II. THE THEORY OF DYNAMICAL SYSTEMS

We shall begin by presenting a brief review of the qualitative theory of ODE [169, 181, 182, 373], with particular emphasis on those aspects of the non-linear theory of importance in cosmology. Let us consider the system of ODE, or more briefly the DE, of the form

$$x' = f(x), \tag{2.1}$$

where $x' \equiv \frac{dx}{dt}$, $x \equiv (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, where we shall assume that f is (at least) C^1 on $\mathbb{R}^n$. Since the right-hand-side of (2.1) does not depend on t explicitly, the DE is called *autonomous*. In general f will be a non-linear function, but if f is linear, i.e.,

$$f(x) = Ax, \tag{2.2}$$

where A is an $n \times n$ matrix of real numbers, then the DE is said to be *linear*. The vector $x \in \mathbb{R}^n$ is called the *state vector* of the system, and $\mathbb{R}^n$ is called the *state space or phase space*. The function f can be interpreted as a *vector field* on $\mathbb{R}^n$, since it associates with each $x \in \mathbb{R}^n$ an element $f(x)$ on $\mathbb{R}^n$, which can be interpreted as a vector $f(x) = \big(f_1(x), \cdots, f_n(x)\big)$ at x. A *solution* of the DE (2.1) is a function $\psi : \mathbb{R} \to \mathbb{R}^n$ which satisfies

$$\psi'(t) = f\big(\psi(t)\big), \tag{2.3}$$

for all $t \in \mathbb{R}$ in the domain of ψ (which may be a finite open interval). The image of the solution curve ψ in $\mathbb{R}^n$ is called an *orbit* of the DE. Equation (2.3) implies that the vector field f at x is tangent to the orbit through x. The evolution of the system in time is described by the motion of a point $x \in \mathbb{R}^n$ representing the state of the physical system along an orbit of the DE in $\mathbb{R}^n$.

In general we cannot hope to find exact solutions of a non-linear DE (2.1) for $n \geq 2$. Consequently qualitative methods, perturbative methods, or numerical methods, must be utilized to deduce the behaviour of the physical system. The aim of *qualitative analysis* is to understand the qualitative behaviour (such as, for example, *the long-term behaviour* as $t \to \infty$) of typical solutions of the DE. Exceptional solutions such as *equilibrium solutions* or *periodic solutions*, and their *stability*, are also of interest and can be important for determining the long-term behaviour of typical solutions.

The zeros of the vector field, or *equilibrium points* of the DE (2.1), are points $a \in \mathbb{R}^n$ such that

$$f(a) = 0. \tag{2.4}$$

Equilibrium points are also referred to as singular points or fixed points. If $f(a) = 0$, then $\psi(t) = a$, for all $t \in \mathbb{R}$, and it is a solution of the DE, since

$$\psi'(t) = f\big(\psi(t)\big) \tag{2.5}$$

is satisfied trivially for all $t \in \mathbb{R}$. A constant solution $\psi(t) = a$ describes an *equilibrium state* of the physical system. In order to study the stability of equilibrium states it is necessary to study the behaviour of the orbits of the DE close to the equilibrium points, and hence we consider the linear approximation of the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ at an equilibrium point. We shall consequently assume that the partial derivatives of f exist and are continuous functions on $\mathbb{R}^n$ (i.e., that the function f is of class $C^1(\mathbb{R}^n)$).

The *derivative matrix* of $f : \mathbb{R}^n \to \mathbb{R}^n$ is the $n \times n$ matrix $\mathrm{D}f(x)$ defined by

$$\mathrm{D}f(x) = \left(\frac{\partial f_i}{\partial x_j}\right), \quad i, j = 1, \cdots, n, \tag{2.6}$$

where the f_i are the component functions of f. The *linear approximation* of f is written in terms of the derivative matrix:

$$f(x) = f(a) + \mathrm{D}f(a)(x - a) + R_1(x, a), \tag{2.7}$$

where $\mathrm{D}f(a)(x-a)$ denotes the $n \times n$ derivative matrix evaluated at a, acting on the vector $(x - a)$, and $R_1(x, a)$ is the *error term* such that if f is of class C^1 then the magnitude of the error $\|R_1(x, a)\|$ tends to zero faster than the magnitude of the displacement $\|x - a\|$ (where the Euclidean norm on $\mathbb{R}^n$ is defined by $\|x\| = \sqrt{x_1{}^2 + \cdots + x_n{}^2}$). If $a \in \mathbb{R}^n$ is an equilibrium point of the DE (2.1), using (2.7) the DE can be written in the form

$$x' = \mathrm{D}f(a)(x - a) + R_1(x, a). \tag{2.8}$$

Defining $u = x - a$, the linear DE

$$u' = \mathrm{D}f(a)u, \tag{2.9}$$

which is called *the linearization of the non-linear DE at the equilibrium point* $a \in \mathbb{R}^n$, can be associated with the non-linear DE. In general solutions of the linear DE approximate the solutions of the non-linear DE near $x = a$ (although in special situations the approximation can fail).

A. Linear Autonomous Differential Equations

The *matrix series*, called the *exponential* of A, is defined by

$$\mathrm{e}^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k, \tag{2.10}$$

where A is an $n \times n$ real matrix, I is the $n \times n$ identity matrix and $A^2 = A\,A$ (matrix product), etc. A matrix series is said to *converge* if the n^2 infinite series corresponding to the n^2 entries converge in $\mathbb{R}$. The exponential matrix, e^A, converges for all $n \times n$ matrices A [181].

A similarity transformation (which corresponds to a change of basis), defined by

$$B = P^{-1}\,A\,P, \tag{2.11}$$

where P is a non-singular matrix, can be effected to simplify A by writing it in a (Jordan) canonical form. Indeed, for any 2×2 real matrix A, there exists a non-singular matrix P such that

$$J = P^{-1}\,A\,P, \tag{2.12}$$

and J is one of the following matrices:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{2.13}$$

Noting that if $B = P^{-1}\,A\,P$ then $\mathrm{e}^B = P^{-1}\,\mathrm{e}^A\,P$, we can now calculate e^A for any matrix. In particular, we now have a complete algorithm for calculating e^A for any 2×2 real matrix A.

Fundamental Theorem for Linear Autonomous DE. Let A be an $n \times n$ real matrix. Then the initial value problem

$$x' = Ax, \qquad x(0) = a \in \mathbb{R}^n, \tag{2.14}$$

has the unique solution

$$x(t) = \mathrm{e}^{tA}a, \qquad \text{for all } t \in \mathbb{R}. \tag{2.15}$$

The unique solution of the DE (2.14) is given by (2.15) for all t. Thus, for each $t \in \mathbb{R}$, the matrix e^{tA} maps

$$a \to \mathrm{e}^{tA}a \tag{2.16}$$

(where a is the state at time $t = 0$ and e^{tA} is the state at time t). The set $\{\mathrm{e}^{tA}\}_{t\in\mathbb{R}}$ is a 1-parameter family of linear maps of $\mathbb{R}^n$ into $\mathbb{R}^n$, and is called the *linear flow* of the DE, denoted by

$$\mathrm{g}^t = \mathrm{e}^{tA}. \tag{2.17}$$

The flow describes the evolution in time of the physical system for all possible initial states. As the physical system evolves in time, one can think of the state vector x as a moving point in state space, its motion being determined by the flow $\mathrm{g}^t = \mathrm{e}^{tA}$. The linear flow $\mathrm{g}^t = \mathrm{e}^{tA}$ satisfies the properties $\mathrm{g}^0 = I$ and $\mathrm{g}^{t_1+t_2} = \mathrm{g}^{t_1} \circ \mathrm{g}^{t_2}$ (which also hold for non-linear flows), which imply that the flow $\{\mathrm{g}^t\}_{t\in\mathbb{R}}$ forms a *group* under composition of maps. The flow g^t of the DE (2.14) partitions the state space $\mathbb{R}^n$ into subsets called *orbits*, defined by

$$\gamma(a) = \{\mathrm{g}^t a | t \in \mathbb{R}\}. \tag{2.18}$$

The set $\gamma(a)$ is called the *orbit of the DE through* a and is the image in $\mathbb{R}^n$ of the solution curve $x(t) = \mathrm{e}^{tA}a$. It follows from the uniqueness of solutions that for $a, b \in \mathbb{R}^n$, either $\gamma(a) = \gamma(b)$ or $\gamma(a) \cap \gamma(b) = \emptyset$.

