The Mathematical Foundations of GR

The viewpoint that will be adopted here is that gravitation is essentially a continuum theory. The concept of continuity more or less demands that the arena for gravitation be a topological space. In order to save time it will implicitely be assumed that all topological spaces discussed here are connected & Hausdorff (i.e if p,q are two distinct points of the space, $\exists$ at least one pair of neighbourhoods of p,q respectively which do not intersect) Physical Laws are usually expressed in terms of differential equations, and so a differentiable structure must be introduced on the space, turning it into a differentiable manifold. Since Vectors and tensors are required they too must be defined. In order to define the derivative of a tensor, additional structure, an affine connection will be needed, and in order to define lengths etc a metric structure must also be given to the space These structures have been listed in order of increasing specialisation. In earlier accounts of the theory, a metric was introduced ab initio In an ideal introduction each mathematical concept would be introduced in parallel with the physical motivation requiring its use. However, for the sake of brevity all the mathematical concepts have been grouped together in chapter 1, and physical ideas are described in chapter.

(1.1) Differentiable Manifolds

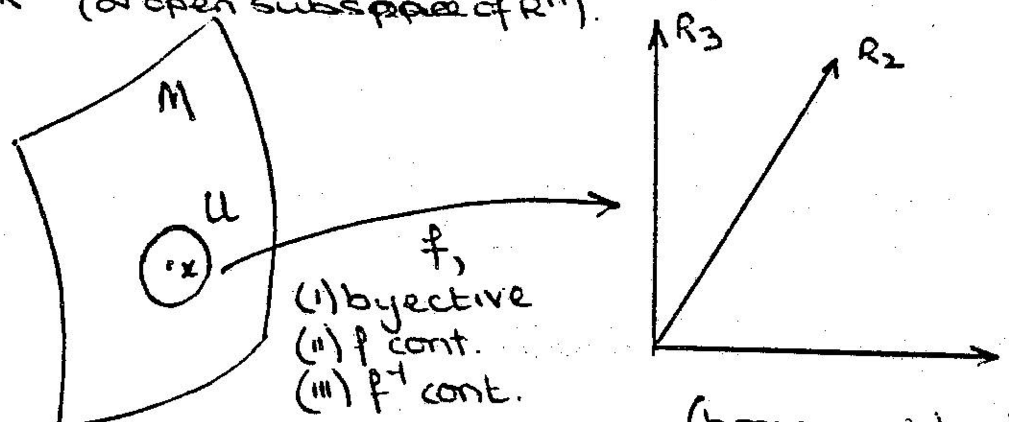
In order to Introduce the idea of differentiability the idea of coordinates are needed. One of the simplest topological spaces is R^n, which automatically comes a global coordinate system. This will not be true for all the spaces discussed here, but it is reasonable to expect that each point there is a coord system which is defined at least locally, and this motivates the Idea of a manifold. An n-dimensional manifold may not necessarily be a Euclidean space, but from the viewpoint of a short sighted observer living in space, locally looks just like such a domain of a

Euclidean space. A case in point is the 2-sphere S^2 (Earth). This cannot be considered a part of the Euclidean plane E^2, but our observer on S^2 sees that he can describe his immediate vicinity by 2 coords and so fails to distinguish between this and a small domain on E^2. S^2 has, obviously, an locally defined coord system attached to it. [S^2 is one of the simplest examples of a manifold].

We need to define a manifold precisely so that we can define functions, tensors and differential forms on such a space. The definition is motivated in this way. Each observer on the manifold has an immediate neighbourhood (local coord neighbourhood) described by n coords. Each point of the space must lie in at least one of these observed neighbourhoods.

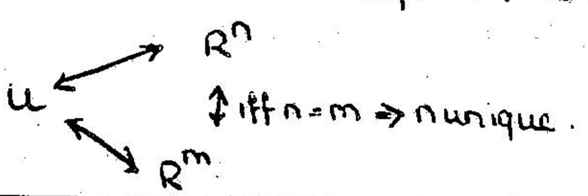
<u>Def^n 1.1.</u>

A <u>manifold M</u> is a topological space with the property that if $x \in M$, $\exists$ an open neighbourhood U of x in M, and an integer $n \geq 0$, such that U is homeomorphic [bijective, map and inverse continuous] to R^n (or open subspace of R^n).



(homeomorphic to R^3 say).

The simplest example of a manifold is R^n itself. The homeomorphism is defined by $R^n : x^i \to x^i$ and is the identity map. Anything homeomorphic to a manifold is itself a manifold. An open subset of a manifold is an <u>open manifold</u>. It should be noted that R^m is not homeomorphic to R^n (unless $m = n$) and so n in def^n 1.1 is unique. The only manifolds of interest are those in which n is the same at all points. The dimension of M is then n (n-Manifold)

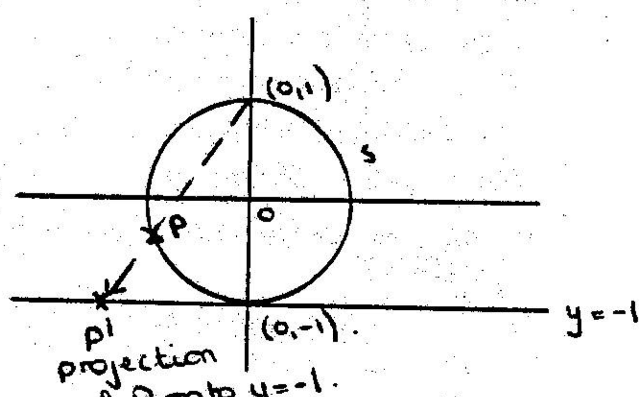


Example.

