3.16 Christoffel symbols.

At any point P of V_n it is possible to find a regular linear transformation of co-ordinates, which may involve complex coefficients, such that

$$g_{ij}\,dx^i dx^j = (dy^1)^2 + (dy^2)^2 + \dots + (dy^n)^2.$$

The y^i will behave like rectangular cartesian co-ordinates in E_n over a small neighbourhood of P and in this neighbourhood Γ^k_{ij} will be of the form (3.29) and so is symmetric in i and j at P. Since this is true at every point of V_n, we take the affinity of a Riemannian space to be symmetric.

Apart from its symmetry, the affinity is still arbitrary. Now the covariant derivative $A_{i;j}$ is obtained from A^i by lowering the suffix and differentiating and, unless these two operations commute, difficulties will arise. Hence, we define the affinity so that these operations commute, i.e.

$$\begin{aligned}(g_{ij}A^j)_{;k} &= g_{ij;k}A^j + g_{ij}A^j{}_{;k} \\ &= g_{ij}A^j{}_{;k},\end{aligned}$$

so that we must have

$$g_{ij;k}A^j = 0.$$

Since this is true for arbitrary A^j, we have

$$g_{ij;k} = 0 \qquad (3.64)$$

This is a set of $\frac{1}{2}n^2(n+1)$ equations and, if the Γ^k_{ij} are symmetric, they also have $\frac{1}{2}n^2(n+1)$ components, so we should be able to find the Γ^k_{ij} in terms of g_{ij} and thus specify the affinity.

Equation (3.64), and the two similar equations obtained by cyclicly permuting i, j, and k, can be written in the form

$$g_{ij,k} - \Gamma^h_{ik}g_{hj} - \Gamma^h_{jk}g_{ih} = 0 \qquad (3.65)$$

$$g_{jk,i} - \Gamma^h_{ji}g_{hk} - \Gamma^h_{ki}g_{jh} = 0 \qquad (3.66)$$

$$g_{ki,j} - \Gamma^h_{kj}g_{hi} - \Gamma^h_{ij}g_{kh} = 0 \qquad (3.67)$$

Remembering that g_{ij} and Γ^h_{ij} are symmetric, (3.66) + (3.67) − (3.65) leads to

$$g_{kh}\Gamma^h_{ij} = [ij, k], \qquad (3.68)$$

where

$$[ij, k] = \tfrac{1}{2}(g_{jk,i} + g_{ki,j} - g_{ij,k}). \qquad (3.69)$$

The quantity $[ij, k]$ is called the Christoffel Symbol of the First Kind.

It is neither a tensor nor an affinity.

Multiplying both sides of (3.68) by g^{hk}, we obtain

$$\Gamma^k_{ij} = \left\{ {k \atop ij} \right\}, \qquad (3.70)$$

where

$$\left\{ {k \atop ij} \right\} = g^{hk}[ij,h] = \tfrac{1}{2} g^{hk}(g_{jh,i} + g_{hi,j} - g_{ij,h}). \qquad (3.71)$$

The quantity $\left\{ {k \atop ij} \right\}$ is called the <u>Christoffel Symbol of the Second Kind</u>. It is an affinity and is symmetric in i and j.

If the affinity is determined by (3.70) then the equation (3.64) is satisfied. It also follows that

$$g^{ij}{}_{;k} = 0 \qquad (3.72)$$

since $g^{ij} g_{kj} = \delta^i_k$.

The affinity in this case is called the metric affinity. Note that in E_n, in rectangular cartesian co-ordinates, the g_{ij} are constants so that the Christoffel symbols are all zero. Hence, covariant and partial derivatives will be the same. In future we shall always use the metric affinity $\left\{ {k \atop ij} \right\}$.

3.17. <u>The Curvature tensor and the Einstein tensor.</u>

The contravariant suffix in $R^h{}_{ijk}$ can be lowered to give the covariant curvature tensor R_{hijk}, i.e.

$$R_{hijk} = g_{hm} R^m{}_{ijk}. \qquad (3.73)$$

By writing out the right side of (3.73) explicitly, using (3.54) with $\Gamma^k_{ij} = \left\{ {k \atop ij} \right\}$, and noting the following results:

$$g_{ij} \left\{ {i \atop kl} \right\}_{,m} = \left[g_{ij} \left\{ {i \atop kl} \right\} \right]_{,m} - \left\{ {i \atop kl} \right\} g_{ij,m} ;$$

$$g_{ij} \left\{ {i \atop kl} \right\} = [kl,j] ;$$

$$g_{ij,k} = [ik,j] + [jk,i] ,$$

it follows that

$$R_{hijk} = \tfrac{1}{2}(g_{hk,ij} + g_{ij,hk} - g_{hj,ik} - g_{ik,hj}) + g_{mn}\left[\left\{ {m \atop ij} \right\}\left\{ {n \atop hk} \right\} - \left\{ {m \atop ik} \right\}\left\{ {n \atop hj} \right\} \right] \qquad (3.74)$$

From (3.74) we find that

$$R_{hijk} = -R_{ihjk} = -R_{hikj} = R_{jkhi}, \qquad (3.75)$$

i.e. R_{hijk} is antisymmetric in the first pair of suffixes, antisymmetric in the last pair of suffixes, and symmetric under interchanges of the first pair with the last pair. Also, from (3.57) we have

$$R_{hijk} + R_{hjki} + R_{hkij} = 0. \qquad (3.76)$$

The Ricci tensor $R_{ij} = R^k{}_{ikj}$ is symmetric and is given by

$$R_{ij} = \left\{{k \atop ij}\right\}_{,k} - \left\{{k \atop ik}\right\}_{,j} + \left\{{k \atop mk}\right\}\left\{{m \atop ij}\right\} - \left\{{k \atop mj}\right\}\left\{{m \atop ik}\right\}. \qquad (3.77)$$

If we contract the Bianchi identity (3.58) we obtain

$$R_{ik;l} + R^k{}_{ikl;h} - R_{il;k} = 0,$$

$$\text{i.e. } g^{ik}\left[R_{ik;l} + R^k{}_{ikl;k} - R_{il;k}\right] = 0,$$

$$R_{,l} - R^k{}_{l;k} - R^k{}_{l;k} = 0,$$

$$\text{i.e. } R_{,l} - 2R^k{}_{l;k} = 0,$$

$$\text{i.e. } \left(R^k{}_l - \tfrac{1}{2}\delta^k{}_l R\right)_{;k} = 0,$$

$$\text{i.e. } G^k{}_{l;k} = 0, \qquad (3.78)$$

where $R = g^{ij}R_{ij}$ is the Ricci scalar and $G^k{}_l = R^k{}_l - \frac{1}{2}\delta^k{}_l R$ is the Einstein tensor. Note that

$$G_{ij} = R_{ij} - \tfrac{1}{2}g_{ij}R \qquad \text{and} \qquad G^{ij} = R^{ij} - \tfrac{1}{2}g^{ij}R$$

On contraction the Einstein tensor gives:

$$G = g^{ij}G_{ij} = G^i_i = R - \tfrac{1}{2}nR$$

$$\text{i.e. } G = -\tfrac{1}{2}(n-2)R \qquad (3.79)$$

Note that equation (3.78) states that the divergence of the Einstein tensor is zero.

