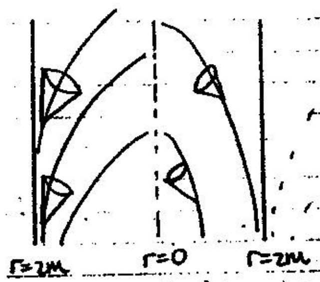
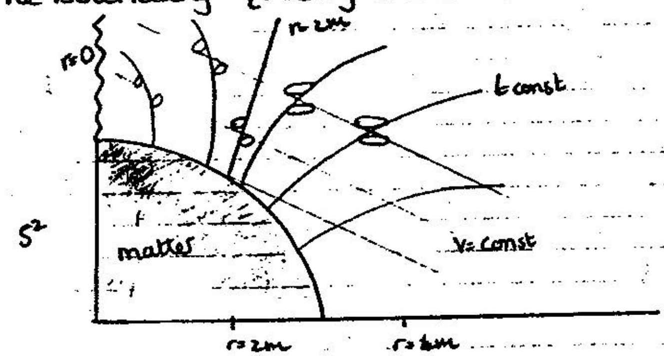
The singularity is not visable from infinity (26)
Since $r = 2m$ is a null surface

$\{ ds^2 = -(1-\frac{2m}{r})dv^2 + 2dv\,dr + r^2 d\Omega^2$
$r = 2m, dr = 0$ $\quad ds^2 = r^2 d\Omega^2$ has intrinsic geometry $(^0_+)$. $\}$
future directed null or timelike curves can only cross $r = 2m$ inwards.

In 2D ($\overset{\theta}{\phi}$ suppressed) the light cones with will, since $r = 2m$ is null surface.



we say that $\{r \leq 2m\}$ is the region inside a black hole.
The Boundary $\{r = 2m\}$ is the event horizon.



Singularity inside matter is also invisible from infinity.

Notes
(1) Shown existence black hole in special case (spherical symmetry) - general case?
Q: Generic collapse possess a) ST singularity b) event horizon surrounding singularity
A: a) Hawking penrose singularity Theorems $\Rightarrow$ a). (with causal cond^{ns}).
b) circumstantial evidence - cosmic censorship hypothesis.
[Failure (b) $\Rightarrow$ physics unknowable inside black holes since do not affect outer regions - physics in "disarray".]
(2) Conservative viewpt. Generic collapse settles down to stationary black holes - characterized by mass, charge and angular momentum (Schwarzschild Riesner Nordstrom, Kerr) (No hair theorems)
(3) Bekenstein - entropy of black hole (Hawking - area $\dot{A} \geq 0$)
(4) Hawking radiation.

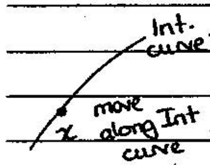
(1)

Schwarzschild solution / spacetime

Spherical symmetry + vacuum ST + asympt flat $\xrightarrow{\text{Birkhoff}}$ Schwarzschild and static

To analyse what happens outside r_0 (radius of star) need to investigate geodesics

A ST has a symmetry transf^n generated by X^i iff



$$\left(\mathcal{L}_X g\right)_{ij} = 0 = 2X_{(i;j)}$$

Then X^i is called Killing vector

$(\mathcal{L}_X g)_{ij} \equiv g_{ij;k} + g_{kj}X^k{}_{;i} + g_{ik}X^k{}_{;j} = X_{j;i} + X_{i;j}$

Symmetries generate conservation laws.

Lemma 1 Suppose T geodesic vector and K Killing vector field (KVF). Then $K \circ T$ is conserved along geodesic $\Rightarrow$

$$\nabla_T (K \circ T) = 0$$

$K_{i;j} + K_{j;i} = 0$
$T^i{}_{;j}T^j = 0$

Proof $\nabla_T(K \circ T) = (K_i T^i)_{;j} T^j = K_{i;j}T^iT^j + T^i{}_{;j}K_i$
$= K_{(i;j)}T^iT^j = 0$

Now consider a radially infalling (Ω invariant) geodesic γ with unit tangent vector T ($T \circ T = -1$ - timelike geodesic). Let τ denote propertime along γ and $\dot{}$ denote $\frac{d}{d\tau}$ $\quad (\dot{f} = \frac{df}{d\tau})$.

$$\left. \begin{aligned} ds^2 &= -F dt^2 + \frac{1}{F}dr^2 + r^2 d\Omega^2 \\ F &= 1 - \frac{2m}{r} \end{aligned} \right\} \quad (1).$$

$$T^i = \frac{dx^i}{d\tau} = \dot{x}^i = (\dot{t}, \dot{r}, 0, 0)$$

Thus $T_i = (-F\dot{t}, \dot{r}/F, 0, 0)$ (2)

$\left[-T\cdot T = -T_i T^i = F\dot{t}^2 - \dot{r}^2/F = 1 \quad \left[= \left(\frac{d\tau}{d\tau}\right)^2 \right] \right]$.

Now the metric is invariant wrt changes in t, $\frac{\partial}{\partial t}$ is a KVF

$$K^i = (1, 0, 0, 0).$$

[check : $X_{i;j} + X_{j;i} = X_{i,j} + \Gamma^k_{ij} X_k + \Gamma^k_{ji} X_k + X_{j,i} = 0$ for $X_i = K_i$ and Γ^i_{jk} constructed from (1)].

Hence, by Lemma1,

$$T_i K^i = -F\dot{t}$$
$$\equiv -E \quad \text{is constant of motion.}$$

Thus $\dot{t} = \frac{E}{F}$ (2)'

and (2) $\Rightarrow$

$$\underbrace{E^2}_{F^2\dot{t}^2} - \dot{r}^2 = F$$

$$\Rightarrow \dot{r} = \underbrace{-}_{\text{take neg sign for infalling geods}} \underbrace{\sqrt{E^2 - 1 + \frac{2m}{r}}}_{E^2>1 \text{ to avoid complex roots}} \quad (3)$$

Note as $r \to 2m$ $F \to 0$

Set $r = 2m(1+\varepsilon) \Rightarrow F = \overbrace{1 - \frac{1}{1+\varepsilon}}^{1-2m/r} = \varepsilon + O(\varepsilon^2)$

Proper time taken to fall from $r_0\,(>2m)$ to $2m$ is $\left(d\tau = \frac{dr}{\dot{r}}\right)$

$$\tau_{2m} - \tau_0 = \int_{2m}^{r_0} \frac{dr}{\left(E^2 - 1 + \frac{2m}{r}\right)^{1/2}} \quad \text{which is always finite}$$

coord time taken is

$$t_{2m} - t_0 = \int \frac{dt}{d\tau}\frac{d\tau}{dr}\, dr \qquad \int \dot{t}\, \frac{dr}{\dot{r}}$$

$$= \int_{2m}^{r_0} \frac{E\, dr}{\underbrace{F}_{\varepsilon} \sqrt{E^2 - 1 + 2m/r}} \quad \text{which has logarithmic singularity}$$

Thus t coord $-\infty < t < \infty$ only covers that part of ST for which $r > 2m$.

What has gone wrong ?