Orbits of a DE can be classified as follows:

1. If $g^t a = a$ for all $t \in \mathbb{R}$, then $\gamma(a) = \{a\}$ and it is called a *point orbit.* Point orbits correspond to equilibrium points.

2. If there exists a $T > 0$ such that $g^T a = a$, then $\gamma(a)$ is called a *periodic orbit.* Periodic orbits describe a system that evolves periodically in time.

3. If $g^t a \neq a$ for all $t \neq 0$, then $\gamma(a)$ is called a *non-periodic orbit.*

We note that non-periodic orbits can be of great complexity for linear DE if $n > 3$ and for non-linear DE if $n > 2$. In addition, since a *solution curve* of a DE is a parameterized curve it *contains information about the flow of time t.* The *orbits* are paths in (or subsets of) state space and orbits which are not point orbits are *directed paths* with the direction defined by increasing time. Hence, the orbits do not provide detailed information about the flow of time. For an autonomous DE, the slope of the solution curves depend only on x and hence the tangent vectors to the solution curves define a vector field $f(x)$ in x–space and infinitely many solution curves may correspond to a single orbit. A non-autonomous DE does not define a flow or a family of orbits.

Given a linear DE $x' = Ax$ in $\mathbb{R}^n$, we can introduce new coordinates by $y = Px$, where P is a non-singular matrix, and a new time variable $\tau = kt$, where k is a positive constant. It follows that $y' = By$, where $B = \frac{1}{k} P\,A\,P^{-1}$, and the linear DE $x' = Ax$ and $x' = Bx$ are said to be *linearly equivalent* if

$$A = kP^{-1}\,B\,P. \tag{2.19}$$

This condition ensures that the linear map P maps each orbit of the flow e^{tA} onto an orbit of the flow e^{tB}. *Two linear flows* e^{tA} and e^{tB} on $\mathbb{R}^n$ are said to be *linearly equivalent* if there exists a non-singular matrix P and a positive constant k such that

$$P\,e^{tA} = e^{ktB}\,P, \qquad \text{for all } t \in \mathbb{R}. \tag{2.20}$$

We can now consider the three cases corresponding to the three Jordan canonical forms for any 2×2 real matrix A.

CASE I: If A has two independent eigendirections, then there exists a matrix P such that $J = P\,A\,P^{-1}$, where

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

It follows that the given DE is linearly equivalent to $y' = Jy$. The flow is

$$e^{tJ} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix},$$

where the eigenvectors are $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$. The solutions are $y(t) = e^{tJ}b$, $b \in \mathbb{R}^2$, i.e., $y_1 = e^{\lambda_1 t}b_1$ and $y_2 = e^{\lambda_2 t}b_2$. By eliminating t, we obtain $\left(\frac{y_1}{b_1}\right)^{\frac{1}{\lambda_1}} = \left(\frac{y_2}{b_2}\right)^{\frac{1}{\lambda_2}}$, if $b_1 b_2 \neq 0$ or $y_1 = 0$ if $b_1 = 0$, $y_2 = 0$ if $b_2 = 0$. These equations define the orbits of the DE $y' = Jy$. The various cases are illustrated in Fig. (1): Ia. $\lambda_1 = \lambda_2 < 0$: Attracting Focus. Ib. $\lambda_1 < \lambda_2 < 0$: Attracting Node. Ic. $\lambda_1 < \lambda_2 = 0$: Attracting Line. Id. $\lambda_1 < 0 < \lambda_2$: Saddle. Ie. $\lambda_1 = 0 < \lambda_2$: Repelling Line (time reverse of Fig. (Ic)). If. $0 < \lambda_1 < \lambda_2$: Repelling Node (time reverse of Fig. (Ib)). Ig. $0 < \lambda_1 = \lambda_2$: Repelling Focus (time reverse of Fig. (Ia)).

CASE II: If A has one eigendirection, then there exists a matrix P such that $J = P\,A\,P^{-1}$, where

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

It follows that the given DE is linearly equivalent to $y' = Jy$. The flow is

$$e^{tJ} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

and the single eigenvector is $e_1 = (1,0)^T$. We note that if $\lambda \neq 0$, the orbits are given by $y_1 = y_2\left[\frac{b_1}{b_2} + \frac{1}{\lambda}\log\frac{y_2}{y_1}\right]$, if $b_2 \neq 0$, or $y_2 = 0$ if $b_2 = 0$. These cases are illustrated in Fig. (2): IIa. $\lambda < 0$: Attracting Jordan Node. IIb. $\lambda = 0$: Neutral Line. IIc. $\lambda > 0$: Repelling Jordan Node (time reverse of Fig. (IIa)).