3.18 Some useful results.

If we contract the covariant derivative $A^i{}_{;j}$ we obtain the scalar $A^i{}_{;i}$, known as the divergence of A^i.

$$A^i{}_{;i} = A^i{}_{,i} + \left\{{i \atop ji}\right\}A^j \qquad (3.80)$$

From (3.71) we find that

$$\left\{{i \atop ji}\right\} = \tfrac{1}{2}g^{il}\left(g_{il,j} + g_{lj,i} - g_{ji,l}\right)$$

$$\text{i.e. } \left\{{i \atop ji}\right\} = \tfrac{1}{2}g^{il}g_{il,j} \qquad (3.81)$$

Now, if g is the determinant $|g_{ij}|$ and Δ^{ij} is the cofactor of g_{ij} in this determinant, then

$$\frac{\partial g}{\partial g_{ij}} = \Delta^{ij} = gg^{ij} \qquad (3.82)$$

$$\text{i.e. } dg = gg^{ij}dg_{ij} = -gg_{ij}dg^{ij} \qquad (3.83)$$

since $d(g_{ij}g^{ij}) = 0$. Hence, we have

$$\frac{\partial g}{\partial x^k} = gg^{ij}g_{ij,k} = -gg_{ij}g^{ij}{}_{,k} \qquad (3.84)$$

Hence, (18.2) becomes

$$\left\{{i \atop ji}\right\} = \tfrac{1}{2}g^{il}g_{il,j} = -\tfrac{1}{2}g_{il}g^{il}{}_{,j}$$

$$= \frac{1}{2|g|}\frac{\partial|g|}{\partial x^j}$$

$$\text{i.e.} \quad \left\{ {i \atop j\, i} \right\} = \frac{1}{\sqrt{|g|}} \frac{\partial \sqrt{|g|}}{\partial x^j} \tag{3.85}$$

$$= \left(\log_e \sqrt{|g|} \right)_{,j} \tag{3.86}$$

Equation (3.80) can thus be written in the form

$$A^i{}_{;i} = A^i{}_{,i} + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}\left(\sqrt{|g|}\right) A^i$$

$$\text{i.e.} \quad A^i{}_{;i} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} A^i \right)_{,i} \tag{3.87}$$

If A_i is the gradient of a scalar, i.e. $A_i = \frac{\partial \phi}{\partial x^i} \equiv \phi_{,i}$, then $A^i = g^{ij} \frac{\partial \phi}{\partial x^j}$. It follows that

$$A^i{}_{;i} = \operatorname{div} \operatorname{grad} \phi = \nabla^2 \phi,$$

$$\text{i.e.} \quad \nabla^2 \phi = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|}\, g^{ij} \frac{\partial \phi}{\partial x^j} \right) \tag{3.88}$$

In E_n this has the form

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^i \partial x^i} \tag{3.89}$$

<u>Example</u>. Express the Laplacian in terms of spherical polar coordinates in E_3.

The metric of E_3 in spherical polars is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\psi^2,$$

so that $g = r^4 \sin^2\theta$. Using (3.88) and the fact that g_{ij} and g^{ij} are diagonal, we obtain

$$\nabla^2 \phi = \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r}\left(r^2 \sin\theta\, g^{11} \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial \theta}\left(r^2 \sin\theta\, g^{22} \frac{\partial \phi}{\partial \theta} \right) + \frac{\partial}{\partial \psi}\left(r^2 \sin\theta\, g^{33} \frac{\partial \phi}{\partial \psi} \right) \right]$$

$$= \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta} \right) + \frac{\partial}{\partial \psi}\left(\operatorname{cosec}\theta \frac{\partial \phi}{\partial \psi} \right) \right]$$

$$= \frac{1}{r^2 \sin\theta} \left[r^2 \sin\theta \frac{\partial^2 \phi}{\partial r^2} + 2r \sin\theta \frac{\partial \phi}{\partial r} + \sin\theta \frac{\partial^2 \phi}{\partial \theta^2} + \cos\theta \frac{\partial \phi}{\partial \theta} + \operatorname{cosec}\theta \frac{\partial^2 \phi}{\partial \psi^2} \right]$$

$$\text{i.e.} \quad \nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r^2} \cot\theta \frac{\partial \phi}{\partial \theta} + \frac{1}{r^2} \operatorname{cosec}^2\theta \frac{\partial^2 \phi}{\partial \psi^2}$$

Consider the divergence of a second rank tensor A^{ij}.

$$A^{ij}{}_{;j} = A^{ij}{}_{,j} + \left\{ {j \atop m\, j} \right\} A^{im} + \left\{ {i \atop m\, j} \right\} A^{mj}$$

$$= \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} A^{ij} \right)_{,j} + \left\{ {i \atop m\, j} \right\} A^{mj} \tag{3.90}$$

If A^{ij} is antisymmetric, the last term is zero and so

$$A^{[ij]}{}_{;j} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} A^{[ij]} \right)_{,j} \tag{3.91}$$

For a symmetric tensor, we have (3.90). For a mixed tensor

$$A_i{}^j{}_{;j} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} A_i{}^j \right)_{,j} - \left\{ {j \atop i\, m} \right\} A_j{}^m, \tag{3.92}$$

and if the corresponding A^{ij} is symmetric, then the last term can be rewritten thus

$$A_i{}^j{}_{;j} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} A_i{}^j \right)_{,j} - \tfrac{1}{2} g_{jk,i} A^{jk}. \tag{3.93}$$

In section 3.5 we saw that the determinant of a second-rank covariant tensor is a relative invariant of weight 2. Hence the transformation for g is

$$g' = J^2 g \tag{3.94}$$

where $J = \left|\frac{\partial x^i}{\partial x^{i'}}\right|$. Hence $\sqrt{|g'|} = J\sqrt{|g|}$ and the n-dimensional volume element $dx^1 \ldots dx^n \equiv d^n x$, which transforms thus

$$d^n x' = J^{-1} d^n x \tag{3.95}$$

satisfies

$$\sqrt{|g'|}\, d^n x' = \sqrt{|g|}\, d^n x. \tag{3.96}$$

Hence $\sqrt{|g|}\, d^n x$ is an invariant (scalar) known as the <u>invariant volume element</u>

3.19 <u>Geodesics</u>

If P and Q are two neighbouring points on a curve C in a V_n and if s is a parameter defined on C such that s and $s+ds$ are the values at P and Q, then the interval ds is given by

$$ds^2 = g_{ij}\, dx^i dx^j.$$

This can be rewritten in the form

$$g_{ij} \frac{dx^i}{ds}\frac{dx^j}{ds} = 1 \tag{3.97}$$

and the quantity $\frac{dx^i}{ds}$, which is thus a unit vector, is defined to be the unit tangent vector to the curve at P.

A geodesic is defined as a curve such that each element of it is a parallel displacement of the preceding element; i.e. the tangents at all its points are parallel.

After parallel displacement from P to Q, the new unit tangent has components

$$\frac{dx^i}{ds} + \delta\left(\frac{dx^i}{ds}\right) = \frac{dx^i}{ds} - \Gamma^i_{jk}\frac{dx^j}{ds} dx^k, \tag{3.98}$$

whereas the unit tangent at Q has components

$$\frac{dx^i}{ds} + d\left(\frac{dx^i}{ds}\right) = \frac{dx^i}{ds} + \frac{d^2x^i}{ds^2} ds. \tag{3.99}$$

The vectors defined by (3.98) and (3.99) are identical if and only if

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{3.100}$$

which is thus the equations for a geodesic.

