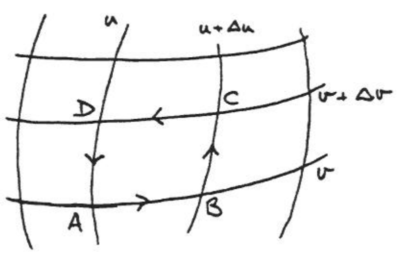
3.13. The Riemann – Christoffel curvature tensor.

If a vector undergoes parallel transport about a closed path in E_n, then the resultant vector is identical with the original one. However, in general, this is not true in a curved space. For example, consider the surface of a sphere on which there is a spherical triangle composed of geodesic curves, i.e. two lines of longitude and the equator. When $\underline{A}$ is parallel – transported it finally becomes $\underline{D} \neq \underline{A}$.



Consider the change in the components of a vector under parallel displacement about an infinestimal closed path defined by four curves of the two parameter set

$$x^i = f^i(u,v).$$



Let the path pass through the four points

$A(u,v)$; $B(u+\Delta u, v)$; $C(u+\Delta u, v+\Delta v)$; $D(u, v+\Delta v)$, so that ABCDA is a complete circuit. Now the absolute change in a vector A_i due to parallel displacement through dx^j is $\delta A_i = dA_i - A_{i;j}dx^j$. But the sum of dA_i round the complete circuit is zero, so we ignore it in what follows. As we go round the circuit we find:

(i) from A to B, the absolute change is $-A_{i;j}\frac{\partial x^j}{\partial u}\Delta u$, calculated for v,

(ii) from B to C, the absolute change is $-A_{i;j}\frac{\partial x^j}{\partial v}\Delta v$, calculated for $u+\Delta u$,

(iii) from C to D, the absolute change is $+A_{i;j}\frac{\partial x^j}{\partial u}\Delta u$, calculated for $v+\Delta v$,

(iv) from D to A, the absolute change is $+A_{i;j}\frac{\partial x^j}{\partial v}\Delta v$, calculated for u.

From (ii) and (iv) the net result is the difference of the changes $A_{i;j}\frac{\partial x^j}{\partial v}\Delta v$ at $u+\Delta u$ and at u, respectively. We require not $\frac{\partial}{\partial u}\left(A_{i;j}\frac{\partial x^j}{\partial v}\Delta v\right)\Delta u$, but the absolute difference

$$-\left(A_{i;j}\frac{\partial x^j}{\partial v}\Delta v\right)_{;k}\frac{\partial x^k}{\partial u}\Delta u$$

$$\text{i.e. } -A_{i;jk}\frac{\partial x^j}{\partial v}\frac{\partial x^k}{\partial u}\Delta u\Delta v, \qquad (3.46)$$

since $dx^j = \frac{\partial x^j}{\partial v}\Delta v$ is the same for (ii) and (iv).

Similarly, (i) and (iii) give

$$+A_{i;jk}\frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}\Delta u\Delta v,$$

so that the total absolute change round the circuit is

$$\Delta A_i = (A_{i;jk} - A_{i;kj})\frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}\Delta u\Delta v \qquad (3.47)$$

In a Euclidean space, the covariant and partial derivatives are identical and (3.47) would be zero, showing that there is no absolute change in the vector on parallel displacement round a circuit. However, in general, we have $A_{i;jk} \neq A_{i;kj}$, so that A_i will be changed when we reach the initial point. Now

$$A_{i;jk} = (A_{i,j} - \Gamma^{\ell}_{ij}A_\ell)_{,k} + \Gamma^m_{ik}(A_{m,j} - \Gamma^{\ell}_{mj}A_\ell) - \Gamma^m_{jk}(A_{i,m} - \Gamma^{\ell}_{im}A_\ell),$$
$$A_{i;kj} = (A_{i,k} - \Gamma^{\ell}_{ik}A_\ell)_{,j} + \Gamma^m_{ij}(A_{m,k} - \Gamma^{\ell}_{mk}A_\ell) - \Gamma^m_{kj}(A_{i,m} - \Gamma^{\ell}_{im}A_\ell),$$

so that

$$A_{i;jk} - A_{i;kj} = (\Gamma^{\ell}_{ik,j} - \Gamma^{\ell}_{ij,k} + \Gamma^m_{ik}\Gamma^{\ell}_{mj} - \Gamma^m_{ij}\Gamma^{\ell}_{mk})A_\ell - (\Gamma^m_{jk} - \Gamma^m_{kj})A_{i;m} \qquad (3.48)$$

Since $\Gamma^m_{jk} - \Gamma^m_{kj}$ is a tensor, it follows that the quantity in the first bracket is a tensor. We write

$$A_{i;jk} - A_{i;kj} = R^{\ell}{}_{ijk}A_\ell - (\Gamma^m_{jk} - \Gamma^m_{kj})A_{i;m} \qquad (3.49)$$

where

$$R^{\ell}{}_{ijk} = \Gamma^{\ell}_{ik,j} - \Gamma^{\ell}_{ij,k} - \Gamma^{\ell}_{mj}\Gamma^m_{ik} - \Gamma^{\ell}_{mk}\Gamma^m_{ij} \qquad (3.50)$$

is a fourth-rank tensor known as the <u>Riemann-Christoffel tensor</u>, or <u>curvature tensor</u>. Note that $R^{\ell}{}_{ijk}$ is anti-symmetric in j and k.