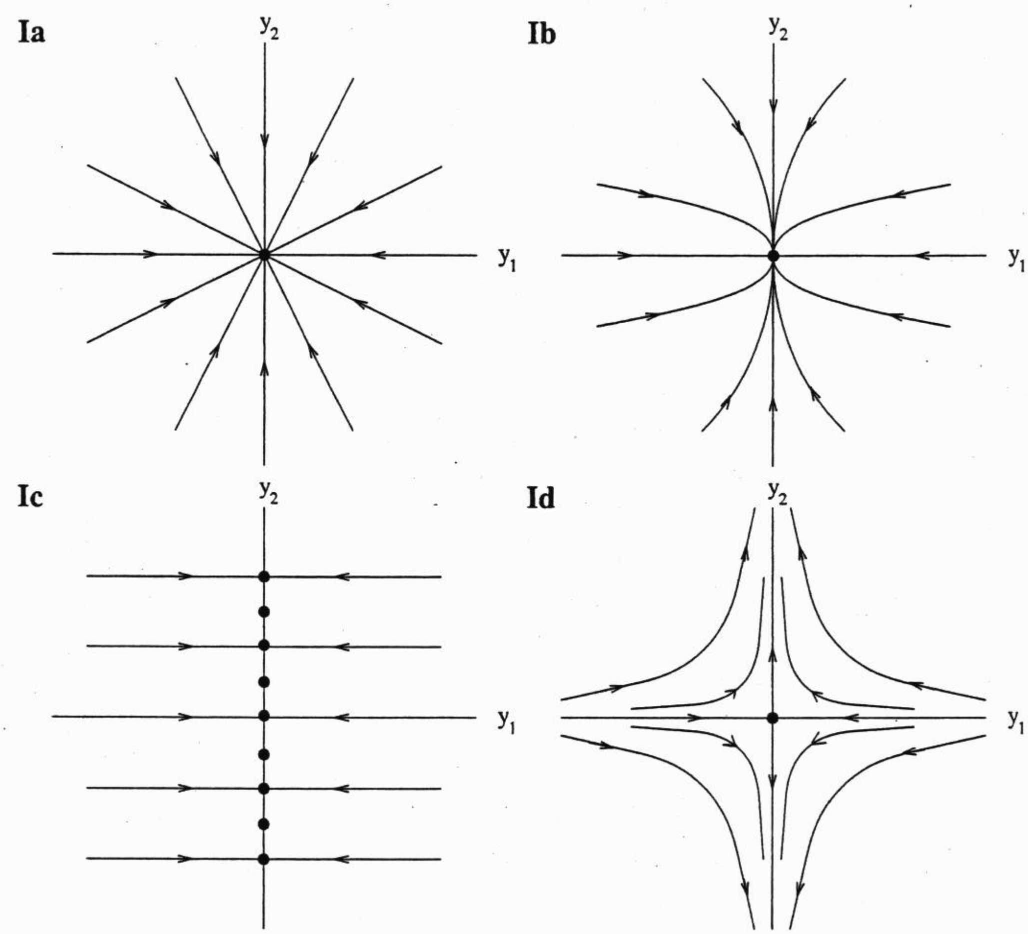

FIG. 1: The phase portrait of the 2-dimensional linear autonomous DE in (Jordan) canonical form which depicts orbits close to the equilibrium point at the origin in case I of two independent eigendirections: (a) Attracting Focus, $\lambda_1 = \lambda_2 < 0$ (the portrait in the case of a Repelling Focus with $0 < \lambda_1 = \lambda_2$ is the time reverse of this portrait). (b). Attracting Node, $\lambda_1 < \lambda_2 < 0$ (the time reverse is the case of a Repelling Node with $0 < \lambda_1 < \lambda_2$). (c) Attracting Line, $\lambda_1 < \lambda_2 = 0$ (the time reverse is the case of a Repelling Line with $\lambda_1 = 0 < \lambda_2$). (d) Saddle, $\lambda_1 < 0 < \lambda_2$.

CASE III: If A has no eigendirections, then there exists a matrix P such that $J = PAP^{-1}$, where

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

It follows that the given DE is linearly equivalent to $y' = Jy$. The simplest way to find the orbits is to introduce polar coordinates (r, θ): $y_1 = r\cos\theta$, and $y_2 = r\sin\theta$. The DE becomes $r' = \alpha r$ and $\theta' = -\beta$. It follows that $\frac{dr}{d\theta} = -\frac{\alpha}{\beta} r$ which can be integrated to yield $r = r_0 e^{-\frac{\alpha}{\beta}(\theta-\theta_0)}$. Without loss of generality, we can assume $\beta > 0$, since the DE is invariant under the changes $(\beta, y_1) \to (-\beta, -y_1)$. Thus $\lim_{t\to\infty}\theta = -\infty$ (counterclockwise rotation as t increases). These cases are illustrated in Fig. (3): IIIa. $\alpha < 0$: Attracting Spiral. IIIb. $\alpha = 0$: Centre. IIIc. $\alpha > 0$: Repelling Spiral (time reverse of Fig. (IIIa)).

1. *Topological Equivalence*

Under linear equivalence, two dimensional flows can be simplified in that they can be parameterized by one real-valued parameter and several discrete parameters (e.g., the number of independent eigenvectors). Linear equivalence thus acts as a filter, which retains only certain essential features of the flow, such as the behaviour of the orbits near

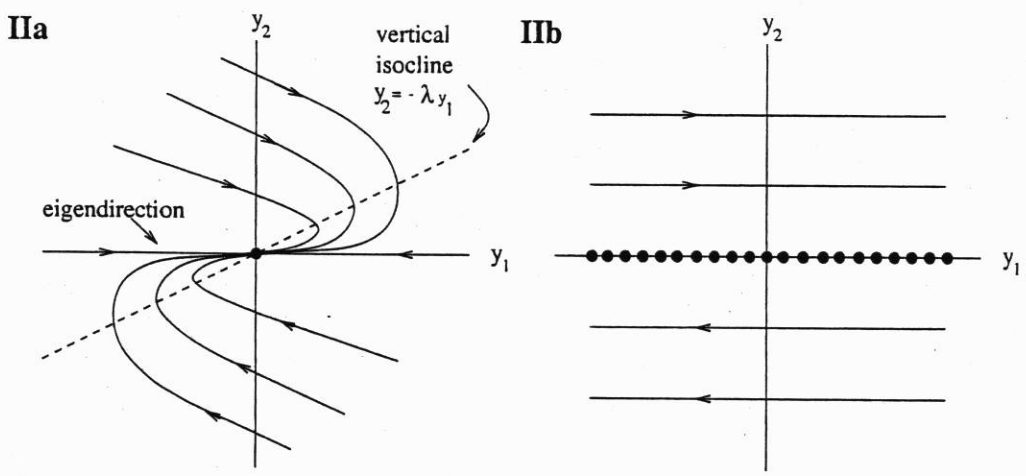

FIG. 2: The phase portrait of the 2-dimensional linear autonomous DE in (Jordan) canonical form in case II of one independent eigendirection: (a) Attracting Jordan Node, $\lambda < 0$ (the case of a Repelling Jordan Node with $\lambda > 0$ is the time reverse of this portrait). (b) Neutral Line, $\lambda = 0$.

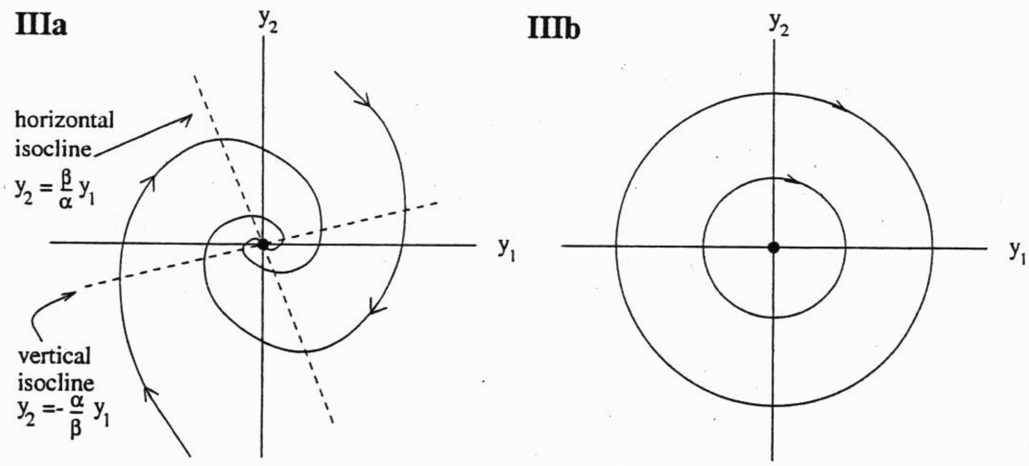

FIG. 3: The phase portrait of the 2-dimensional linear autonomous DE in (Jordan) canonical form in case III with no independent eigendirections: (a) Attracting Spiral, $\alpha < 0$ (the case of a Repelling Spiral with $\alpha > 0$ is the time reverse of this portrait). (b) Centre, $\alpha = 0$.

the equilibrium point $(0,0)$. On the other hand, if one is primarily interested in long-term behaviour, one can use a finer filter, which eliminates more features, and hence leads to a much simpler (but coarser classification). This is the notion of *Topological Equivalence* of linear flows.

For example, cases Ia, Ib, IIa, and IIIa have the common characteristic that all orbits approach the origin (an equilibrium point) as $t \to \infty$. We would like these flows to be "equivalent" in some sense. For all of these flows, *the orbits of one flow can be mapped onto the orbits of the simplest flow* Ia, using a (non-linear) map $h : \mathbb{R}^2 \to \mathbb{R}^2$: which is a *homeomorphism* (i.e., h is one-to-one and onto, h is continuous, and h^{-1} is continuous) on $\mathbb{R}^2$. Two linear flows e^{tA} and e^{tB} on $\mathbb{R}^n$ are said to be *topologically equivalent* if there exists a homeomorphism h on $\mathbb{R}^n$ and a positive constant k such that

$$h\left(e^{tA}x\right) = e^{ktB}h(x), \qquad \text{for all } x \in \mathbb{R}^n \text{ and for all } t \in \mathbb{R}. \tag{2.21}$$

In addition, a *hyperbolic* linear flow in $\mathbb{R}^2$ is one in which the real parts of the eigenvalues are all non-zero (i.e., $\Re e(\lambda_i) \neq 0, \quad i = 1, 2$). The following result is well-known: *Any hyperbolic linear flow in* $\mathbb{R}^2$ *is topologically equivalent to the linear flow* e^{tA}, *where* A *is one of the following matrices:*

1. $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$*; standard sink.* *2.* $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$*; standard source.* *3.* $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$*; the standard saddle.*

For *non-hyperbolic* linear flows in $\mathbb{R}^2$, it is clear that none of the 5 canonical flows [i.e., the centre, the attracting and repelling lines, the neutral line and the neutral 2–space ($A = 0$)] are topologically equivalent (their asymptotic behaviour as $t \to \infty$ differs). Thus, two non-hyperbolic linear flows in $\mathbb{R}^2$ are topologically equivalent if and only if they are linearly equivalent. Therefore, *any non-hyperbolic linear flow in $\mathbb{R}^2$ is linearly (and hence topologically) equivalent to the flow e^{tA}, where A is one of the following matrices:*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.22}$$

These five flows are topologically inequivalent.