The class of curves for which $ds = 0$, which exist only if the metric is indefinite, are called <u>null curves</u>. In finding the equation of a null geodesic we can have equation (3.100) since this requires ds in the denominators. In this case we define some other parameter λ on the curve assuming that $\frac{dx^i}{d\lambda}$ exists at each point of the curve, where $\frac{dx^i}{d\lambda}$, the zero tangent is a null vector. In this case the equation of the null geodesic is

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0 \quad , \quad ds = 0. \tag{3.101}$$

In equations (3.100) and (3.101) we shall take $\Gamma^i_{jk} = \left\{ {i \atop jk} \right\}$.

This definition of a geodesic corresponds in E_3 to a straight line, the shortest distance between two points, and suggests that, in the general case, a geodesic can be defined as a curve of extremal length. We require the condition for the integral $\int ds$ to have a stationary value, i.e.

$$\delta\int ds = 0,$$

where δ means a variation corresponding to a change in co-ordinates and $\frac{dx^i}{dt}$ as we go from one possible path to another

$$\text{i.e. } \delta\int\sqrt{g_{ij}\,dx^i dx^j} = 0$$

$$\text{i.e. } \delta\int\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\,dt = 0,$$

where t is some parameter, which may be s (except for null geodesics).

Put $L^2 = g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}$, then $\delta\int L\,dt = 0$. Now

$$\delta\int L\,dt = \int\left[\frac{\partial L}{\partial x^i}\delta x^i + \frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)}\delta\left(\frac{dx^i}{dt}\right)\right]dt,$$

and, using $\delta\left(\frac{dx^i}{dt}\right) = \frac{d}{dt}(\delta x^i)$, the second term may be rewritten as

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)}\delta x^i\right\} - \frac{d}{dt}\left\{\frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)}\right\}\delta x^i$$

The first term vanishes on integration since $\delta x^i = 0$ the end points. Hence, we have

$$\delta\int L\,dt = \int\left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left\{\frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)}\right\}\right]\delta x^i\,dt = 0, \tag{3.102}$$

and, since this vanishes for arbitrary displacements δx^i, the integrand must vanish identically, i.e.

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)}\right\} - \frac{\partial L}{\partial x^i} = 0 \tag{3.103}$$

Now $\frac{\partial L}{\partial x^i} = \frac{1}{2}\left(g_{hk}\frac{dx^h}{dt}\frac{dx^k}{dt}\right)^{-1/2}\frac{\partial g_{mn}}{\partial x^i}\frac{dx^m}{dt}\frac{dx^n}{dt}$ and

$$\frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)} = \left(g_{hk}\frac{dx^h}{dt}\frac{dx^k}{dt}\right)^{-1/2} g_{ij}\frac{dx^j}{dt},$$

so equation (3.103) can be written as

$$\frac{d}{dt}\left\{\frac{g_{ij}\frac{dx^j}{dt}}{\frac{ds}{dt}}\right\} - \frac{g_{mn,i}\frac{dx^m}{dt}\frac{dx^n}{dt}}{2\frac{ds}{dt}} = 0.$$

i.e. $$g_{ij,k}\frac{dx^k}{dt}\frac{dx^j}{dt} + g_{ij}\frac{d^2x^j}{dt^2} - g_{ij}\frac{dx^j}{dt}\frac{\frac{d^2s}{dt^2}}{\frac{ds}{dt}} - \frac{1}{2}g_{mn,i}\frac{dx^m}{dt}\frac{dx^n}{dt} = 0$$

Multiply by g^{ki}, then
$$\frac{d^2x^k}{dt^2} + \frac{1}{2}g^{ik}(g_{ij,k} + g_{ik,j} - g_{kj,i})\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{dx^k}{dt}\cdot\frac{\frac{d^2s}{dt^2}}{\frac{ds}{dt}} \qquad (3.104)$$

Now choose the parameter t to be the arc length of the curve, i.e. set
$$\frac{ds}{dt} = \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}} = 1.$$

Then equation (3.104) simplifies to
$$\frac{d^2x^k}{ds^2} + \left\{ {k \atop jk} \right\}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \qquad (3.105)$$

which are the equations of the geodesics. Hence geodesics may be defined either as curves such that each element is a parallel displacement of the previous element, or as curves of extremal length. Either definition gives rise to equation (3.105) or (3.100).

3.20. Absolute derivative.

In section 3.6 we saw that the quantity
$$dA_i = \frac{\partial A_i}{\partial x_j}dx^j$$
is not a tensor, but the quantity
$$dA_i - \delta A_i = A_{i;j}dx^j$$
is a tensor. We write
$$DA_i = dA_i - \delta A_i = A_{i;j}dx^j, \qquad (3.106)$$
and DA_i is called the absolute differential, or intrinsic differential, of A_i

If t is any scalar, then the absolute derivative of A_i with respect to t is defined by
$$\frac{DA_i}{Dt} = A_{i;j}\frac{dx^j}{dt}. \qquad (3.107)$$
This is a vector and obeys the usual rules of differentiation. Similarly, the absolute derivative of any tensor is given by
$$\frac{D}{Dt}T^{k\ldots i}_{j\ldots k} = T^{k\ldots i}_{j\ldots k;l}\frac{dx^l}{dt} \qquad (3.108)$$
Note that the equations of geodesics (3.105) can be written in the form
$$\frac{D}{Ds}t^k = 0, \qquad (3.109)$$
where $t^k \equiv \frac{dx^k}{ds}$ is the unit tangent vector.

3.21 <u>The permutation tensor and dual tensors.</u>

The permutation symbol $\varepsilon^{ij\cdots k}$ is a tensor density, i.e. a relative tensor of weight 1, so it follows that its covariant counterpart $\varepsilon_{ij\cdots k}$ is a relative tensor of weight -1. (Note that the suffixes on the permutation symbol are raised and lowered by using the metric tensor of the locally flat Euclidean space, i.e. the tangent space, expressed in cartesian co-ordinates). Since $\sqrt{|g|}$ is a relative invariant of weight 1, it follows that the quantity

$$\eta_{ij\cdots k} = \pm\sqrt{|g|}\,\varepsilon_{ij\cdots k} \qquad (3.110)$$

is a tensor whose values are given by

$$\left.\begin{aligned} \eta_{ij\cdots k} &= +\sqrt{|g|}\,\varepsilon_{12\cdots n}\ , \text{ if } i,j,\ldots,k \text{ is an even perm of } 1,2,\ldots,n \\ \eta_{ij\cdots k} &= -\sqrt{|g|}\,\varepsilon_{12\cdots n}\ , \text{ if } i,j,k \text{ is an odd perm of } 1,2,\ldots,n \\ \eta_{ij\cdots k} &= 0, \text{ if any two suffixes are the same.} \end{aligned}\right\} \quad (3.111)$$

The ambiguity in sign in equation (3.110) is due to the fact that we define

$$\eta^{ij\cdots k} = \frac{1}{\sqrt{|g|}}\,\varepsilon^{ij\cdots k} \qquad (3.112)$$

and the positive (negative) sign in equation (3.110) arises according as g is positive or negative.