(i) $g^{tt}, g_{rr} \to \infty$ as $r \to 2m$. Is this real or coord singularity ?
(i.e. has singularity arisen due to choice of coords or from intrinsic physics)

calculating $\underset{0}{\underset{\parallel}{R}}$, $\underset{\text{finite}}{\underset{|}{R_{abcd}R^{abcd}}}$ near $r = 2m$ (scalars coord independent).

we see that coords faulty.

Analytic Continuation

It is clear that problem lies with 2-metric

$$ds^2 = -F\,dt^2 + \frac{1}{F}\,dr^2$$

Notice that as $r \to 2m$, KVF K becomes null

$[g(K,K) = g_{ab}K^aK^b = g_{11} = -F \to 0$ as $r \to 2m$ $(K_i = (1,0,0,0))$ — $g(K,K) \to 0 \to K$ null vector$]$ Normal to surfaces r = constant (i.e. $R_i = \frac{\partial r}{\partial x^i}$) also becomes null.

Thus: examine null rays $m_i = (1, f, 0, 0)$

$$g^{ij}m_i m_j = 0 \Rightarrow -\frac{1}{F} + f^2 F = 0 \Rightarrow f = \pm\frac{1}{F}$$

$$L_i = (1, -\tfrac{1}{F}, 0, 0) \qquad N_i = (1, \tfrac{1}{F}, 0, 0)$$

radial ingoing null ray — radial outgoing one

These vectors satisfy $m_{[i;j]} = 0 \Rightarrow m_i = u_{;i}$ for some scalar u.

Lemma 2

Any null vector m satisfying $m_{[i;j]} = 0$ is a geodesic.

Proof:

$$m_{i;j}m^j = m_{j;i}m^j = \tfrac{1}{2}(m_j m^j)_{;i}$$
$$= \tfrac{1}{2}\underbrace{(m_j m^j)}_{\text{scalar}}{}_{,i} = 0$$

Note

L, N are geodesics.

Let $u_{,i} = L_i = (1, -\frac{1}{F}, 0, 0) = \delta_i^{\,t} - \delta_i^{\,r}/F$
$= t_{,i} - r_{,i}/F$

Then $u = t - \int \frac{dr}{F}$

$\int \frac{dr}{F} = \int \frac{dr}{(1-2m/r)}$ Note that $\frac{d}{dr}(r + 2m\log(r-2m) + \overset{\text{constant}}{K})$
$= 1 + \frac{2m}{r-2m} = \frac{r}{r-2m} = \frac{1}{1-\frac{2m}{r}}$

Hence $\int \frac{dr}{F} = r + 2m\log(r-2m) + K$.

$u = t - r - 2m\log|r-2m|$ — K ignored at mom. - does not affect differential forms which will initially be discussed.

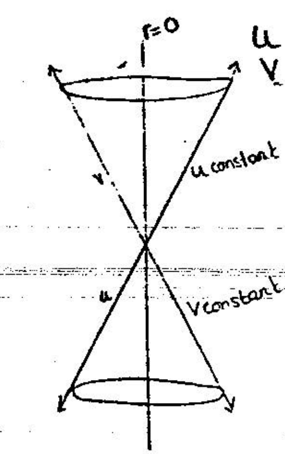
Similarly $N_i = v_{,i}$ yields $v = t + \int \frac{dr}{F}$

$v = t + r + 2m\log|r-2m|$

What do u, v mean?

Minkowski Space: No matter present $m = 0$.

$u = t - r$ retarded time
$v = t + r$ advanced time



One can use u or v as new coord to replace t, or both u, v to replace t, r.

$du = dt - dr$
$dv = dt + dr$

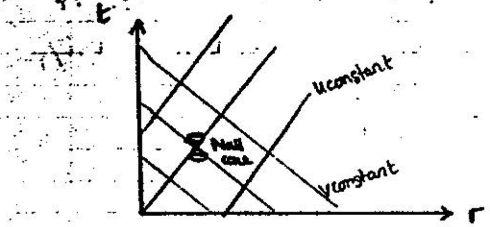
$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$ for $m = 0$

$dt = du + dr \Rightarrow ds^2 = -(du + dr)^2 + dr^2 + r^2 d\Omega^2$
$= -du^2 - 2du\,dr + r^2 d\Omega^2$ – retarded Null cones

$dt = dv - dr \Rightarrow ds^2 = -dv^2 + 2dv\,dr + r^2 d\Omega^2$ – advanced Null cones

Alternatively both. $dt = \frac{du + dv}{2}$, $dr = \frac{dv - du}{2} \Rightarrow ds^2 = -2du\,dv + r^2 d\Omega^2$ – double Null cones

Suppress θ, ϕ coords.



Each pt represents a sphere of radius r.

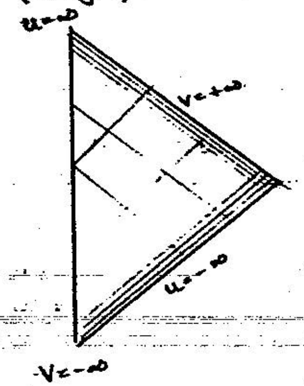
$r > 0 \Rightarrow v > 0$

we can make this diagram finite.

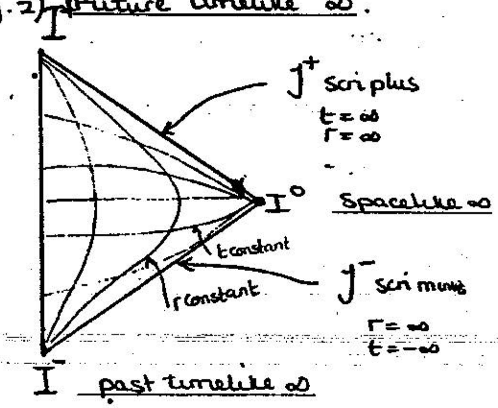
$$u = \tan \chi \qquad -\infty < u < v < \infty$$
$$v = \tan \gamma \qquad -\pi/_2 < \chi < \gamma < \infty.$$

[- see sheet for diagrams]

(diag. 1)



(diag. 2)



triangle Representation of minkowski space.

<u>Summary</u> $\quad ds^2 = -F dt^2 + \frac{dr^2}{F} + r^2 d\Omega^2$

$m=0 \quad ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$

$$\left.\begin{array}{l} u = t - r \\ v = t + r \end{array}\right\} \Rightarrow \quad \begin{aligned} ds^2 &= -du^2 - 2du\,dr + r^2 d\Omega^2 \\ &= -dv^2 + 2dv\,dr + r^2 d\Omega^2 \\ &= -2du\,dv + r^2 d\Omega^2. \end{aligned}$$

Causal Structure

<u>Chronological future</u> (past) $I^{\mp}(p)$ of $p \in M$ is set of pts $q \in M$ which can be joined to p by a future (past) directed timelike curve from p to q.

<u>Causal future</u> (past) $J^{\mp}(p)$ of $p \in M$ is set of all pts $q \in M$ which can be joined to p by by future (past) directed timelike or null curve from p to q.