2. Linear Stability

It is important to determine whether a physical system that is disturbed from an equilibrium state *remains close to* (stable) or *approaches* (asymptotically stable) the equilibrium state as time evolves ($t \to \infty$). First, we need the following definitions

1. The equilibrium point 0 of a linear DE $x' = Ax$ in $\mathbb{R}^n$ is *stable* if for all neighbourhoods U of 0, there exists a neighbourhood V of 0 such that $g^t V \subseteq U$ for all $t \geq 0$, where $g^t = e^{tA}$, is the flow of the DE.
2. The equilibrium point 0 of a linear DE $x' = Ax$ in $\mathbb{R}^n$ is *asymptotically stable* if it is stable and if, in addition, for all $x \in V$, $\lim_{t\to\infty} \|g^t x\| = 0$.

If $A \in M_n(\mathbb{R})$, it follows that

$$\lim_{t\to\infty} e^{tA} a = 0, \qquad \text{for all } a \in \mathbb{R}^n, \tag{2.23}$$

if and only if $\Re e(\lambda) < 0$ *for all eigenvalues of A.* This means that if $\Re e(\lambda) < 0$ then *all* solutions $x(t)$ of the DE $x' = Ax$ approach the equilibrium point 0 in the long term; i.e., $x(t) \to 0 \in \mathbb{R}^n$, as $t \to \infty$. Thus if $A \in M_n(\mathbb{R})$ is such that $\Re e(\lambda) < 0$ for all eigenvalues, then we say that the equilibrium point 0 of the DE $x' = Ax$ is a *sink* in $\mathbb{R}^n$. If we replace A by $-A$ and t by $-t$, we obtain the time reverse. Thus if $A \in M_n(\mathbb{R})$ is such that $\Re e(\lambda) > 0$ for all eigenvalues, then we say that the equilibrium point 0 of the DE $x' = Ax$ is a *source* in $\mathbb{R}^n$. Finally, we note that if $A \in M_n(\mathbb{R})$ and *if there exists a constant k such that all eigenvalues of A satisfy* $\Re e(\lambda) < -k < 0$, *then there exists a positive constant M such that*

$$\left\| e^{tA} x \right\| \leq M e^{-kt} \|x\|, \qquad \text{for all } x \in \mathbb{R}^n, \quad \text{all } t \geq 0. \tag{2.24}$$

We note that if the equilibrium point $0 \in \mathbb{R}^n$ is a sink of the DE $x' = Ax$, then 0 is an asymptotically stable equilibrium point.

B. Non-Linear Differential Equations

For non-linear DE, the flow usually cannot be written down explicitly. Indeed, the aim of dynamical systems is to describe the qualitative properties of a non-linear flow *without knowing the flow explicitly.* Let us first state the standard existence-uniqueness theorem for the initial value problem (IVP) for a DE in $\mathbb{R}^n$.

Theorem (Existence-Uniqueness) [181] . Consider the initial value problem

$$x' = f(x), \qquad x(0) = a \in \mathbb{R}^n. \tag{2.25}$$

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class $C^1(\mathbb{R}^n)$, then for all $a \in \mathbb{R}^n$, there exists an interval $(-\delta, \delta)$ and a unique function $\psi_a : (-\delta, \delta) \to \mathbb{R}^n$ such that

$$\psi_a'(t) = f\big(\psi_a(t)\big), \qquad \psi_a(0) = a. \tag{2.26}$$

We note that if the hypothesis that f be of class C^1 is weakened, then uniqueness may fail. The existence-uniqueness theorem is a local result — it guarantees existence of a solution in some interval $(-\delta, \delta)$ centered at $t = 0$. Since we are interested in the long-term behaviour of solutions, we would like the solutions to be defined for all $t \geq 0$. We can extend the interval of definition of the solution $\psi_a(t)$ by successively reapplying the theorem, and in this way obtain a *maximal interval of definition* of the solution $\psi_a(t)$. We shall denote this maximal interval by (α, β). We say that the solution $\psi_a(t)$ has *finite escape time* β_a if $\|\psi_a(t)\| \to +\infty$ as $t \to \beta_a^-$.

Theorem (Maximality). Let $\psi_a(t)$ be the unique solution of the DE $x' = f(x)$, where $f \in C^1(\mathbb{R}^n)$, which satisfies, $\psi_a(0) = a$, and let (α_a, β_a) denote the maximal interval on which $\psi_a(t)$ is defined. If β_a is finite, then [181]

$$\lim_{t \to \beta^-{}_a} \|\psi_a(t)\| = +\infty. \tag{2.27}$$

Therefore, for the DE (2.1), $f \in C^1(\mathbb{R}^n)$, if a solution $\psi_a(t)$ is bounded for $t \geq 0$, then the solution is defined for all $t \geq 0$.

A given DE $x' = f(x)$, $x \in \mathbb{R}^n$, and $f \in C^1(\mathbb{R}^n)$, can always be modified so that the orbits are unchanged, but such that all solutions are defined for all $t \in \mathbb{R}$, by re-scaling the vector field f (the velocity of the state point x):

$$f(x) \to \lambda(x) f(x), \tag{2.28}$$

where $\lambda(x) : \mathbb{R}^n \to \mathbb{R}$ is a C^1-function (a scalar) which is *positive* on $\mathbb{R}^n$ (in order to preserve the direction of time). This rescaling does not change the direction of the vector field, hence the orbits are unchanged. However, one can choose λ so that $\|\lambda f\|$ is bounded; e.g., $\lambda(x) = (1 + \|f(x)\|^{-1})$ [284].

Let us consider the DE $x' = f(x)$, where f is of class C^1, and the unique maximal solution which satisfies $\psi_a(0) = a$. The *flow* of the DE is defined to be the one-parameter family of maps $\{g^t\}_{t \in \mathbb{R}}$ such that $g^t : \mathbb{R}^n \to \mathbb{R}^n$ and $g^t a = \psi_a(t)$ for all $a \in \mathbb{R}^n$. The flow $\{g^t\}$ is defined in terms of the solution function $\psi_a(t)$ of the DE by

$$g^t a = \psi_a(t). \tag{2.29}$$

It is important to understand the difference between $\psi_a(t)$ and $g^t a$: *For a fixed* $a \in \mathbb{R}^n$, $\psi_a : \mathbb{R}^n \to \mathbb{R}^n$ gives the state of the system $\psi_a(t)$ *for all* $t \in \mathbb{R}$, with $\psi_a(0) = a$ initially. *For a fixed* $t \in \mathbb{R}$, $g^t : \mathbb{R}^n \to \mathbb{R}^n$ gives the state of the system $g^t a$ at time t *for all initial states* a.