The tensor $\eta^{ij\cdots k}$ is called the permutation tensor. Note that both $\eta^{ij\cdots k}$ and $\eta_{ij\cdots k}$ have zero covariant derivative.

Note that if $T^{ij\cdots k}$ is a tensor antisymmetric in all its suffixes and of rank less than the dimension of the space, then we define the dual tensor $^{*}T^{ij\cdots}$ by.

$$^{*}T^{i_1 i_2\cdots i_m} = \frac{1}{m!}\,\eta^{i_1 i_2\cdots i_m\cdots i_n}\,T_{i_{m+1}\cdots i_n} \qquad (3.113)$$

For example in a V_4, the dual of an antisymmetric tensor T_{ij} is

$$^{*}T_{ij} = \tfrac{1}{2}\eta_{ijkl}T^{kl}, \qquad (3.114)$$

$$^{*}T^{ij} = \tfrac{1}{2}\eta^{ijkl}T_{kl}. \qquad (3.115)$$

Note that the dual of the dual is given by

$$\begin{aligned} ^{**}T_{ij} &= \tfrac{1}{2}\eta_{ijkl}\,{}^{*}T^{kl} \\ &= \tfrac{1}{4}\eta_{ijkl}\,\eta^{klmn}T_{mn} \\ &= \pm T_{mn}, \text{ depending on the signature.} \end{aligned}$$

$$\text{i.e.}\quad ^{**}T_{ij} = (\mathrm{sgn})\,T_{mn} \qquad (3.116)$$

where (sgn) is the sign of g.

Also in V_4, the dual of a vector A_i is a third-rank tensor

$*A^{jkl}$ and vice-versa.

The tensor $\delta^{k\dots i}_{j\dots k}$ is defined by

$$\delta^{k\dots i}_{j\dots k} = \det\begin{bmatrix}\delta^k_j & \cdots & \delta^i_j \\ \vdots & & \vdots \\ \delta^k_k & & \delta^i_k\end{bmatrix} \tag{3.117}$$

is identically zero if there are more than n suffixes (upper or lower). In a V_4 with negative g we find

$$\delta^{ki}_{jk} = -\tfrac{1}{2}\eta^{kimn}\eta_{jlmn}. \tag{3.118}$$

3.22 <u>Orthogonal ennuples.</u>

In a V_n we call n mutually orthogonal non-null vector fields an orthogonal ennuple. If the n vectors are denoted by h^i_α, the latin suffix is the tensor suffix, which is raised and lowered by the metric tensor g_{ij}, and the greek suffix is the numbering suffix which denotes the n vectors and is raised and lowered by the metric tensor $\eta_{\alpha\beta}$ of the locally flat (Euclidean) space in cartesian co-ordinates. The ennuple satisfies the following relations

$$\left.\begin{aligned} h_\alpha{}^i h^\alpha{}_j &= \delta^i_j , \quad h_{\alpha i} h^\alpha{}_j = g_{ij} \\ h_\alpha{}^i h^\beta{}_i &= \delta_\alpha^\beta , \quad h_\alpha{}^i h_{\beta i} = \eta_{\alpha\beta} \end{aligned}\right\} \tag{3.119}$$

The ennuple components for a diagonal metric are just the square roots of the metric tensor. For example, in an E_2 with polar co-ordinates the metric is

$$ds^2 = dr^2 + r^2 d\theta^2$$

and the ennuple components are

$$\begin{aligned} h_{(1)i} &= (1,0) , \quad h_{(1)}{}^i = (1,0), \\ h_{(2)i} &= (0,r) , \quad h_{(2)}{}^i = (0,\tfrac{1}{r}), \end{aligned}$$

and the ennuple suffixes (1),(2) are raised by the E_2 cartesian metric tensor, i.e. by 1's.

If we take a space which does not have positive definite signature, e.g.

$$ds^2 = e^{2\mu}dx^2 + e^{2\nu}dy^2 - e^{2\lambda}dz^2,$$

then the ennuple is

$$\begin{aligned} h_{(1)}{}^i &= (e^{-\mu},0,0) , \quad h_{(2)}{}^i = (0,e^{-\nu},0) , \quad h_{(3)}{}^i = (0,0,e^{-\lambda}) \\ h_{(1)i} &= (e^{\mu},0,0) , \quad h_{(2)i} = (0,e^{\nu},0) , \quad h_{(3)i} = (0,0,e^{\lambda}) \end{aligned}$$

but, since the local flat space is

$$ds^2 = dx^2 + dy^2 - dz^2,$$

the ennuple suffixes (1), (2), (3) are raised and lowered by +1, +1, −1, respectively.

Given a vector A_i, the four scalar A_α defined by the expression

$$A_\alpha = h_\alpha{}^i A_i, \tag{3.120}$$

are the components of A_i referred to the local cartesian co-ordinate system and are called the local components of A_i. Similarly, the local components of a mixed tensor $B_i{}^j$ are

$$B_\alpha{}^\beta = h_\alpha{}^i h^\beta{}_j B_i{}^j. \tag{3.121}$$

The local components are sometimes called the physical components; the reason for this is illustrated by the following example: Consider an E_2 and the velocity vector u^i defined by

$$u^i = \frac{dx^i}{dt}$$

where t is time, a scalar. In cartesians

$$u^i = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}\right) = (\dot{x}, \dot{y}).$$

Changing to polar co-ordinates, $x^{i'} = (r, \theta)$, then

$$u^{i'} = \frac{dx^{i'}}{dt} = (\dot{r}, \dot{\theta}).$$

However, these are not the usual components of velocity in polar co-ordinates — the usual ones are the physical components found by taking

$$h^\alpha{}_{i'} u^{i'} = (1.u^{1'}, r u^{2'}) = (\dot{r}, r\dot{\theta}).$$

3.23. Riemannian curvature and spaces of constant curvature.

Let λ^i, μ^i be two contravariant vector fields at a point P in V_n. Then, in a neighbourhood of P, the geodesics through P having tangent vectors of the form

$$\xi^i = \alpha\lambda^i + \beta\mu^i,$$

where α and β are real parameters, form a portion S of a two-dimensional surface. The Gaussian curvature K of S at P is called the Riemannian curvature of V_n at P in the two-dimensional direction determined by λ^i and μ^i. It can be shown that this curvature is given by the expression

$$K(g_{aj}g_{ik} - g_{ak}g_{ij})\lambda^a\mu^i\lambda^j\mu^k = R_{aijk}\lambda^a\mu^i\lambda^j\mu^k. \qquad (3.121)$$

Schur's Theorem.

If the Riemannian curvature of a V_n is the same for every pair of vectors at every point, then the curvature is a constant at every point of V_n.

Proof. From equation (3.121) we see that if K is independent of the choice of λ^i and μ^i, then

$$R_{aijk} = K(g_{aj}g_{ik} - g_{ak}g_{ij}). \qquad (3.122)$$

Taking the covariant derivative and cyclicly permuting suffixes, we find

$$R_{aijk;l} + R_{aikl;j} + R_{ailj;k} = K_{,l}(g_{aj}g_{ik} - g_{ak}g_{ij}) + K_{,j}(g_{ak}g_{il} - g_{al}g_{ik}) + K_{,k}(g_{al}g_{ij} - g_{aj}g_{il})$$

The left-hand side is zero from the Bianchi identities, and inner multiplication of the right-hand side by $g^{aj}g^{ik}$ leads to

$$(n-1)(n-2)\,K_{,l} = 0,$$

so that, assuming $n \geq 3$, K is a constant, thus proving the theorem.