If the affinity is symmetric then

$$A_{i;jk} - A_{i;kj} = R^{\ell}{}_{ijk}A_\ell \qquad (3.51)$$

In rectangular cartesian co-ordinates $A_{i;jk} - A_{i;kj} = A_{i,jk} - A_{i,kj} = 0$ so $R^{\ell}{}_{ijk} = 0$ for Euclidean space, and, since it is a tensor, it will be zero whatever co-ordinate system is used in this space. The vanishing of $R^{\ell}{}_{ijk}$ guarantees that an arbitrary vector will not change under parallel displacement. It follows that $R^{\ell}{}_{ijk} = 0$ is a necessary and sufficient condition for the space to be flat (Euclidean) and, on the other hand, it provides a measure of the curvature of non-flat spaces.

From (3.47) we see that the absolute change in A_i round a closed circuit is

$$\Delta A_i = R^{\ell}{}_{ijk}A_\ell\frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}\Delta u\Delta v. \qquad (3.52)$$

Since this expression contains only first order terms in $\Delta u, \Delta v$, it is more precise to write

$$\lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \frac{\Delta A_i}{\Delta u \Delta v} = R^\ell{}_{ijk} A_\ell \frac{\partial x^j}{\partial u} \frac{\partial x^k}{\partial v} \qquad (3.53)$$

3.14 Properties of the curvature tensor.

(i) If $R^\ell{}_{ijk}$ is contracted with respect to ℓ and j, we obtain the Ricci tensor $R_{ik} \equiv R^\ell{}_{i\ell k} = -R^\ell{}_{ik\ell}$, where

$$R_{ik} = \Gamma^\ell_{ik,\ell} - \Gamma^\ell_{i\ell,k} + \Gamma^\ell_{m\ell}\Gamma^m_{ik} - \Gamma^\ell_{mk}\Gamma^m_{i\ell}. \qquad (3.54)$$

Contraction of $R^\ell{}_{ijk}$ with respect to ℓ and i produces the tensor S_{jk} given by

$$S_{jk} \equiv R^\ell{}_{\ell jk} = \Gamma^\ell_{\ell k,j} - \Gamma^\ell_{\ell j,k}. \qquad (3.55)$$

(ii) Using (3.50) and cyclically permuting the lower suffixes, we obtain

$$R^\ell{}_{jki} = \Gamma^\ell_{ji,k} - \Gamma^\ell_{jk,i} + \Gamma^\ell_{mk}\Gamma^m_{ji} - \Gamma^\ell_{mi}\Gamma^m_{jk},$$
$$R^\ell{}_{kij} = \Gamma^\ell_{kj,i} - \Gamma^\ell_{ki,j} + \Gamma^\ell_{mi}\Gamma^m_{kj} - \Gamma^\ell_{mj}\Gamma^m_{ki},$$

and adding these to (3.50) yields

$$R^\ell{}_{ijk} + R^\ell{}_{jki} + R^\ell{}_{kij} = A^\ell_{ik,j} + A^\ell_{ji,k} + A^\ell_{kj,i} + \Gamma^\ell_{mj}A^m_{ik} + \Gamma^\ell_{mk}A^m_{ji} + \Gamma^\ell_{mi}A^m_{kj}, \qquad (3.56)$$

where $A^\ell_{ik} = \Gamma^\ell_{ik} - \Gamma^\ell_{ki} = 2\Gamma^\ell_{[ik]}$. Note that if Γ^ℓ_{ij} is symmetric, then

$$R^\ell{}_{ijk} + R^\ell{}_{jki} + R^\ell{}_{kij} = 0. \qquad (3.57)$$

(iii) Choose a geodesic co-ordinate system at some point. Then, at this point Γ^ℓ_{ij}, but not necessarily its derivatives, vanish, so that (3.50) becomes

$$R^\ell{}_{ijk} = \Gamma^\ell_{ik,j} - \Gamma^\ell_{ij,k},$$

$$\text{i.e. } R^\ell{}_{ijk;l} = R^\ell{}_{ijk,l} = \Gamma^\ell_{ik,jl} - \Gamma^\ell_{ij,kl}.$$

Writing down two similar expressions obtained by cyclically permuting j, k, l, we find

$$R^\ell{}_{ijk;l} + R^\ell{}_{ikl;j} + R^\ell{}_{ilj;k} = 0 \qquad (3.58)$$

Since this is a tensor equation and valid in the geodesic frame, it must be valid in all frames. Also, the chosen point can be any point of L_n, so it valid at all points. The expression (3.58) is the Bianchi identity.

Example.

Using plane polar co-ordinates in E_2 show that the curvature tensor vanishes.