The solution function $\psi_a(t)$ satisfies $\psi_a'(t) = f(\psi_a(t))$, $\psi_a(0) = a$. Hence $\psi_a'(0) = f(a)$. By definition of the flow, it follows that

$$\left.\frac{d}{dt}(g^t a)\right|_{t=0} = f(a), \tag{2.30}$$

which is simply a restatement of the fact that the vector field f is tangent to the orbits of the DE. If $\{g^t\}$ is the flow of a DE $x' = f(x)$, then the following two properties are satisfied:

$$\begin{aligned} g^0 &= I && \text{(identity map)} \\ g^{t_1+t_2} &= g^{t_1} \circ g^{t_2} && \text{(composition)} \end{aligned} \tag{2.31}$$

Theorem (Smoothness of a Flow)[181]. If $f \in C^1(\mathbb{R}^n)$, then the flow $\{g^t\}$ of the DE $x' = f(x)$ consists of C^1 maps.

Therefore, the solutions of the DE depend smoothly on the initial state. The *orbit* through a, denoted $\gamma(a)$, is defined to be

$$\gamma(a) = \left\{x \in \mathbb{R}^n \middle| x = g^t a, \text{ for all } t \in \mathbb{R}\right\}. \tag{2.32}$$

As in the linear case, orbits are classified as *point orbits*, *periodic orbits*, and *non-periodic orbits*. Sometimes it is convenient to work with the *positive orbit through a* denoted $\gamma^+(a)$ and defined by

$$\gamma^+(a) = \left\{x \in \mathbb{R}^n \middle| x = g^t a, \text{ for all } t \geq 0\right\}. \tag{2.33}$$

Let us consider a physical system with initial state vector $x \in \mathbb{R}^n$, whose evolution is described by a DE $x' = f(x)$ which determines a flow $\{g^t\}_{t \in \mathbb{R}}$. It is of interest to determine the *long-term behaviour* of the system as $t \to \infty$, starting at an initial state a when $t = 0$. The simplest behaviour is (i) the system, starting at state a, *approaches an equilibrium state* as $t \to \infty$; i.e., $\lim_{t\to\infty} g^t a = p$. The next simplest behaviour is (ii) the system, starting at state a, approaches periodic evolution; i.e., the orbit approaches a periodic orbit γ. In this situation, $\lim_{t\to\infty} g^t a$ does not exist, since the orbit does not approach a unique point. However, for any point $p \in \gamma$, we can choose a sequence of times $\{t_n\}$, with $\lim_{n\to\infty} t_n = \infty$, such that $\lim_{n\to\infty} g^{t_n} a = p$. This motivates the important notion of a *limit set*.

Consider the DE (2.1) in $\mathbb{R}^n$, and the associated flow $\{g^t\}_{t\in\mathbb{R}}$. Given an initial point $a \in \mathbb{R}^n$, a point $p \in \mathbb{R}^n$ is said to be an *ω-limit point* of a if there exists a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that $\lim_{n\to\infty} g^{t_n}a = p$. The set of all ω-points of a is called *the ω-limit set of* a, denoted by $\omega(a)$. In cases (i) and (ii) above the ω-limit sets of the initial point a is the equilibrium point p and the periodic orbit γ, i.e., $w(a) = \{p\}$ and $w(a) = \gamma$, respectively. Determining the subsets of $\mathbb{R}^n$ which can be ω-limit sets for a flow $\{g^t\}$ is a difficult question, and is unsolved if $n > 2$. The following two results are useful for identifying $\omega(a)$: *(1) An ω-limit set $\omega(a)$ of a flow $\{g^t\}$ is a whole orbit of the flow, or is the union of more than one whole orbit. (2) If the positive orbit through a, $\gamma^+(a) = \{g^t a | t \geq 0\}$ is bounded, then $\omega(a) \neq \emptyset$.*

1. Liapunov Theory

Given a DE $x' = f(x)$ in $\mathbb{R}^n$, a set $S \subseteq \mathbb{R}^n$ which is the union of whole orbits of the DE, is called an *invariant set* for the DE. For example, for a Hamiltonian DE in $\mathbb{R}^2$ the level sets $H(x_1, x_2) = k$ are invariant sets, since H is constant along any orbit. More generally, a function $H : \mathbb{R}^n \to \mathbb{R}$ of class C^1, that is not constant on any open subset of $\mathbb{R}^n$, is called a *first integral* of the DE $x' = f(x)$ if H is constant on every orbit; i.e.,

$$\frac{d}{dt}H\big(x(t)\big) = \nabla H\big(x(t)\big) \cdot f\big(x(t)\big) = 0, \tag{2.34}$$

for all t (using the chain rule and the DE). It follows that $H(x)$ is a first integral of the DE $x' = f(x)$ if and only if $\nabla H(x) \cdot f(x) = 0$, for all $x \in \mathbb{R}^n$, and $H(x)$ is not identically constant on any open subset of $\mathbb{R}^n$. If there is a first integral (e.g., a Hamiltonian function) then the orbits of the DE are contained in the one-parameter family of level sets $H(x) = k$.

It sometimes happens that there is a function $F : \mathbb{R}^n \to \mathbb{R}$ such that *only a particular level set of F is an invariant set.* Given a DE $x' = f(x)$ in $\mathbb{R}^n$, and a function $G : \mathbb{R}^n \to \mathbb{R}$ of class C^1, if $\nabla G(x) \cdot f(x) = 0$ for all x such that $G(x) = k$, then the level set $G(x) = k$ is an *invariant* set of the DE. These invariant sets play a major role in determining the portrait of the orbits.

Given a DE $x' = f(x)$ in $\mathbb{R}^n$, with flow g^t, a subset $S \subset \mathbb{R}^n$ is said to be a *trapping* set of the DE if it satisfies: (i) S is a closed and bounded set, and (ii) $a \in S$ implies $g^t a \in S$ for all $t \geq 0$. Hence if an orbit enters a trapping set S it *never leaves it*, and for all $a \in S$, the ω-limit set $\omega(a)$ is non-empty and is contained in S. Let $V : \mathbb{R}^n \to \mathbb{R}$ be $C^1(\mathbb{R}^n)$. We can calculate the rate of change of V along a solution of the DE (2.1) by:

$$\frac{d}{dt}V\big(x(t)\big) = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial V}{\partial x_n}\frac{dx_n}{dt} = \nabla V\big(x(t)\big) \cdot f\big(x(t)\big) \equiv \dot{V}(x), \tag{2.35}$$

where $(\cdot)$ is the scalar product in $\mathbb{R}^n$. Suppose that $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$. Then for any orbit $\gamma(a)$ in a trapping set S, $V(x)$ will keep decreasing along $\gamma(a)$ until the orbit approaches its ω-limit set $\omega(a)$. Strong restrictions on the possible ω-limit sets can therefore be obtained.

Theorem (Global Liapunov Theorem). Consider the DE $x' = f(x)$ in $\mathbb{R}^n$, and let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. If $S \subset \mathbb{R}^n$ is a trapping set, and $\dot{V}(x) \leq 0$ for all $x \in S$, then for all $a \in S$, $\omega(a) \subseteq \left\{x \in S | \dot{V}(x) = 0\right\}$ [169].

A function $V : \mathbb{R}^n \to \mathbb{R}$ which satisfies the above theorem for $x \in S \subset \mathbb{R}^n$ is called a *Liapunov function on S.* In applying the theorem, we note that we simply have to find *whole orbits* that are contained in the set $\left\{x \in S | \dot{V}(x) = 0\right\}$ to obtain the ω-limit set $\omega(a)$.

The stability of an equilibrium point can be determined by studying the linearization of the DE. The basic definitions are the same as in the linear case, with the linear flow e^{tA} being replaced by g^t:

1. The equilibrium point $\bar{x}$ of a DE $x' = f(x)$ in $\mathbb{R}^n$ is *stable* if for all neighbourhoods U of $\bar{x}$, there exists a neighbourhood V of $\bar{x}$ such that $g^t V \subseteq U$ for all $t \geq 0$, where g^t is the flow of the DE.