A V_n satisfying equation (3.122) is called a space of constant curvature.

A V_n is called an Einstein space if its Ricci tensor is proportional to its metric tensor, i.e.

$$R_{ij} = \lambda g_{ij}, \qquad (3.123)$$

where λ is a scalar. In fact, λ must be a constant because, contracting (3.123), we have $R = n\lambda$, so that the Einstein tensor G_{ij} is given by

$$G_{ij} \equiv R_{ij} - \tfrac{1}{2}Rg_{ij} = (1-\tfrac{n}{2})\lambda g_{ij},$$

and since $G^{ij}{}_{;j} \equiv 0$, it follows that $\lambda_{,i} = 0$. Contracting equation (3.122) with g^{aj}, we obtain

$$R_{ik} = K(ng_{ik} - g_{ik}) = K(n-1)g_{ik}, \qquad (3.124)$$

so that a space of constant curvature is a special case of an Einstein space. Note that $R = n(n-1)K$, so that the Ricci scalar of a space of constant curvature is also a constant.

3.24 Conformal spaces.

If two spaces V_n, $\bar{V}_n$ are such that their metric tensors are related by an expression of the form

$$\bar{g}_{ij} = e^{2\sigma} g_{ij}, \tag{3.125}$$

where σ is a scalar function of the co-ordinates, it follows that the magnitudes of the displacement vectors at points of V_n and $\bar{V}_n$ with the same co-ordinates are proportional, and the angles between two corresponding directions at corresponding points are equal. The spaces V_n, $\bar{V}_n$ are said to be in conformal correspondence to each other, and the mapping given by equation (3.125) maps V_n conformally onto $\bar{V}_n$.

From equation (3.125) we have

$$\bar{g}^{ij} = e^{-2\sigma} g^{ij}, \tag{3.126}$$

and the Christoffel symbols in V_n and $\bar{V}_n$ are related by

$$\overline{\left\{ {}^{k}_{ij} \right\}} = \left\{ {}^{k}_{ij} \right\} + \delta^k_i \sigma_{,j} + \delta^k_j \sigma_{,i} - g_{ij} g^{kl} \sigma_{,l}. \tag{3.127}$$

If A_{ij} denotes the quantity

$$A_{ij} \equiv \sigma_{;ij} - \sigma_{,i}\sigma_{,j} \quad , \tag{3.128}$$

where the semi-colon denotes covariant differentiation using $\left\{ {}^{k}_{ij} \right\}$ and not $\overline{\left\{ {}^{k}_{ij} \right\}}$, then we find that the curvature tensors of V_n and $\bar{V}_n$ are related by

$$e^{-2\sigma} \bar{R}_{hijk} = R_{hijk} + g_{hk} A_{ij} + g_{ij} A_{hk} - g_{hj} A_{ik} - g_{ik} A_{hj} + (g_{hk} g_{ij} - g_{hj} g_{ik}) \sigma_{,m} \sigma_{,}{}^{m}, \tag{3.129}$$

where $\sigma_{,}{}^{m} = g^{mn}\sigma_{,n}$. Remembering that suffixes on barred and unbarred quantities are raised and lowered by $\bar{g}_{ij}$ and g_{ij}, respectively, the expression (3.129) can be rewritten as.

$$\bar{R}^h{}_{ijk} = R^h{}_{ijk} + \delta^h_k A_{ij} + g_{ij} A^h{}_k - \delta^h_j A_{ik} - g_{ik} A^h{}_j + (\delta^h_k g_{ij} - \delta^h_j g_{ik}) \sigma_{,m} \sigma_{,}{}^{m}. \tag{3.130}$$

The components of the Ricci tensor are given by

$$\bar{R}_{ik} \equiv \bar{R}^h{}_{ihk} = R_{ik} + A_{ik} + A_{ik} - n A_{ik} - g_{ik} A^j{}_j + g_{ik}(1-n)\sigma_{,m}\sigma_{,}{}^{n}$$

$$\text{i.e. } \bar{R}_{ik} = R_{ik} - (n-2)(\sigma_{;ik} - \sigma_{,i}\sigma_{,k}) - (n-2) g_{ik} (\sigma_{;}{}^{j}{}_{j} + \sigma_{,j}\sigma^{,j}); \tag{3.131}$$

where $\sigma_{;}{}^{j}{}_{j} = g^{kj}\sigma_{;kj}$.

The Ricci scalars, R and $\bar{R}$, are related by

$$e^{2\sigma} \bar{R} = R - (n-1)(n-2)\sigma_{,j}\sigma_{,}{}^{j} - 2(n-1)\sigma_{;}{}^{j}{}_{j} \tag{3.132}$$

The <u>Weyl conformal curvature tensor</u> $C^h{}_{ijk}$ is defined by

$$C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}\left(\delta^h_j R_{ik} - \delta^h_k R_{ij} + g_{ik} R^h{}_j - g_{ij} R^h{}_k\right) - \frac{R}{(n-1)(n-2)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right). \tag{3.133}$$

From equations (3.130) to (3.133) it follows that two spaces are in conformal correspondence with each other if and only if their Weyl tensors are identical, i.e.

$$\bar{C}^{k}_{ijk} = C^{k}_{ijk} \tag{3.134}$$

The expression (3.133) does not exist when $n = 2$. However, it can be shown that any V_2 metric, i.e. a quadratic differential form in two variables, can be reduced to the form

$$ds^2 = \lambda(x^1, x^2)\left[(dx^1)^2 \pm (dx^2)^2\right] \tag{3.135}$$

so that any V_2 is conformal to any other V_2. In the case of a V_3, it can be shown that C^{k}_{ijk} is identically zero.

If a V_n is conformal to an E_n (which, of course, will have the same signature as the V_n) then, since the Weyl tensor of the E_n is zero, it follows that V_n satisfies

$$C^{k}_{ijk} = 0. \tag{3.136}$$

A V_n ($n \geq 4$) satisfying this condition is said to be <u>conformally flat</u>. As can be seen from equation (3.135), any V_2 is conformally flat. However, although any V_3 satisfies the condition (3.136), this does not imply that a V_3 is necessarily conformally flat. The necessary and sufficient condition for a V_3 to be conformally flat is that the <u>Cotton-York tensor</u>

$$Y^{ij} = 2\eta^{imn}\left(R^{j}{}_{m} - \tfrac{1}{4}\delta^{j}_{m}R\right)_{;n} \tag{3.137}$$

vanishes. This condition can also be written in the form

$$R_{ij;k} - R_{ik;j} + \tfrac{1}{4}\left(g_{ik}R_{,j} - g_{ij}R_{,k}\right) = 0 \tag{3.138}$$

$$\text{i.e.} \qquad R_{i[j;k]} + \tfrac{1}{4}g_{i[k}R_{,j]} = 0.$$

Thus we have the following results concerning conformally flat spaces:

(i) Any V_2 is conformally flat.

(ii) A V_3 is conformally flat if and only if the Cotton-York tensor vanishes.

(iii) A V_n ($n > 3$) is conformally flat if and only if the Weyl tensor vanishes.