Now consider $m \neq 0$

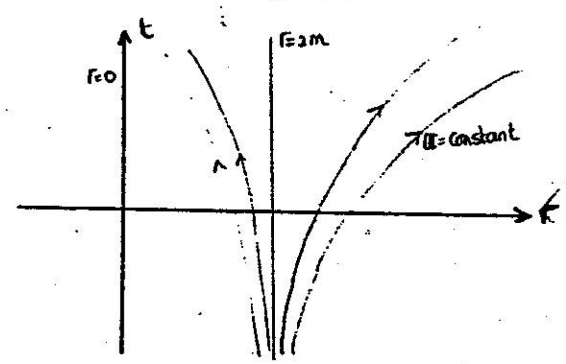
Again $u_{,i} = t_{,i} - \frac{r_{,i}}{F}$

$$u = t - \int \frac{dr}{F}$$

We wish to consider lines of constant u. Along such curves

$$t = \text{const} + \int \frac{dr}{F}$$

$$\left(\frac{dt}{dr}\right)_u = \frac{1}{F} \quad - \text{eq}^n \text{ of } u = \text{constant}.$$



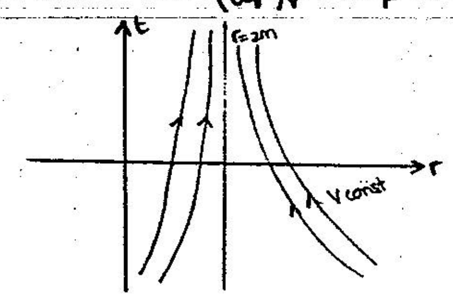
Note that $\left(\frac{dt}{dr}\right)_u \to \infty$ as $r \to 2m$

$\to 1$ as $r \to \infty$

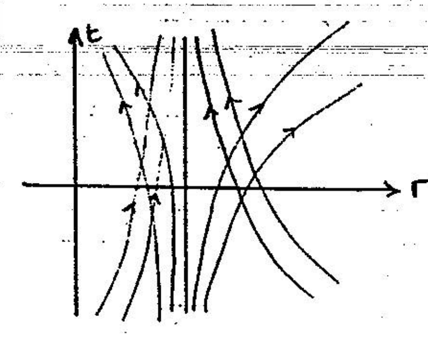
Also $v_{,i} = t_{,i} + \frac{r_{,i}}{F}$.

Along v = constant

$$\left(\frac{dt}{dr}\right)_v = -\frac{1}{F}$$



Hence



The diagram is very confused near $r = 2m$. Transform to advanced Null coords (v, r, ϕ, θ).

— Eddington-Finkelstein coords; $dt = dv - \frac{dr}{F}$

These coords are regular for $0 < r < \infty$ $\quad -\infty < v < \infty$.

This coord system cover $r = 2m$, i.e we have analytically continued the original manifold.

$$dt = dv - \frac{dr}{F} \quad ; \text{ Eddington - Finkelstein}$$

$$dt^2 = \left(dv - \frac{dr}{F}\right)^2 = dv^2 - 2\frac{dv\,dr}{F} + \frac{dr^2}{F^2}.$$

Hence $ds^2 = -F dt^2 + \frac{1}{F} dr^2 + r^2 d\Omega$

becomes $\underline{ds^2 = -F dv^2 + 2 dv\,dr + r^2 d\Omega}$ — No coord. singularity now at $r = 2m$

Deviation; Alternatively we could have transformed to Double Null coords

$$\left.\begin{aligned} v &= t + r + 2m\log|r-2m| + k_1 \\ u &= t - r - 2m\log|r-2m| + k_2 \end{aligned}\right\}$$

$$\Rightarrow \quad \left.\begin{aligned} dv &= dt + \frac{dr}{F} \\ du &= dt - \frac{dr}{F} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} dt &= \frac{du + dv}{2} \\ dr &= \frac{dv - du}{2} \end{aligned}$$

Hence $ds^2 = -F dt^2 + F dr^2 + r^2 d\Omega^2 = -F\,du\,dv + r^2 d\Omega$.

Since $g_{ij} \to \eta_{ij}$ as $r \to \infty$ we try to draw v = constant lines at $-\pi/4$ slope

Define new verticle coord $\underline{y = v - r}$, ($\neq t$ note)

Hence lines of v = constant have slope $\left(\frac{dy}{dr}\right)_v = -1$ since $y = \text{const} - r$.

Consider lines of constant u.

Note that $u = t - r - 2m\log|r-2m| + k_1$ $\quad$ i.e $t = u + r + 2m\log|r-2m| + k_1$

$v = t + r + 2m\log|r-2m| + k_2$

$\therefore$ Sub value of t into 2nd eq^n $\quad v = u + 2r + 4m\log|r-2m| + k_3$ $\quad (k_3 = k_2 - k_1)$

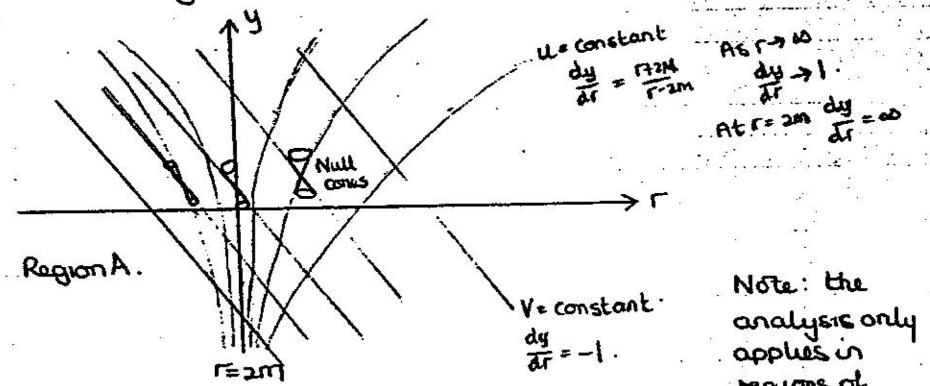
$$y = (v - r) = u + r + 4m\log|r-2m|$$

Lines of constant u are

$$y = \text{constant} + r + 4m\log|r-2m|$$

$$\left(\frac{dy}{dr}\right)_u = 1 + \frac{4m}{(r-2m)} = \frac{r+2m}{r-2m}$$

Hence diagram can be drawn.



Note: the analysis only applies in regions of Schwartzchild geometry (i.e does not hold inside star).

Black Holes.

Examine timelike geodesics

$$v = t + r + 2m\log|r-2m| + K$$

$$\mathring{v} = \mathring{t} + \mathring{r} + \frac{2m}{(r-2m)}\mathring{r} = \mathring{t} + \left(1 + \frac{2m}{r-2m}\right)\mathring{r}$$