2. The equilibrium point $\bar{x}$ of a DE $x' = f(x)$ in $\mathbb{R}^n$ is *asymptotically stable* if it is stable and if, in addition, for all $x \in V$, $\lim_{t\to\infty} \|g^t x - \bar{x}\| = 0$

Now, consider a non-linear DE (2.1) in $\mathbb{R}^n$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be $C^1(\mathbb{R}^n)$. The rate of change of V along a solution of the DE is given by $\dot{V} = \nabla V\big(x(t)\big) \cdot f\big(x(t)\big)\dot{V}(x)$. Thus, if $\dot{V}(x) < 0$ *for all t then $V(x)$ decreases with time along the corresponding orbit.* From a geometrical point of view, the orbits cut the level sets $V(x) = k$ in the direction away from $\nabla V(x)$. Suppose that $\bar{x}$ is an equilibrium point of the DE. If $V(\bar{x}) = 0$ and $V(x) > 0$ for all $x \in U - \{\bar{x}\}$, where U is a neighbourhood of $\bar{x}$, then we expect the level sets of V in U to be concentric curves (n=2) or concentric spheres (n=3); consequently when $\dot{V} < 0$ for all $x \in U - \{\bar{x}\}$, any orbit in $U - \{\bar{x}\}$ will cut the level sets of V in the inward direction, and we expect that this will continue until the orbit is forced to approach the equilibrium point $\bar{x}$ as $t \to \infty$, showing that the equilibrium point is *asymptotically stable.* If, instead, $\dot{V} \leq 0$ for all $x \in U - \{\bar{x}\}$, then U may contain periodic orbits, and we only obtain the weaker conclusion that $\bar{x}$ is *stable.* Finally, if $\dot{V} > 0$ for all $x \in U - \{\bar{x}\}$, then the orbits are forced away from $\bar{x}$, which is thus an *unstable* equilibrium point.

Theorem (Liapunov Stability Theorem). Let $\bar{x}$ be an equilibrium point of the DE $x' = f(x)$ in $\mathbb{R}^n$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that $V(\bar{x}) = 0$, $V(x) > 0$ for all $x \in U - \{\bar{x}\}$, where U is a neighbourhood of $\bar{x}$.

1. If $\dot{V}(x) < 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is asymptotically stable.
2. If $\dot{V} \leq 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is stable.
3. If $\dot{V}(x) > 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is unstable.

A function $V : \mathbb{R}^n \to \mathbb{R}$ which satisfies $V(\bar{x}) = 0$, $V(x) > 0$ for all $x \in U - \{\bar{x}\}$, and $\dot{V}(x) \leq 0$ (respectively < 0) for all $x \in U - \{\bar{x}\}$, is called a *Liapunov function* (respectively, a *strict Liapunov function*) for the equilibrium point $\bar{x}$. Hence we obtain the following Criterion for Asymptotic Stability: Let $\bar{x}$ be an equilibrium point of the DE $x' = f(x)$ in $\mathbb{R}^n$. If all of the eigenvalues of the derivative matrix $\mathrm{D}f(\bar{x})$ satisfy $\Re e(\lambda) < 0$, then the equilibrium point $\bar{x}$ is asymptotically stable.

2. *Linearization and the Hartman-Grobman Theorem*

In general the linearizations give a reliable description of the non-linear orbits *near the equilibrium points.*

Theorem (Hartman-Grobman). Let $\bar{x}$ be an equilibrium point of the DE $x' = f(x)$ in $\mathbb{R}^n$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1. If all of the eigenvalues of the matrix $\mathrm{D}f(\bar{x})$ satisfy $\Re e(\lambda) \neq 0$, then there is a homeomorphism $h : U \to \bar{U}$ of a neighbourhood U of O onto a neighbourhood $\bar{U}$ of $\bar{x}$ which maps orbits of the linear flow $\mathrm{e}^{t\mathrm{D}f(\bar{x})}$ onto orbits of the non-linear flow g^t of the DE, preserving the parameter t [172].

Two flows g^t and $\tilde{\mathrm{g}}^t$ on $\mathbb{R}^n$ are said to be *topologically equivalent* if there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ which maps orbits of g^t onto orbits of $\tilde{\mathrm{g}}^t$, and preserves the direction of the parameter t. An equilibrium point $\bar{x}$ of a non-linear DE is said to be *hyperbolic* if *all* eigenvalues of the matrix $\mathrm{D}f(\bar{x})$ satisfy $\Re e(\lambda) \neq 0$. The Hartman-Grobman Theorem can therefore be stated more concisely: if $\bar{x}$ is a *hyperbolic equilibrium point,* then the flow of the DE $x' = f(x)$ and the flow of its linearization $u' = \mathrm{D}f(\bar{x})u$, are *locally* topologically equivalent.

An equilibrium point $\bar{x}$ of a DE (2.1) in $\mathbb{R}^n$ is a *saddle point* if the real parts of the eigenvalues of the matrix $\mathrm{D}f(\bar{x})$ are all non-zero, and not all of one sign (i.e., a saddle point is a hyperbolic (all $\Re e(\lambda) \neq 0$) equilibrium point which is neither a sink (all $\Re e(\lambda) < 0$) nor a source (all $\Re e(\lambda) > 0$)). If $\bar{x}$ is a saddle point of the DE (2.1) in $\mathbb{R}^n$, and U is a neighbourhood of $\bar{x}$, then the *local stable manifold* of $\bar{x}$ in U is defined by

$$W^s(\bar{x}, U) = \left\{ x \in U \lim_{t\to\infty} \mathrm{g}^t x = \bar{x}, \mathrm{g}^t x \in U \text{ for all } t \geq 0 \right\}. \tag{2.36}$$

Theorem (Stable Manifold Theorem). Let $\bar{x}$ be a saddle point of $x' = f(x)$ in $\mathbb{R}^n$, where f is of class C^1, and let E^s be the stable subspace of the linearization at $\bar{x}$. Then there exists a neighbourhood U of $\bar{x}$ such that the local stable manifold $W^s(\bar{x}, U)$ is a smooth (C^1) curve which is tangent to E^s at $\bar{x}$.

We can define in an analogous way the local unstable manifold of $\bar{x}$ in U, denoted $W^u(\bar{x}, U)$, and similarly there is an "Unstable Manifold Theorem."

Let us give a more detailed description of the local behaviour of non-linear orbits near a non-linear sink. Suppose that $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2$ is an asymptotically stable equilibrium point of the DE $x' = f(x)$. In order to describe the orbits near $\bar{x}$, we can introduce polar coordinates $x_1 - \bar{x}_1 = r\cos\theta$, $x_2 - \bar{x}_2 = r\sin\theta$. Since $\bar{x}$ is asymptotically stable,

$$\lim_{t\to+\infty} r(t) = 0, \tag{2.37}$$

if $r(0)$ is sufficiently close to zero. We say that the equilibrium point $\bar{x}$ is a *non-linear spiral* if

$$\lim_{t\to+\infty} \theta(t) = \pm\infty, \tag{2.38}$$

for any solution $\big(r(t), \theta(t)\big)$ for which (2.37) holds. A number of results are known [90]. Consider the DE

$$x' = f(x), \tag{2.39}$$

in $\mathbb{R}^2$, where f is of class C^1, and consider the linearization

$$u' = \mathrm{D}f(\bar{x})u, \tag{2.40}$$

at the equilibrium point $\bar{x}$. If O is an attracting spiral point of the linearization of the DE, then $\bar{x}$ is an attracting spiral point of the non-linear DE. Similarly, if O is an attracting node (or Jordan node) of the linearization of the DE then $\bar{x}$ is an attracting node (respectivley, Jordan node) of the non-linear DE [90]. An asymptotically stable equilibrium point $\bar{x}$ is said to be an *attracting non-linear focus* if all orbits sufficiently close to $\bar{x}$ approach $\bar{x}$ in a definite direction as $t \to \infty$, and given any direction there exists an orbit which tends to $\bar{x}$ in this direction. Note that if O is a focus of the linearization of the DE, it does not necessarily follow in general that $\bar{x}$ is a non-linear focus of the non-linear DE. Finally, suppose that the vector field f is of class C^2. If O is an attracting focus of the linearization of the DE then $\bar{x}$ is an attracting focus of the non-linear DE [90]. A stable equilibrium point $\bar{x}$ is said to be a *non-linear centre* if in some neighbourhood of $\bar{x}$, the orbits are periodic orbits which enclose $\bar{x}$. Recall that the Hartman-Grobman theorem does not apply if O is a *centre* of the linearization of the DE; i.e., *one cannot conclude that $\bar{x}$ is a centre of the non-linear DE.* However, if O is a centre of the linearization of the DE, then $\bar{x}$ is either a centre, an attracting spiral, or a repelling spiral of the non-linear DE [90].