From equations (3.122) and (3.133) it follows that any space of constant curvature is conformally flat, but the converse is not necessarily true.

3.25 Isometries and Killing vectors.

Let $\xi^i(x^j)$ be a contravariant vector field defined in V_n with metric

$$ds^2 = g_{ij}\,dx^i dx^j.$$

Suppose that each point of the V_n is subjected to an infinitesimal transformation

$$x'^i = x^i + \xi^i \delta t \tag{3.139}$$

$$\text{i.e.} \quad \delta x^i = \xi^i \delta t$$

This transformation leads to

$$\delta(dx^i) = d(\delta x^i) = \xi^i{}_{,j}\, dx^j \delta t \tag{3.140}$$

$$\delta(g_{ij}) = g_{ij,k}\,\delta x^k = g_{ij,k}\,\xi^k \delta t \tag{3.141}$$

In general, the effect of such a transformation would be to distort the metric, but if the field ξ^i is such that the metric remains invariant under the transformation (3.139), then this transformation is called an isometry or motion of the V_n. In this case we have

$$\delta(ds^2) = 0$$

$$\text{i.e.} \quad \delta g_{ij}\,dx^i dx^j + g_{ij}\,\delta(dx^i)dx^j + g_{ij}\,dx^i \delta(dx^j) = 0$$

$$\text{i.e.} \quad g_{ij,k}\,\xi^k \delta t\, dx^i dx^j + g_{ij}\,\xi^i{}_{,k}\,dx^k dx^j \delta t + g_{ij}\,dx^i \xi^j{}_{,k}\,dx^k \delta t = 0.$$

Rearranging this equation we obtain

$$g_{ij,k}\,\xi^k + g_{ik}\,\xi^k{}_{,j} + g_{jk}\,\xi^k{}_{,i} = 0 \tag{3.142}$$

Since $g_{ij;k} = 0$, we have $g_{ij,k} = \left\{ {l \atop ik} \right\} g_{lj} + \left\{ {l \atop jk} \right\} g_{il}$. Using this, equation (3.142) becomes

$$\left\{ {l \atop ik} \right\} g_{lj}\,\xi^k + \left\{ {l \atop jk} \right\} g_{il}\,\xi^k + g_{ik}\,\xi^k{}_{,j} + g_{jk}\,\xi^k{}_{,i} = 0.$$

$$\text{i.e.} \quad g_{il}\,\xi^l{}_{;j} + g_{jl}\,\xi^l{}_{;i} = 0$$

$$\text{i.e.} \quad \xi_{i;j} + \xi_{j;i} = 0 \tag{3.143}$$

These equations (or equations (3.142)) are called Killing's equations and any solutions ξ_i are known as Killing vectors. These vectors define the possible infinitesimal isometries of the form (3.139).

In V_n the maximum number of independent solutions of the equations (3.143) is $\frac{1}{2}n(n+1)$ and the V_n will admit this maximum number if and only if V_n is a space of constant curvature. If the V_n admits $r \le \frac{1}{2}n(n+1)$ independent Killing vectors, then the V_n is said to admit an r-parameter group of motions (or isometries). If V_n admits the maximum number of Killing vectors then it is said to be maximally symmetric.

Examples.

1. Find the independent Killing vectors of E_2.

$$ds^2 = dx^2 + dy^2.$$

Since the Christoffel symbols are zero, équations (3.143) become

$$\xi_{i,j} + \xi_{j,i} = 0,$$

with i,j taking the values 1, 2, i.e.

$$\xi_{1,1} = 0,$$
$$\xi_{1,2} + \xi_{2,1} = 0,$$
$$\xi_{2,2} = 0.$$

The first equation shows that $\xi_1 = f(y)$ and the third equation shows that $\xi_2 = g(x)$. The second equation then gives

$$f_{,2} = -g_{,1} = k,$$

where k is a constant, since the left side is a function of y only and the right side is a function of x only. Hence

$$\xi_1 = f = ky + h,$$
$$\xi_2 = g = -kx + l,$$

where h and l are constants. Thus, there are three constant parameters in this solution giving three independent Killing vectors, viz.

(i) Put $k = l = 0$, $h = 1$,

$$\underset{(1)}{\xi}_i = \underset{(1)}{\xi}^i = (1, 0) \qquad \text{(Translation in } x\text{-direction)}$$

(ii) Put $k = h = 0$, $l = 1$

$$\underset{(2)}{\xi}_i = \underset{(2)}{\xi}^i = (0, 1) \qquad \text{(Translation in } y\text{-direction)}$$

(iii) Put $h = l = 0$, $k = 1$

$$\underset{(3)}{\xi}_i = \underset{(3)}{\xi}^i = (y, -x) \qquad \text{(Rotation in plane)}.$$

Since E_2 is a space of constant (zero) curvature, it admits the maximum number, 3, of Killing vectors.

2. Find the independent Killing vectors of the unit sphere.

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

The only non-zero Christoffel symbols are

$$\left\{ {1 \atop 22} \right\} = -\sin\theta\cos\theta, \quad \left\{ {2 \atop 12} \right\} = \cot\theta,$$

and the Killing's equations have the form

$$\xi_{1,1} = 0 \qquad (3.144)$$

$$\xi_{1,2} + \xi_{2,1} = 2\left\{ {2 \atop 12} \right\}\xi_2 = 2\cot\theta\, \xi_2 \qquad (3.145)$$

$$\xi_{2,2} = \left\{ {1 \atop 22} \right\}\xi_1 = -\sin\theta\cos\theta\, \xi_1 \qquad (3.146)$$

Differentiate equation (3.145) with respect to r and use (3.144) to obtain

$$\xi_{2,11} - 2\cot\theta\, \xi_{2,1} + 2\csc^2\theta\, \xi_2 = 0. \qquad (3.147)$$

To solve this we put $\xi_2 = u\sin\theta$ where $u = u(\theta, \phi)$. Equation (3.147) then becomes

$$u_{,11} + u = 0$$

$$\text{i.e. } u = A(\phi)\cos\theta + B(\phi)\sin\theta$$

$$\text{i.e. } \xi_2 = A(\phi)\cos\theta\sin\theta + B(\phi)\sin^2\theta \qquad (3.148)$$

Substituting for ξ_2 in equation (3.146) we obtain

$$-\sin\theta\cos\theta\, \xi_1 = A_{,2}\cos\theta\sin\theta + B_{,2}\sin^2\theta$$

$$\text{i.e. } \xi_1 = -A_{,2} - B_{,2}\tan\theta \qquad (3.149)$$

Substituting for ξ_1 in equation (3.144) shows that $B_{,2} = 0$, i.e. $B = \beta$, a constant.
Substituting for ξ_1 and ξ_2 in equation (3.145) gives

$$-A_{,22} + A(\cos^2\theta - \sin^2\theta) + 2\beta\sin\theta\cos\theta = 2A\cos^2\theta + 2\beta\sin\theta\cos\theta$$

$$\text{i.e. } A_{,22} + A = 0$$

$$\therefore \quad A = \alpha\cos\phi + \gamma\sin\phi,$$

where α, γ are constants. Hence, we have

$$\xi_1 = \alpha\sin\phi - \gamma\cos\phi$$

$$\xi_2 = (\alpha\cos\phi + \gamma\sin\phi)\sin\theta\cos\theta + \beta\sin^2\theta.$$

Putting $\alpha = 1, \beta = \gamma = 0$; $\beta = \alpha = 0, \gamma = 1$; $\beta = 1, \alpha = \gamma = 0$, respectively, we obtain the three independent Killing vectors

$$\underset{(1)}{\xi}{}_i = (\sin\phi, \cos\phi\cos\theta\sin\theta)$$

$$\underset{(2)}{\xi}{}_i = (-\cos\phi, \sin\phi\cos\theta\sin\theta)$$

$$\underset{(3)}{\xi}{}_i = (0, \sin^2\theta)$$

and the corresponding contravariant components are

$$\underset{(1)}{\xi}{}^i = (\sin\phi, \cos\phi\cot\theta)$$

$$\underset{(2)}{\xi}{}^i = (-\cos\phi, \sin\phi\cot\theta)$$

$$\underset{(3)}{\xi}{}^i = (0, 1)$$

The sphere is a space of constant positive curvature and again admits the maximum number of Killing vectors.