$$= \mathring{t} + \frac{1}{(1-\frac{2m}{r})}\mathring{r}$$

Put $F = (1 - \frac{2m}{r})$

$$\therefore \quad \mathring{v} = \mathring{t} + \frac{1}{F}\mathring{r}$$

But eq (2)' $\mathring{t} = \frac{E}{F}$ eqⁿ (2) $\mathring{r} = -(E^2-F)^{1/2}$

$$\therefore \quad \dot{v} = \frac{E - (E^2-F)^{1/2}}{F}$$

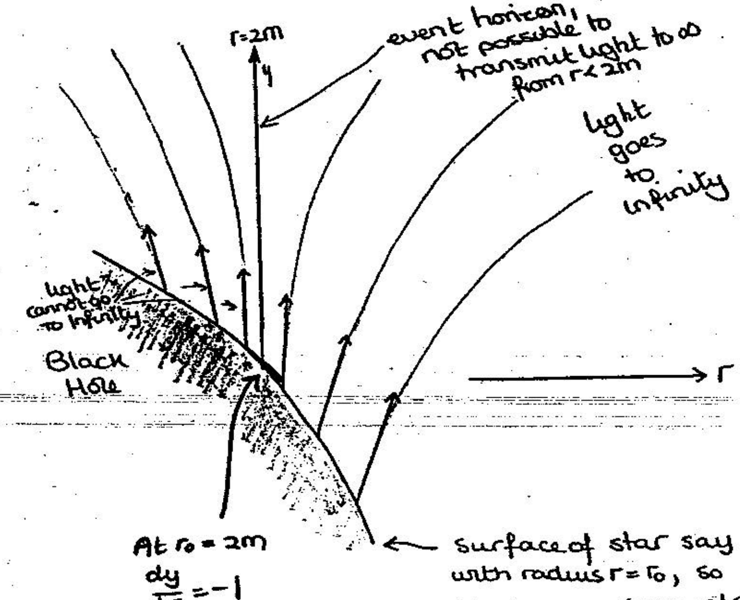
But $y = v - r$ $\qquad \dot{y} = \dot{v} - \dot{r}$

And $\frac{dy}{dr} = \frac{\dot{y}}{\dot{r}} = \frac{\dot{v}}{\dot{r}} - 1 = \dfrac{\frac{E-(E^2-F)^{1/2}}{F}}{-(E^2-F)^{1/2}} - 1$

Hence $$\frac{dy}{dr} = -\frac{\left[\frac{E}{(E^2-F)^{1/2}} - \frac{2m}{r}\right]}{F}$$ Note $(F = 1-\frac{2m}{r})$

As $r \to 2m$ $\quad F \to 0 \quad [\;] \to F$ and $\frac{dy}{dr} = -1$

and so curve is perfectly regular.

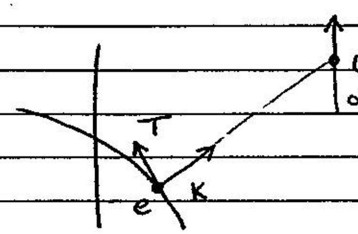


surface of star say with radius $r = r_0$, so that curve represents 'world line' of surface of star.

dust constituting star follows timelike geodesics. As star contracts – as $r \to 2m$ $\frac{dy}{dr} \to -1$ → slope at $r_0 = 2m$ is $\frac{dy}{dr} = -1$

Consider star, whose radius r_0 contracts from $r = r_0$ to $r_0 = 2m$. Thus any star that collapses through $r_0 = 2m$ must hit singularity $r = 0$ (null cone "directed" towards singularity)

Q: What does an observer at ∞ see?



$u^i = (1,0,0,0)$ $\qquad$ $L_i = (1, -\frac{1}{F}, 0, 0)$

Light ray k^i

Know that

$$\underset{\text{Killing vector}}{k^i} u_i = \text{const} = \underset{\text{frequency of energy (as seen at } \infty)}{-\nu_0}$$

Frequency as seen by an observer with 4velocity V^i is $-V_i k^i$

In particular, frequency as seen at emitter is

$$\nu_e = -k_i T^i$$

$$\left.\begin{aligned} \text{Set } \; k_i &= \lambda L_i = \lambda(1, -\tfrac{1}{F}, 0, 0) \\ T^i &= (\tfrac{E}{F}, \dot{r}, 0, 0) \\ u^i &= (1,0,0,0) \end{aligned}\right\} \quad \begin{aligned} \nu_e &= -k_i T^i = -\frac{\lambda(E - \dot{r})}{F} \\ \nu_0 &= -k_i u^i = -\lambda \end{aligned}$$

$$\frac{\nu_e}{\nu_0} = \frac{-\lambda(E - \dot{r})}{-\lambda \cdot F} = \frac{E - \dot{r}}{F}$$

$$\text{s.t.} \qquad \frac{\nu_0}{\nu_e} = \frac{F}{E - \dot{r}} \longrightarrow 0 \text{ as } r \to 2m$$

Thus, as seen at infinity, constant ν_e signals become fainter and fainter.

$v = t + r + 2m \log|r - 2m|$
$u = t - r - 2m \log|r - 2m|$

Penrose diagram and Maximal Analytic Extensions.

Use tan angle transf^n from afore.

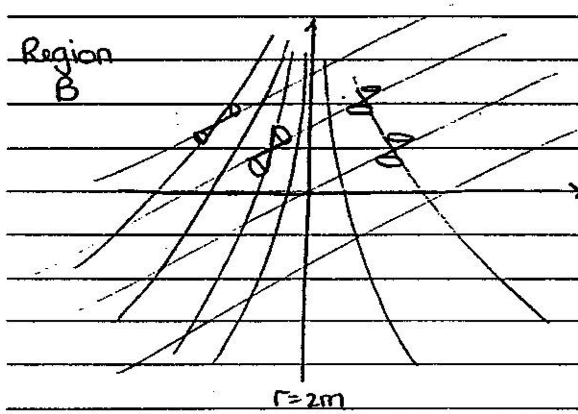
$$u = \tan \chi \qquad v = \tan \psi$$

In region A; $\quad r \to \infty$, v finite $\Rightarrow$ $\left.\begin{aligned} u &\Rightarrow -\infty \\ t &\Rightarrow -\infty \end{aligned}\right\}$ $\begin{aligned} \chi &\to -\pi/2 \\ \psi &\to \pi/2 \end{aligned}$

~~[$r \to 2m$ $\log(r-2m) \to -\infty$, $t \to \infty$, $u \to \infty$, $\chi \to \pi/2$]~~

Just as making v = const have slope $-\pi/4$ yields diags 3,5 and black holes, so making u = const have slope $\pi/4$ yields diag 7. (define $z = u + r$)

Region B



{Note that u=const / $v$const slope $-\pi/4$ and v=const / u=const with slope $\pi/4$ are both coord patches of full ST manifold – this is hidden in Schwarzschild coords – so we have analytically continued}

No information can cross $r = 2m$ from outside, info can travel out

— White hole

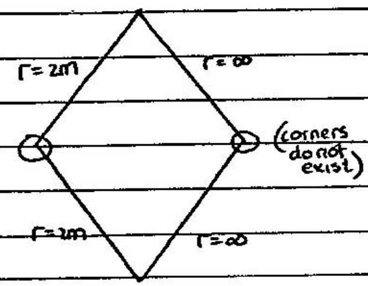
(Comparison diags 4,7 yields perfect symmetry).