C. Periodic Orbits and The Poincaré-Bendixson Theorem in the Plane

A linear DE can admit a family of periodic orbits. Of greater interest is the case where a DE admits an *isolated periodic orbit,* i.e., the orbit has a neighbourhood U which contains no other periodic orbits, which is only possible for a non-linear DE. In this situation, the periodic orbit γ may attract neighbouring orbits, thereby describing a physical system which has an *oscillatory steady state* which is stable. The question of the existence of periodic orbits is a difficult one. However, a criterion for excluding periodic orbits for a DE in $\mathbb{R}^2$ was given by Dulac based on Green's theorem.

Dulac's Criterion. If $D \subseteq \mathbb{R}^2$ is a simply connected open set and $\mathrm{div}(Bf) = \frac{\partial}{\partial x_1}(Bf_1) + \frac{\partial}{\partial x_2}(Bf_2) > 0$ (< 0), for all $x \in D$ where B is a C^1 function, then the DE $x' = f(x)$ where $f \in C^1$ has no periodic orbit which is contained in D. The function $B(x_1, x_2)$ is called a *Dulac function* for the DE in the set D.

A second criterion for excluding periodic orbits, which is valid in $\mathbb{R}^n$, $n \geq 2$, follows from the observation that if *a function $V(x)$ is monotone decreasing along an orbit of a DE, then that orbit cannot be periodic.*

Monotone Criterion. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. If $\dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0$ on a subset $D \subseteq \mathbb{R}^n$, then any periodic orbit of the DE $x' = f(x)$ which lies in D, belongs to the subset $\left\{x | \dot{V}(x) = 0\right\} \cap D$.*

An isolated periodic orbit γ of a DE $x' = f(x)$ in $\mathbb{R}^2$, is called a *stable limit cycle* if there exists a neighbourhood U of γ such that $\omega(a) = \gamma$ for all $a \in U$.

A *local section* of the flow of a DE in $\mathbb{R}^2$ is a smooth curve segment Σ such that the vector field f of the DE satisfies $n \cdot f \neq 0$ on Σ, where n is normal to Σ. Note that this implies that no equilibrium points of f lie on Σ, and by continuity, that orbits pass through Σ in one direction only. If x is a *regular* (i.e., non-equilibrium) *point* of the flow (i.e., $f(x) \neq 0$) and Σ is a local section through x, then a *flow-box* for x is a neighbourhood of x of the form $N = \left\{\mathrm{g}^t\Sigma \middle| \, |t| < \delta\right\}$ for some $\delta > 0$. Finally we recall the following properties of ω-limit sets: (i) $\omega(a)$ is the union of whole orbits, (ii) $\omega(a)$ is a closed set, (iii) if $\omega(a)$ is bounded, then $\omega(a)$ is connected (i.e., is not the union of disjoint sets). Then we obtain the following Lemma (which is not valid for a flow in $\mathbb{R}^3$).

Lemma (Fundamental Lemma on ω-limit sets in $\mathbb{R}^2$). Let $\omega(a)$ be an ω-limit set of a DE in $\mathbb{R}^2$. If $y \in \omega(a)$, then the orbit through y, $\gamma(y)$, cuts any local section Σ in at most one point [169, 181].

Theorem (Poincaré-Bendixson). Let $\omega(a)$ be a non-empty ω-limit set of the DE $x' = f(x)$ in $\mathbb{R}^2$, where $f \in C^1$. If $\omega(a)$ is a bounded subset of $\mathbb{R}^2$ and $\omega(a)$ contains no equilibrium points, then $\omega(a)$ is a periodic orbit [181].

In applications it is often convenient to use the following Corollary of the Poincaré-Bendixson theorem.

Corollary. Let K be a positively invariant subset of the DE $x' = f(x)$ in $\mathbb{R}^2$, where $f \in C^1$. If K is a closed and bounded set, then K contains either a periodic orbit or an equilibrium point.

The fundamental property of an ω-limit set is that it consists of one whole orbit, or that it is the union of more than one whole orbit. The simplest situations are: (i) $\omega(a)$ is an equilibrium point; i.e., the system approaches an equilibrium state as $t \to +\infty$, (ii) $\omega(a)$ is a periodic orbit; i.e., the system approaches an oscillatory steady state as $t \to +\infty$. Other examples of invariant sets (i.e., unions of orbits) can arise as ω-limit sets in $\mathbb{R}^2$, such as, for example, cycle graphs. Let $\alpha(x)$ denote the α-limit set of the point x, which is the set of past limit points of x; i.e.,

$$\alpha(x) = \left\{y \middle| y = \lim_{n\to\infty} \mathrm{g}^{t_n}x, \text{ and } t_n \to -\infty\right\}. \tag{2.41}$$

A *cycle graph* S of a DE $x' = f(x)$ in $\mathbb{R}^2$ is a connected union of orbits such that: (i) For all $x \in S$, $\omega(x) = \{p\}$ and $\alpha(x) = \{q\}$, where p and q are equilibrium points in S. (ii) For all equilibrium points $p \in S$, there exists points $x, y \in S$ such that $\omega(x) = \{p\}$, $\alpha(y) = \{q\}$, and the number of equilibrium points in S is finite. (iii) The orientations of the orbits define a continuous closed path in S. If we consider the DE $x' = f(x)$ in $\mathbb{R}^2$ and let $a \in \mathbb{R}^2$ be an initial point such that $\{g^t a | t \geq 0\}$ lies in a closed bounded subset $K \subset \mathbb{R}^2$, it then follows that if K contains only a finite number of equilibrium points then one of the following holds: 1. $\omega(a)$ is an equilibrium point. 2. $\omega(a)$ is a periodic orbit. 3. $\omega(a)$ is a cycle graph [169, 227]. Unfortunately, this theorem does not generalize to DE in $\mathbb{R}^n$, $n \geq 3$, or to DE on the 2–torus. Indeed, the problem of describing all possible ω-limit sets in $\mathbb{R}^n$, $n \geq 3$, is presently unsolved.

D. More General Non-Linear Behaviour

The motion of an undamped symmetric 2-mass oscillator, whose orbits lie in invariant 2-tori, depends on the values of two constants ω_1 and ω_2, which are the two natural frequencies of oscillation of the physical system. If $m\omega_1 = n\omega_2$, where m, n are positive integers without common factors, the solutions are periodic with period $T = \frac{2\pi m}{\omega_1} - \frac{2\pi n}{\omega_2}$. The corresponding orbit on one of the invariant tori is thus periodic, and eventually closes up as it winds around the torus. If, on the other hand, $\frac{\omega_1}{\omega_2}$ is *irrational*, then the orbits are not periodic, and hence do not close up as they wind around the invariant tori. As the orbit winds around the torus, it passes arbitrarily close to each point of the torus, and the orbit is said to be *everywhere dense on the torus* [11]. In this latter case, $\omega(a)$ is the union of an uncountable infinity of whole orbits, including the orbit through a, and the invariant 2-tori do not attract neighbouring orbits. However, an attracting 2-torus can be created, giving rise to so-called quasiperiodic motion, in a similar fashion to that in which non-linearity can be used to create an attracting periodic orbit. This example illustrates the richness of the possible dynamical behaviour in non-linear systems and motivates the idea of attracting sets in describing long-term behaviour. Given a DE $x' = f(x)$ in $\mathbb{R}^n$, a closed invariant set $A \subset \mathbb{R}^n$ is said to be an *attracting set* if there exists a neighbourhood U of A such that $g^t U \subseteq U$ for all $t \geq 0$ and $\omega(a) \subseteq A$ for all $a \in U$, where g^t is the flow of the DE and $\omega(a)$ is the ω-limit set of the point a. Intuitively, an attracting set is a generalization of an asymptotically stable equilibrium point or periodic orbit. The *basin of attraction* of an attracting set A is the subset of $\mathbb{R}^n$ defined by $\rho(A) = \{x \in \mathbb{R}^n | \omega(x) \subseteq A\}$.