$r = x^1$, $\theta = x^2$. Non-zero Γ's are $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$, $\Gamma^1_{22} = -r$.
Curvature tensor components must be of the form $R^\ell{}_{i12}$ since it is antisymmetric in the last two suffixes.

$$R^\ell{}_{i12} = \Gamma^\ell_{i2,1} - \Gamma^\ell_{i1,2} + \Gamma^\ell_{m1}\Gamma^m_{i2} - \Gamma^\ell_{m2}\Gamma^m_{i1}.$$

If $k = i = 1$ or $k = i = 2$, components are zero since we then have Γ's with two 1's or all 2's. Only other components are

$$R^1{}_{212} = \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^1_{m1}\Gamma^m_{22} - \Gamma^1_{m2}\Gamma^m_{21}$$
$$= \Gamma^1_{22,1} - 0 + 0 - \Gamma^1_{22}\Gamma^2_{21}$$
$$= -1 - (-r)(\tfrac{1}{r}) = 0.$$
$$R^2{}_{112} = \Gamma^2_{12,1} - \Gamma^2_{11,2} + \Gamma^2_{m1}\Gamma^m_{12} - \Gamma^2_{12}\Gamma^m_{11}$$
$$= \Gamma^2_{12,1} - 0 + \Gamma^2_{21}\Gamma^2_{12} - 0$$
$$= -\tfrac{1}{r^2} + \tfrac{1}{r^2} = 0.$$

Hence, all components of $R^k{}_{ijl} = 0$.

3.15. Riemannian space.

We now further specify the space L_n by introducing the idea of 'distance' between neighbouring points. This is given by

$$ds^2 = g_{ij}\,dx^i\,dx^j, \qquad (3.59)$$

where the n^2 coefficients g_{ij} are specified at every point of L_n in some co-ordinate system. Such a space is called a Riemannian space and is denoted by V_n. The expression (3.59) is called the metric of V_n and corresponds to the first fundamental form for a two-dimensional surface. The quantity ds is regarded as an invariant for any two neighbouring points, so it follows that g_{ij} is a second-rank tensor which may be taken to be symmetric, without loss of generality. It is called the fundamental tensor, or metric tensor, of V_n.

As in section 3.4 (vi), we can form the conjugate tensor g^{ij}, i.e. the contravariant form of g_{ij}, which exists provided that $|g_{ij}| \equiv g \neq 0$ which is assumed to be true.

If A^j is a contravariant tensor (vector) defined at a point of V_n, then $A_i = g_{ij}A^j$ is a covariant vector at the same point; A_i and A^i are regarded as the covariant and contravariant components of the same vector.

Similarly, we have $A^i = g^{ij}A_j$, so that the suffix can be raised. Any suffix or suffixes can be raised in the same way, e.g.

$$A^{ij}{}_k = g^{il}A_l{}^j{}_k = g^{il}g_{km}A_l{}^{jm} = g^{il}g_{km}g^{jn}A_{ln}{}^m.$$

If a suffix of g_{ij} is raised we have

$$g^i{}_j = g^{ik}g_{kj} = \delta^i_j,$$

so that the Kronecker delta is the mixed component of the fundamental tensor.

The inner product of two vectors A^i, B_i is

$$A^i B_i = g^{ij} A_j B_i = A_j B^j = A_i B^i. \qquad (3.60)$$

For rectangular cartesian co-ordinates in E_n, the metric is

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2,$$

$$\text{i.e.} \quad ds^2 = g_{ij} dx^i dx^j,$$

where

$$g_{ij} = 1 \ (i = j) \ ; \ g_{ij} = 0 \ (i \neq j)$$

Since $g = 1$, we have

$$g^{ij} = 1 \ (i = j) \ ; \ g^{ij} = 0 \ (i \neq j).$$

If A_i is a vector in E_n, then $A^i = g^{ij} A_j$, so that, in E_n with rectangular cartesian co-ordinates, the covariant and contravariant components are the same.

Note that if the metric of V_n is diagonal, i.e.

$$g_{ij} = 0 \ , \text{ if } i \neq j,$$

then we find

$$g^{ij} = 0 \ (i \neq j) \ ; \ g^{ij} = \frac{1}{g_{ij}} \ (i = j). \qquad (3.61)$$

The squared magnitude of a vector is defined as the invariant $(A)^2$ given by

$$(A)^2 = A^i A_i = g_{ij} A^i A^j = g^{ij} A_i A_j. \qquad (3.62)$$

Note that A_i, A^i have the same magnitudes. If $A^i A_i = 0$, then A^i is a <u>null vector</u>. Such a vector can only exist if the metric is indefinite.

The angle, θ, between two vectors is, by analogy with E_3, given by

$$AB\cos\theta = A^i B_i$$

$$\text{i.e.} \quad \cos\theta = \frac{A^i B_i}{\sqrt{(A^k A_k)(B^m B_m)}}. \qquad (3.63)$$

If $\theta = \frac{\pi}{2}$, then $A_i B^i = 0$, and the two vectors are said to be <u>orthogonal</u>.