Similarly, in region B $\quad r \to \infty$, u finite $\to$ $\left.\begin{aligned} v &\to \infty \\ t &\to \infty \end{aligned}\right\}$ $\psi \to \pi/2$

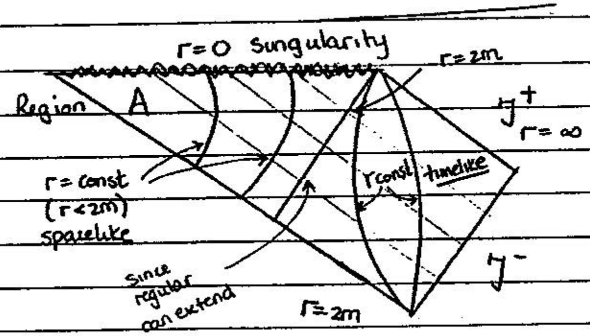
$r \to 2m$, u finite, $\quad \begin{aligned} v &\to -\infty \\ t &\to -\infty \\ \psi &\to -\pi/2 \end{aligned}$

(13)

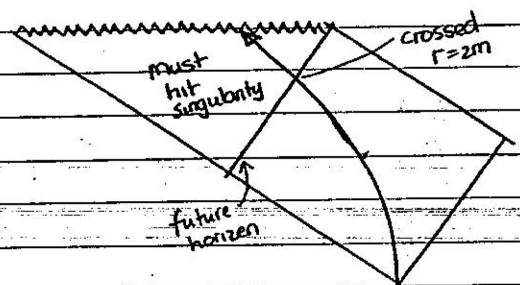
For $r>2m$ these regions overlap, as shown in diags 8, 9.



lines of u, v; t, r constant given in figs 8, 9.



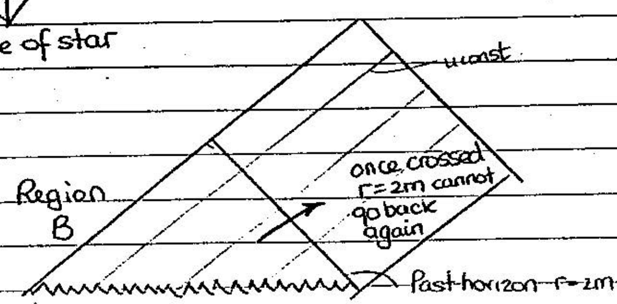
What happens at $r=0$?



collapse of star

A geometric proof of singularity theorem. A star which collapses across $r=2m$ must hit singularity.

Whitehole.



(14)

* Note: In diags: Each pt represents an S^2 (θ, ϕ varying) so that topology of ST is $\mathbb{R}^2 \times S^2$

Situation	Conventional picture	Penrose diag.	
Minkowski	—	1, 2	
Schwarzschild (t, r)	3	9	− t doesn't cover 2m
" (v, r)	4	10	Black Hole*
Collapsing star (t, r)	5 [y axis is v−r]	11	Black Hole*
" (v, r)	6		
Schwarzschild (u, r)	7 [y axis is u+r]	12	White Hole*

where $u = t - r - 2m\log|r-2m| + k_1$

$v = t + r + 2m\log|r-2m| + k_2$

Note * Black preferred to white in nature because

(a) Astrophysical interpretation

(b) Thermodynamics suggest it

[Bekenstein $\dot{A} \geq 0 \Leftrightarrow$ entropy increasing $\Leftrightarrow$ 1st law thermodyn ~~i.e. laws st go from low entropy~~ to high entropy as t increases].

(c) Cosmic Censor (Penrose)

~~– all singularities must be closed~~

(cannot predict what happens at naked singularity – white holes excluded by cosmic censorship

Diag 10

"vertical" – timelike

horizon – null

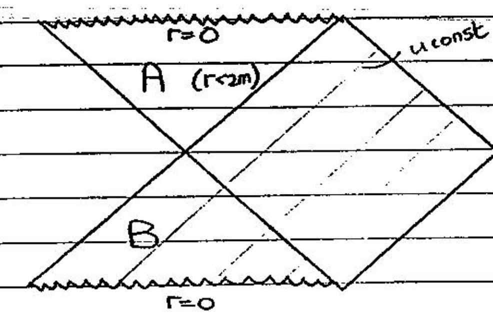
"horizontal" – spacelike

$$ds^2 = -F\,dt^2 + F^{-1}dr^2 + r^2 d\Omega^2$$

sig. $(-+++)$ $r>2m$ $(F>0)$

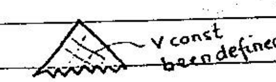
sig $(+-++)$ $r<2m$ $(F<0)$

Fig 13 – superposition of figs 10 and 12

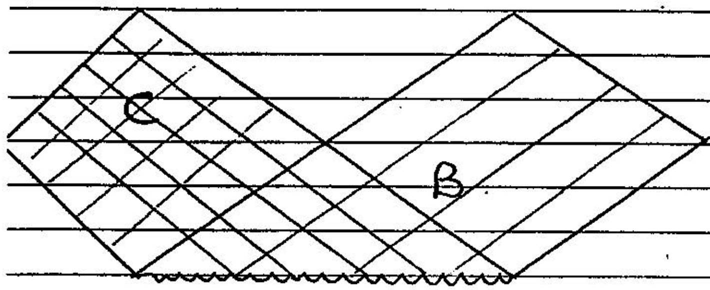


u, r coords cover B.

Inside B one can define new coord by

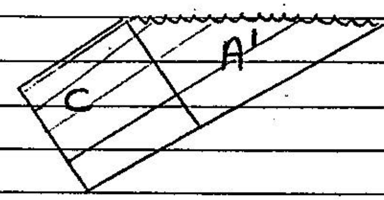


(having defined new coords in B can extend them to new region C] ⑮



New region C can be constructed. Can continue into new region C isomorphic with original region.

Extend C by same argument



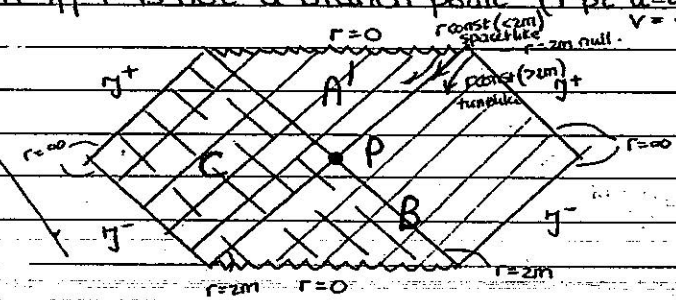
Q: Is $A' = A$ or not? (if not could continue to construct new regions B', C', A'' by same procedure).

One can identify A, A' iff P is not a branch point (P pt $u = \infty$, $v = -\infty$)

Use new coords

Kruskel coords

$$U = -e^{-u/4m}$$
$$V = e^{v/4m}$$



Result In these coords a) $U = -\left|\frac{r}{2m} - 1\right|^{1/2} e^{r/4m} e^{-t/4m}$

$V = \left|\frac{r}{2m} - 1\right|^{1/2} e^{r/4m} e^{t/4m}$ [assignment ?]