If a DE has an attracting set A, then for all initial states a in the basin of attraction the physical system approaches some kind of "steady state". The nature of the steady state is determined by the orbits which form the attractor. In the case that the attracting set is an equilibrium point or a periodic orbit, the long term steady state behaviour is an equilibrium state or periodic motion, respectively. In the case that the attracting set is an invariant 2-torus (or k-torus in general) with dense orbits the long-term steady state behaviour is 2-quasiperiodic (k-quasiperiodic). Finally, there is the possibility of "strange" attractors and chaotic behaviour which is the subject of current research. A strange attractor is not a piecewise smooth surface, and can have a structure like that of a Cantor set. Chaotic behaviour occurs when neighbouring orbits diverge (separate) from each other at an exponential rate, while remaining bounded, a phenomenon that is referred to as "sensitive dependence on initial conditions" [14, 275].

1. *Higher Dimensions*

There are many of new features possible in higher dimensions. Although, in general, a qualitative analysis is much more difficult there are some cases, such as conservative and gradient systems, which have special characteristics that make their analysis possible (e.g., limit sets of orbits in gradient systems are necessarily part of the set of equilibria). In addition, Hamiltonian systems often occur in physical applications. Completely integrable systems can be analyzed successfully. However, an analysis of general Hamiltonian systems for $n \geq 4$ is currently out of reach.

Unlike a linear DE, a non-linear system allows for singular structures which are more complicated than that of equilibrium points, fixed lines or periodic orbits, particularly in higher dimensions ($n > 2$). These structures include, but are not limited to, such things as heteroclinic and/or homoclinic orbits and non-linear invariant sub-manifolds [373]. Sets of non-isolated equilibrium points often occur in applications (and particularly in cosmology) and therefore their stability needs to be examined more carefully. A set of non-isolated equilibrium points is said to be *normally hyperbolic* if the only eigenvalues with zero real parts are those whose corresponding eigenvectors are tangent to the set. Since by definition any point on a set of non-isolated equilibrium points will have at least one eigenvalue which is zero, all points in the set are *non-hyperbolic*. The stability of a set which is normally hyperbolic can, however, be completely classified by considering the signs of the eigenvalues in the remaining directions (i.e., for a curve, in the remaining $n - 1$ directions) [13].

The local dynamics of an equilibrium point may depend on one or more arbitrary parameters. parameter When small continuous changes in a parameter results in dramatic changes in the dynamics, the equilibrium point is said to undergo a *bifurcation* [181]. The values of the parameter(s) which result in a bifurcation at the equilibrium point can often be located by examining the linearized system. Equilibrium point bifurcations will only occur if one (or more) of the eigenvalues of the linearized system are a function of a parameter and the bifurcations are located at the parameter values for which the real part of an eigenvalue is zero.

There are a variety of possible future and past asymptotic states of a non-linear system. In the case of a plane system the possible asymptotic states can be given explicitly via the Poincare-Bendixson Theorem due to the limited degrees of freedom and the fact that the flows (or orbits) in any 2-dimensional phase space cannot cross. This theorem has a very important consequence in that if the existence of a closed (i.e., periodic, heteroclinic or homoclinic) orbit can be ruled out it follows that all asymptotic behaviour is located at an equilibrium point. As noted earlier the existence of a closed orbit can be ruled out by many methods. When the phase space is of a higher dimension (than two) the requirement that orbits cannot cross does not result and the decisive Poincare-Bendixson theorem does not follow. The behaviour in such higher-dimensional spaces is very complicated, with the possibility of phenomena such as recurrence and strange attractors occurring [165]. For this reason the analysis of non-linear systems in spaces of three or more dimensions cannot in general progress much further than the local analysis of the equilibrium points (or non-isolated equilibrium sets). However, one tool which does allow for some progress in the analysis of higher dimensional systems is the possible existence of monotone functions.

Theorem (LaSalle Invariance Principle) [332]. *Consider a DE $\dot{x} = f(x)$ on $\mathbb{R}^n$. Let S be a closed, bounded and positively invariant set of the flow, and let Z be a C^1 monotonic function. Then for all $x_0 \in S$,*

$$w(x_0) \subset \{x \in S | \dot{Z} = 0\},$$

where $w(x_0)$ is the w-limit set for the orbit with initial value x_0.

This principle has been generalized to the following result:

Theorem (Monotonicity Principle) [226]. Let ϕ_t be a flow on $\mathbb{R}^n$ with S an invariant set. Let $Z : S \rightarrow \mathbb{R}$ be a C^1 function whose range is the interval (a, b), where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{\infty\}$ and $a < b$. If Z is decreasing on orbits in S, then for all $x \in S$,

$$\omega(x) \subseteq \{s \in \bar{S} \setminus S | \lim_{y \rightarrow s} Z(y) \neq b\},$$
$$\alpha(x) \subseteq \{s \in \bar{S} \setminus S | \lim_{y \rightarrow s} Z(y) \neq a\},$$

where $\omega(x)$ and $\alpha(x)$ are the w- and α-limit sets of x, respectively.

As noted earlier, in most cases the eigenvalues of the linearized DE will have eigenvalues with both positive, negative and/or zero real parts. In these cases it is important to identify which orbits are attracted to the equilibrium point, and which are repelled away, as the independent variable tends to infinity. For a linear DE (2.2), the phase space $\mathbf{R}^n$ is spanned by the eigenvectors of A. These eigenvectors divide the phase space into three distinct subspaces; namely, the *stable subspace* $E^s = \text{span}(s_1, s_2, ...s_{ns})$, the *unstable subspace* $E^u = \text{span}(u_1, u_2, ...u_{nu})$, and the *centre subspace* $E^c = \text{span}(c_1, c_2, ...c_{nc})$, where s_i are the eigenvectors who's associated eigenvalues have negative real part, u_i those who's eigenvalues have positive real part, and c_i those who's eigenvalues have zero eigenvalues. Flows (or orbits) in the stable subspace asymptote in the future to the equilibrium point, and those in the unstable subspace asymptote in the past to the equilibrium point.

In the non-linear case, the topological equivalence of flows allows for a similar classification of the equilibrium points. The equivalence only applies in directions where the eigenvalue has non-zero real parts. In these directions, since the flows are topologically equivalent, there is a flow *tangent* to the eigenvectors. The phase space is again divided into stable and unstable subspaces (as well as centre subspaces). The *stable manifold* W^s of an equilibrium point is a differential manifold which is tangent to the stable subspace of the linearized system (E^s). Similarly, the *unstable manifold* is a differential manifold which is tangent to the unstable subspace (E^u) at the equilibrium point. The *centre manifold*, W^c, is a differential manifold which is tangent to the centre subspace E^c. It is important to note, however, that unlike the case of a linear system the centre manifold W^c will contain all the dynamics not classified by linearization (i.e., in the non-hyperbolic directions). In particular, this manifold may contain regions which are stable, unstable or neutral. The classification of the dynamics in this manifold can only be determined by utilizing more sophisticated methods, such as the Centre Manifold Theorem or the theory of normal forms [373].