Note that if there exists a co-ordinate system in which a Killing vector has the form $\xi^i = (0, \ldots, 0, 1, 0, \ldots, 0)$, i.e. $\xi^i = \delta^i_m$, so that the m^{th} component of the vector is 1 and all other components are zero, the equation (3.142) yields $g_{ij,m} = 0$, which implies that the metric tensor is independent of x^m in this co-ordinate system. In example 1, two of the Killing vectors are (1,0) and (0,1) and the metric tensor is independent of x and y. In example 2, the third Killing vector shows that the metric tensor is independent of ϕ.

3.26 <u>Lie derivatives.</u>

The directional derivative of a scalar f along a given vector ξ^i is defined as the inner product of ξ^i with the gradient of f, i.e.

$$\nabla_\xi f = f_{,i} \xi^i \tag{3.150}$$

In fact, the quantity $\nabla_\xi f . \delta t$ is the change in f under the infinitesimal transformation (3.139).

The <u>Lie derivative</u> of a scalar function with respect to the vector field ξ^i is defined to be the directional derivative, i.e.

$$\mathcal{L}_\xi f = \nabla_\xi f = f_{,i} \xi^i \tag{3.151}$$

For a vector field A^i, we define the Lie derivative with respect to ξ^i as the commutator of ξ^i and A^i, i.e.

$$\begin{aligned} \mathcal{L}_\xi A^i &= [\xi^i, A^i] = \nabla_\xi A^i - \nabla_A \xi^i \\ &= A^i{}_{;j} \xi^j - \xi^i{}_{;j} A^j \end{aligned} \tag{3.152}$$

It is easily seen that this can be rewritten in the form

$$\mathcal{L}_\xi A^i = A^i{}_{,j} \xi^j - \xi^i{}_{,j} A^j. \tag{3.153}$$

Using the fact that $A_i A^i$ is a scalar and also equations (3.150) and (3.152), we find that the Lie derivative of a covariant vector is (assuming the Leibnitz rules for differentiation):

$$\mathcal{L}_\xi A_i = A_{i;j} \xi^j + \xi^j{}_{;i} A_j, \tag{3.154}$$

$$\text{i.e.} \quad \mathcal{L}_\xi A_i = A_{i,j} \xi^j + \xi^j{}_{,i} A_j. \tag{3.155}$$

Similarly, for a tensor $T^{ij}{}_k$, the Lie derivative is found to be

$$\mathcal{L}_\xi T^{ij}{}_k = T^{ij}{}_{k;m} \xi^m - T^{mj}{}_k \xi^i{}_{;m} - T^{im}{}_k \xi^j{}_{;m} + T^{ij}{}_m \xi^m{}_{;k}, \quad (3.156)$$

and, again, the covariant derivatives can be replaced by partial derivatives.

In particular, the Lie derivative of the metric tensor g_{ij} is

$$\mathcal{L}_\xi g_{ij} = g_{ij;m} \xi^m + g_{mj} \xi^m{}_{;i} + g_{im} \xi^m{}_{;j} \quad (3.156)$$

Using the fact that $g_{ij;m} = 0$, this becomes

$$\mathcal{L}_\xi g_{ij} = \xi_{j;i} + \xi_{i;j} \tag{3.157}$$

If ξ_i is a Killing vector, then equation (3.143) shows that the right-hand side of the above equation is zero, so the Lie derivative of the metric tensor with respect to a Killing vector is zero. This just means that there is no change in the metric tensor in the direction of the Killing vector, which is in accord with the definition of an isometry.

Another definition of the Lie derivative $\Phi^A[x^i(P)]$ (A represents all the tensor suffixes and $x^i(P)$ are the co-ordinates of the point P) is as follows: Make an infinitesimal of the form (3.139), i.e. $P \to P'$ by $x^i(P) = x^i(P') + \xi^i(P')$, where ξ^i is infinitesimal and so can be evaluated either at P or P'. Also make an infinitesimal co-ordinate transformation that makes the numerical values of the co-ordinates of P' the same as those of P in the original co-ordinates, i.e.

$$x'^i(P') = x^i(P)$$

Then we define

$$\mathcal{L}_\xi \Phi^A(P) = \lim_{\xi \to 0} \left[\Phi^A(P) - \Phi'^A(P')\right]. \tag{3.158}$$

It can be shown that this definition is equivalent to those of equations (3.150), (3.152), (3.154), and (3.156).

3.27. The generalized Kronecker delta.

The generalized Kronecker delta $\delta^{i_1 \dots i_r}_{j_1 \dots j_r}$ is defined as

$$\delta^{i_1 \dots i_r}_{j_1 \dots j_r} = \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_2}_{j_1} & \dots & \delta^{i_r}_{j_1} \\ \delta^{i_1}_{j_2} & \delta^{i_2}_{j_2} & \dots & \delta^{i_r}_{j_2} \\ \vdots & \vdots & & \vdots \\ \delta^{i_1}_{j_r} & \delta^{i_2}_{j_r} & \dots & \delta^{i_r}_{j_r} \end{vmatrix} \tag{3.159}$$

When $r = 1$, this is the usual Kronecker delta δ^i_j. When $r = 2$, we have

$$\delta^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}. \tag{3.160}$$

In general, $\delta^{i_1 \dots i_r}_{j_1 \dots j_r}$ is the sum of $r!$ terms, each of which is the product of r Kronecker deltas. Since δ^i_j is a (1,1) tensor, it follows that $\delta^{i_1 \dots i_r}_{j_1 \dots j_r}$ is a (r, r) tensor.