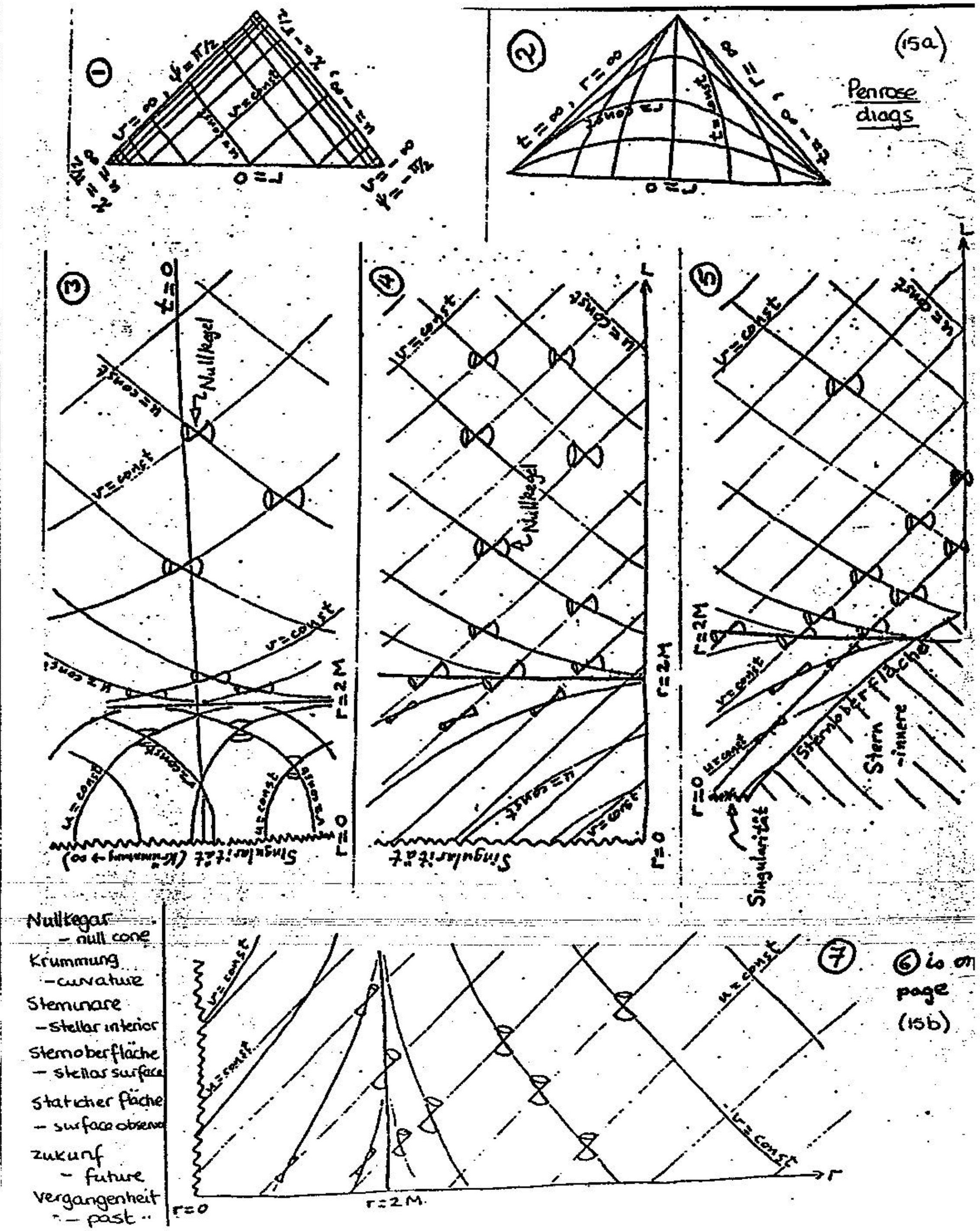
b) $ds^2 = \frac{-32m^3}{r} e^{-r/2m} dU dV + r^2 d\Omega^2$]

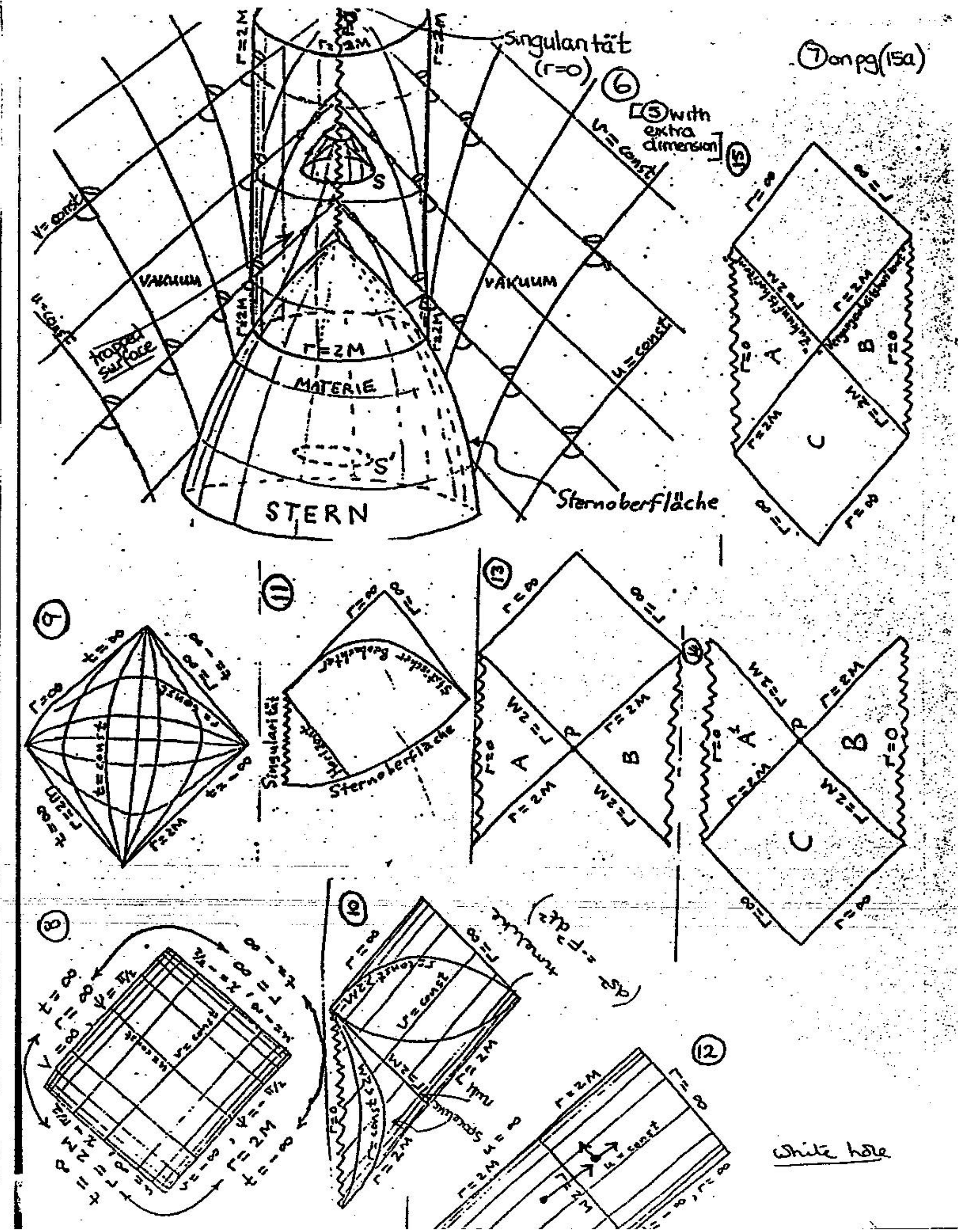
Clearly metric is analytic at P. P is not a branch point. [A' is the same as A].

Fig ⑮ is maximal analytic extension of schwarzschild.

(15a)

Penrose diags

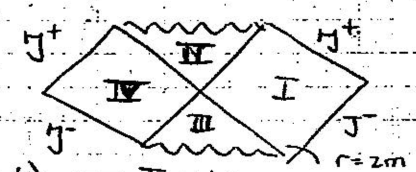




Notes

(1) Extension maximal – no geodesic can be continued (geod either runs into infinite region or singularity – cannot extend geods so we really do have whole ST)

(2) Certain coords only cover parts of ST (complete ST).



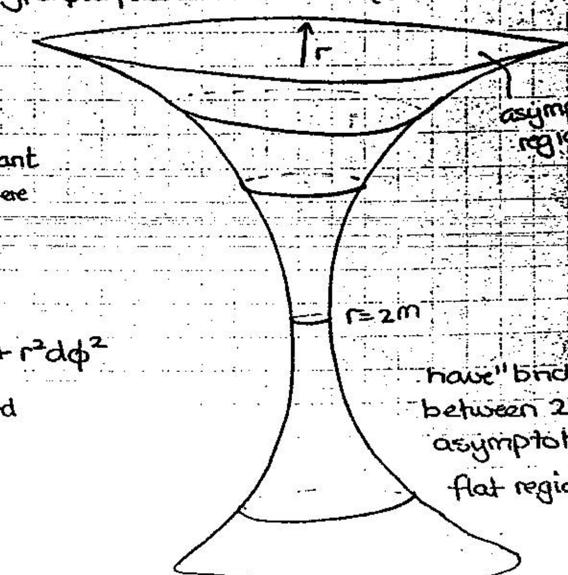
Penrose diag

(i) Schwartschild coords (t, r, θ, ϕ) — I only
(ii) (v, r, θ, ϕ) — I and II
(iii) (u, r, θ, ϕ) — I and III
(iv) Full extension — Kruskal coords (U, V, θ, ϕ) — all (sing. at r=0)

3 Have 2 asymptotic regions joined by a bridge (regions I and IV connected by area of very high curvature).

Consider spacelike hypersurface $\{t'=0$ — we are considering $t=0$ in I, we have to analytically continue $t=0$ into region IV – call this $t'=0\}$. Topology of this hypersurface is $R^1 \times S^2$ (t = const suppressed)

Visualize equation $\theta = \pi/2$ and embed this in $\mathbb{R}^3$ (wish to represent in diagram). Each circle of constant $\bar{r} = (\bar{x}^2+\bar{y}^2)^{1/2}$ replaced with sphere of surface area $4\pi\bar{r}^2$

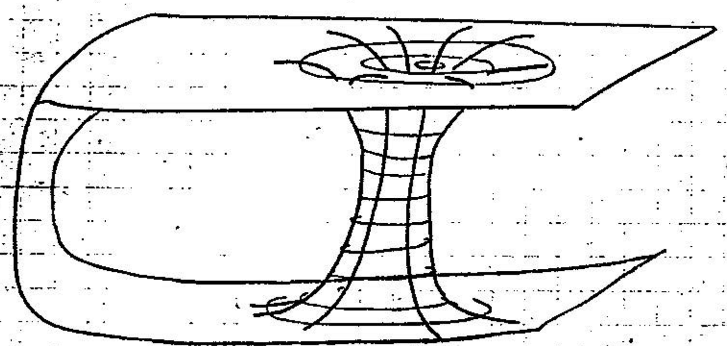


have "bridge" between 2 asymptotically flat regions.

Intrinsic geometry

$$ds^2 = \frac{16m^2}{r}\exp\left(-\frac{r}{2m}\right)dr'^2 + r^2 d\phi^2$$

(Kruskel coord r' – well behaved $-\infty < r' < \infty$).



MTW pg 837

Embedding of schwarzschild geom at "time" $t=0$, which is geometrically identical to that on pg (16) – but which is topologically different. Here schwarzschild "wormhole" connects two (distinct) regions of a single asymptotically flat universe

[Note EFE's fix geometry – they do not fix topology].

Particle motion in Schwarzschild ST

Lemma: Geodesic motion described by Lagrangian

$$\left(\dot{x}^a = \frac{dx^a}{d\tau}\right) \qquad L = \tfrac{1}{2} g_{ab}(x)\, \dot{x}^a \dot{x}^b \qquad (3).$$

Proof: Take L as above, Euler Lagrange eq^{n}s are geodesic eq^n $\left\{ \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \right\}$ Note: geodesic – minimizes distance.

Now apply to Schwarzschild ST $\quad (ds^2 = -F dt^2 + F^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2))$

$$L = \tfrac{1}{2}\left[-F\dot{t}^2 + \frac{\dot{r}^2}{F} + r^2\dot{\theta}^2 + r^2\sin^2\theta\, \dot{\phi}^2 \right]$$

$$\frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = r^2 \sin\theta \cos\theta\, \dot{\phi}^2$$

Thus "θ eq^n" $\Rightarrow \quad \frac{d}{d\tau}\left(r^2\dot{\theta}\right) = r^2 \sin\theta\cos\theta\, \dot{\phi}^2$

Choose θ, ϕ st $\theta = \pi/2$, $\dot{\theta} = 0$ initially $\Rightarrow \ddot{\theta} = 0 \Rightarrow \boxed{\theta = \pi/2}$

[That is, motion is confined to plane of constant θ, "choose this plane to be $\theta = \frac{\pi}{2}$"]

Other eq^{n}s $\quad \frac{\partial L}{\partial \dot{t}} = -F\dot{t}, \quad \frac{\partial L}{\partial t} = 0 \Rightarrow -F\dot{t} = $ const motion $\equiv E$

(same as KVF const) energy at infinity

$$\frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi}, \quad \frac{\partial L}{\partial \phi} = 0 \qquad r^2\dot{\phi} = \text{const motion}$$

$\equiv h$ (ang. mom per unit mass)

Consts of motion (first integrals) $\boxed{-F\dot{t} = E,\ r^2\dot{\phi} = h}$

Note a constant of motion $0 = g_{ab}\dot{x}^a\dot{x}^b - \mu$ (mag. 4 momentum; mass particle.)

[Rest mass remains constant]

$$\Rightarrow \quad g_{ab}\dot{x}^a\dot{x}^b = \mu = \begin{cases} -1 & \text{particles} \\ 0 & \text{photons.} \end{cases}$$

(for particle – choose units to "normalize"). (mass photon zero)

$$\Rightarrow \quad -F\dot{t}^2 + \frac{\dot{r}^2}{F} + r^2\dot{\phi}^2 = \mu$$

Using other constants of motion we can write this as

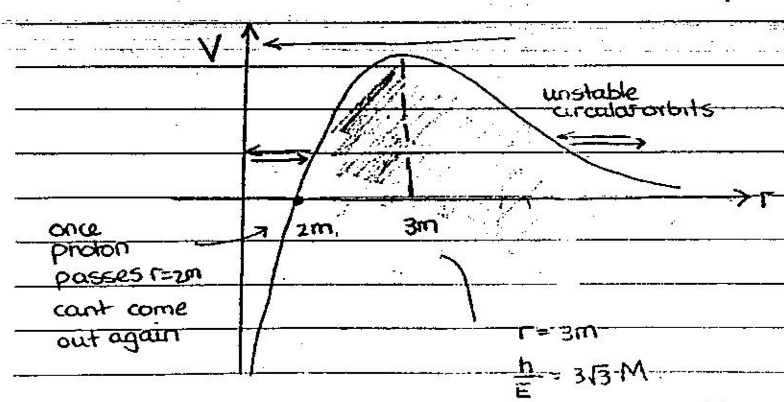
$$\boxed{\dot{r}^2 = E^2 + F\left(\mu - \frac{h^2}{r^2}\right)}$$

[Note that we do not need to consider "r eq^n" in Euler-Lagrange system since we have found a fourth constant of motion (typical technique in Lagrangian theory). Thus eq^n (above) is 4th indep eq^n we seek and will be equivalent to the first integral of the "r eq^n"].