Since the sign of a determinant changes on interchange of any pair of rows or of columns, it follows that the generalized Kronecker delta is antisymmetric in both the upper and the lower sets of suffixes. Thus, if $r > n$, the dimension of the V_n, there must be at least one repeated suffix in each of the upper and lower sets, so that

$$\delta^{i_1 \dots i_r}_{j_1 \dots j_r} = 0 \quad \text{if} \quad r > n. \tag{3.161}$$

The generalized Kronecker delta in V_n is related to the permutation tensor $\eta^{i_1 \dots i_n}$, defined by equations (3.110) – (3.112), by the following relations

$$\eta^{i_1 \dots i_n} \eta_{j_1 \dots j_n} = \pm \delta^{i_1 \dots i_n}_{j_1 \dots j_n}, \tag{3.162}$$

where the ambiguity in sign corresponds to that in equation (3.110). These relations can be written in the equivalent form

$$\varepsilon^{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n} = \delta^{i_1 \dots i_n}_{j_1 \dots j_n}. \tag{3.163}$$

To prove this result consider the tensor

$$A^{i_1 \dots i_n}_{j_1 \dots j_n} = \varepsilon^{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n} - \delta^{i_1 \dots i_n}_{j_1 \dots j_n}. \tag{3.164}$$

This tensor is clearly antisymmetric in both sets of suffixes, so the only possible non-zero components occur when $i_1 \dots i_n$ and $j_1 \dots j_n$ are each a permutation of $1, 2, \dots, n$, i.e. the only possible non-zero components are of the form $\pm A^{12 \dots n}_{12 \dots n}$. However, from equation (3.164), we see that

$$A^{12 \dots n}_{12 \dots n} = \varepsilon^{12 \dots n} \varepsilon_{12 \dots n} - \delta^{12 \dots n}_{12 \dots n} = 1 - 1 = 0.$$

Hence, $A^{i_1 \dots i_n}_{j_1 \dots j_n} = 0$ and equations (3.163) and (3.162) follow.

It can be shown that

$$\eta^{i_1 \dots i_n} \eta_{i_1 \dots i_n} = \pm n!, \tag{3.165}$$

and that

$$\eta^{i_1 \dots i_r i_{r+1} \dots i_n} \eta_{j_1 \dots j_r i_{r+1} \dots i_n} = \pm (n-r)! \, \delta^{i_1 \dots i_r}_{j_1 \dots j_r}. \tag{3.166}$$

In particular, in a V_4 with signature $+2$, i.e. $(-,+,+,+)$, these results are:

$$\eta^{hijk} \eta_{hijk} = -24, \tag{3.167}$$

$$\eta^{hijk} \eta_{hijm} = -6 \delta^k_m, \tag{3.168}$$

$$\eta^{hijk} \eta_{himn} = -2 \delta^{jk}_{mn}, \tag{3.169}$$

$$\eta^{hijk} \eta_{hmnp} = - \delta^{ijk}_{mnp}, \tag{3.170}$$

$$\eta^{hijk} \eta_{mnpq} = - \delta^{hijk}_{mnpq}. \tag{3.171}$$

Expanding the determinant (3.159) in terms of the elements of the last row, we obtain

$$\delta^{i_1\cdots i_r}_{j_1\cdots j_r} = \delta^{i_r}_{j_r}\delta^{i_1\cdots i_{r-1}}_{j_1\cdots j_{r-1}} - \delta^{i_{r-1}}_{j_r}\delta^{i_1\cdots i_{r-2} i_r}_{j_1\cdots j_{r-1}} + \delta^{i_{r-2}}_{j_r}\delta^{i_1\cdots i_{r-3} i_{r-1} i_r}_{j_1\cdots j_{r-1}} + \cdots + (-1)^{r+1}\delta^{i_1}_{j_r}\delta^{i_2\cdots i_r}_{j_1\cdots j_{r-1}}. \tag{3.172}$$

Contracting over i_r and j_r, we obtain

$$\begin{aligned}\delta^{i_1\cdots i_{r-1} j_r}_{j_1\cdots j_r} &= n\,\delta^{i_1\cdots i_{r-1}}_{j_1\cdots j_{r-1}} - \delta^{i_1\cdots i_{r-1}}_{j_1\cdots j_{r-1}} + \delta^{i_1\cdots i_{r-3} i_{r-1} i_{r-2}}_{j_1\cdots j_r} + \cdots + (-1)^{r+1}\delta^{i_2\cdots i_{r-1} i_1}_{j_1\cdots j_{r-1}} \\ &= (n-r+1)\,\delta^{i_1\cdots i_{r-1}}_{j_1\cdots j_{r-1}}.\end{aligned} \tag{3.173}$$

Contracting over i_{r-1}, j_{r-1} leads to

$$\delta^{i_1\cdots i_{r-2} j_{r-1} j_r}_{j_1\cdots j_r} = (n-r+2)(n-r+1)\,\delta^{i_1\cdots i_{r-2}}_{j_1\cdots j_{r-2}}, \tag{3.174}$$

and, in general,

$$\delta^{i_1\cdots i_s j_{s+1}\cdots j_r}_{j_1\cdots j_r} = \frac{(n-s)!}{(n-r)!}\,\delta^{i_1\cdots i_s}_{j_1\cdots j_s} \tag{3.175}$$

In particular

$$\delta^{i_1\cdots i_r}_{j_1\cdots j_r} = \frac{n!}{(n-r)!} \tag{3.176}$$

<u>Examples.</u>

1. $$\delta^{\ell ij}_{mnp} = \delta^{\ell}_{m}\left(\delta^{i}_{n}\delta^{j}_{p} - \delta^{j}_{n}\delta^{i}_{p}\right) - \delta^{i}_{m}\left(\delta^{\ell}_{n}\delta^{j}_{p} - \delta^{j}_{n}\delta^{\ell}_{p}\right) + \delta^{j}_{m}\left(\delta^{\ell}_{n}\delta^{i}_{p} - \delta^{i}_{n}\delta^{\ell}_{p}\right)$$

So
$$\begin{aligned}\delta^{\ell ij}_{mnp} R^{n}{}_{kij} &= \delta^{\ell}_{m} R^{n}{}_{knp} - \delta^{\ell}_{m} R^{n}{}_{kpn} - R^{\ell}{}_{kmp} + \delta^{\ell}_{p} R^{n}{}_{kmn} + R^{\ell}{}_{kpm} - \delta^{\ell}_{p} R^{n}{}_{knm} \\ &= 2\delta^{\ell}_{m} R_{kp} - 2R^{\ell}{}_{kmp} - 2\delta^{\ell}_{p} R_{km}\end{aligned}$$

In a V_2, $\delta^{\ell ij}_{mnp} = 0$, so in a V_2 the above equation leads to

$$R^{\ell}{}_{kmp} = \delta^{\ell}_{m} R_{kp} - \delta^{\ell}_{p} R_{km}$$

Hence, in a V_2 the curvature tensor is completely specified by the Ricci tensor.

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2. In a V_3 consider the expression $\delta^{kij}_{mnp} A_{kij}$

$$\delta^{kij}_{mnp} A_{kij} = A_{mnp} - A_{mpn} - A_{nmp} + A_{pmn} + A_{npm} - A_{pnm}.$$

If A_{kij} is a completely antisymmetric tensor, then this gives

$$\delta^{kij}_{mnp} A_{kij} = 6A_{mnp}.$$

In fact, for any $r \le n$, we have

$$\delta^{i_1\cdots i_r}_{j_1\cdots j_r} A_{i_1\cdots i_r} = r!\,A_{j_1\cdots j_r} \tag{3.177}$$

provided that $A_{i_1\cdots i_r}$ is antisymmetric in all of its suffixes. Similarly, if $B^{i_1\cdots i_r}$ is antisymmetric in all of its suffixes, then

$$\delta^{i_1\cdots i_r}_{j_1\cdots j_r} B^{j_1\cdots j_r} = r!\,B^{i_1\cdots i_r} \tag{3.178}$$