Rewriting $\qquad \dot{r}^2 - F\left(\mu - \frac{h^2}{r^2}\right) = E^2 \qquad (4).$

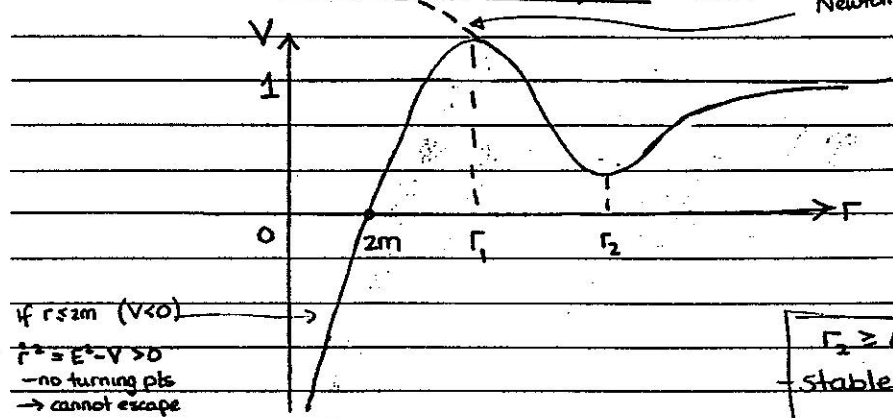
— motion in the potential $V(r) = -F(\mu - h^2/r^2)$.

case 1 : photons $\quad \mu = 0 \quad V(r) = \frac{Fh^2}{r^2}$



case 2 : particles $\quad \mu = -1$.

$V = F(1 + h^2/r^2)$.



[plot V as fn of r.]

r_1, r_2 depend on h, E for particular particle

$r_2 \geq 6m$
– stable circular orbits
$3m < r_1 \leq 4m$
– unstable circular orbits

Features

MTW pg 662 for details

(1) for large r – Keplerian orbits with perihelion shifts

(2) for small h/m particle pulled into black hole (nearly radial motion).

(3) closed orbits in well (depends on h, E for individual particle). Circular at V_{min} (stable).

Notes

Value of r where $V = E^2$(energy) called turning pt $\quad (\dot{r} = 0)$

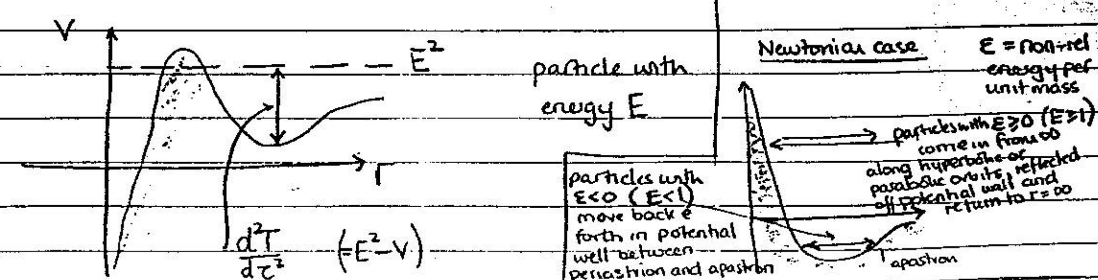
(particle moving to larger values of r, once arrived at turning pt, turns to move toward smaller r

" " " smaller " " ", ———, " " larger r).

[Note : turning pt. is not pt. equilibrium – (stone thrown up does not sit at pt. of equilibrium at top of its flight]. However, if $E^2 - V$ has double root (not single)

→ then one has pt equilibrium — called stable if it occurs at V_{min}

unstable " V_{max}



Perihelion shift Mercury

(for particle $\mu = -1$) eqⁿ (4) becomes

$$\mathring{r}^2 + F + \frac{Fh^2}{r^2} = E^2 \qquad (5).$$

Using $\quad \frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{dr}{d\phi}\dot{\phi} \Rightarrow$

$$\left(\frac{h}{r^2}\right)^2\left(\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} = E^2 - 1 + \frac{2m}{r}\left(1 + \frac{h^2}{r^2}\right)$$

define $u = 1/r$, then we obtain

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{E^2-1}{h^2} + \frac{2m}{h^2}u + 2mu^3 \qquad (6).$$

Diff wrt u (1st order nonlinear ODE $\Rightarrow$ 2nd order linear) – $u' \equiv \frac{du}{d\phi}$

$$u'' + u = \frac{m}{h^2} + 3mu^2$$

[For Mercury around sun $\quad \frac{3mu^2}{m/h^2} = \frac{3h^2}{r^2} \sim 7.7 \times 10^{-8}$]

(Put $u = u_0 + u_1 + \ldots$)

First approx $\qquad u_0'' + u_0 = m/h^2$

$$\Rightarrow \text{sol}^n \quad u_0 = \frac{m}{h^2}\left(1 + \underbrace{e}_{\text{eccentricity}}\cos\phi\right)$$

[Usual Keplerian elliptical orbit]

2nd Approx $\qquad u_1'' + u_1 = 3mu_0^2$

$$= \frac{3m^3}{h^4} + \frac{6m^3e\cos\phi}{h^4} + \frac{3m^3e^2(1+\cos2\phi)}{h^4}$$

Put $\quad u_1 = {}_1u + {}_2u + {}_3u$

Where $\quad {}_1u'' + {}_1u = \frac{3m^2}{h^4}$ —— Just makes small corrections to constants in u_0.

$${}_2u'' + {}_2u = \frac{3m^2e^2(1+\cos2\phi)}{h^4}$$ — usually $e \ll 1$ – small contribution

$${}_3u'' + {}_3u = \frac{6m^3e\cos\phi}{h^4}$$

Solⁿ ${}_3u \quad \rightarrow \quad {}_3u = \frac{3m^3}{h^4}e\phi\sin\phi \qquad$ (ignoring complementary function).

Thus

$$u \simeq \underbrace{\frac{m}{h^2}(1 + e\cos\phi)}_{u_0} + \underbrace{\frac{3m^3}{h^4}e\phi\sin\phi}_{\sim u_1}$$

(Taylor Series exp)

$$\simeq \frac{m}{h^2}\left[1 + e\cos[\phi - \delta\phi]\right]$$

$$\delta\phi = \frac{3m^2}{h^2}\phi$$

Thus: Change in $\delta\phi$ in one orbit is $(\phi = 2\pi)$

$$\Delta = \frac{6\pi m^2}{h^2} = \left[\frac{6\pi mG}{c^2a(1-e^2)}\right]$$

$(h^2 = Gma(1-e^2)$ — a semimajor axis).

Orbit of mercury not closed – perihelion is processing

[Effect not present in Newtonian Gravity].

Planet	Einstein prediction	observation
mercury	43·03	43·11 ± ·45
Venus	8·6	8·4 ± 4·8
Earth	3·8	5·0 ± 1·2

(secs arc / century)