The simplest non trivial example of a manifold is the unit circle

$$S^1 = \{ (x,y) \in R^2 : x^2 + y^2 = 1 \}$$

The function $f : (0, 2\pi) \to S^1 ; \theta \to (\cos\theta, \sin\theta)$ is a homeomorphism, but it is not $(1,1)$ on $[0, 2\pi]$. Another homeomorphism can be found ~~from~~ by projecting from the pt. $(0,1)$ onto the line $y=-1$. This is homeomorphism of all points of S^1 except $(0,1)$ onto R^1. The point $(0,1)$ can be taken care of by projecting from $(0,-1)$ onto the line $y=1$.



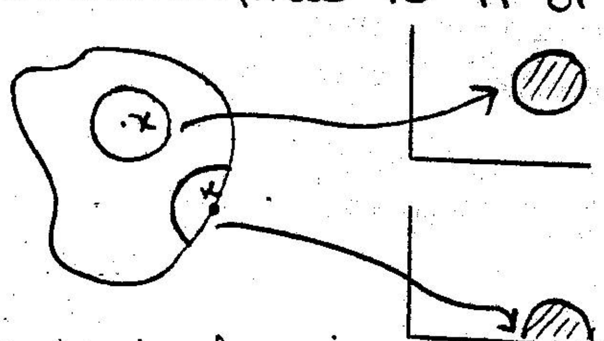
(i.e $\exists$ homeomorphism that maps a ngbhd of $(0,1)$ to the line $y=1$)

The closed half space H^n is defined by

$$H^n = \{ (x^1, \ldots, x^n) \in R^n : x^n \geq 0 \}.$$

Defn 1.1a.

An n-closed manifold (manifold with boundary) is a topological space with property that if $x \in M$, $\exists$ a nhbd U of x, (which can be closed) and an integer $n \geq 0$ so that U is homeomorphic to H^n or R^n. (or closed/open subspace of R^n)



[In this case U can be closed $\therefore$ U homeomorphic to either H^n or R^n (but not both)

$\updownarrow$ not homeomorphic | Several homeomorphs required since $\to R^n$, somewhat

The subset of points x, for which U is homeomorphic to H^n (and $x^n = 0$) is called the boundary ∂M of M.
[pts on boundary of M map to points on boundary of H^n)

A feature common to most manifolds is that there is no single homeomorphism mapping the whole manifold into R^n eg the example S^1 involved at least 2 homeomorphisms. This produces a slight complication when the concept of a differentiable function is introduced. Because M is a topological space a continuous map $f: M \to R$ is well defined, but is it differentiable? if U is a nhbd of pt $p \in M$, such that $\phi: U \to R^n$ is a homeomorphism, a provisional def^n might be that f is differentiable if $f \circ \phi^{-1}: R^n \to R$ is diffble. This definition clearly depends on the choice of ϕ [i.e choose $\psi: U \to R^n$ ψ a homeomorphism, (V nhbd of p) is this differentiable?].

$f: M \to R$
Is f differentiable?
$\phi: U \to R^n$
$\psi: V \to R^n$

$$\begin{tikzcd}
 & R^n \arrow[dr, "f \circ \psi^{-1}"] \arrow[dd, bend right=90, "\phi \circ \psi^{-1} \text{ on } U \cap V"'] & \\
\begin{matrix}(V)\\ M\\ (U)\end{matrix} \arrow[ur, "\psi"] \arrow[rr, "f"] \arrow[dr, "\phi"'] & & R \\
 & R^n \arrow[ur, "f \circ \phi^{-1}"'] &
\end{tikzcd}$$

Suppose V is another nhbd of p, and ψ a different homeomorphism of V to R^n. Then since $f \circ \psi^{-1} = f \circ \phi^{-1} \circ (\phi \circ \psi^{-1})$, $f \circ \psi^{-1}$ will be differentiable iff the map

$$\phi \circ \psi^{-1}: R^n \to R^n$$

is differentiable.

This motivates the follow def^{n}s. if U, V are subsets of M, 2 homeomorphism $\phi: U \to R^n$ $\psi: V \to R^n$ are <u>C^∞ related</u> if the maps

$$\phi \circ \psi^{-1}: \psi(U \cap V) \to \phi(U \cap V)$$
$$\psi \circ \phi^{-1}: \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞ (i.e differentiable an infinite no. of times)

<u>Def^n 1.2.</u>

A family of mutually C^∞ related homeomorphisms whose domains cover M is an <u>atlas A for M</u>. A particular member of an atlas (ϕ, U) is called a <u>chart</u> or <u>coordinate system on U</u>.

If A is an atlas, any map $\psi : M \to R^n$ which is C^∞ related to all members of A may be adjoined to A, and so a (unique) maximal atlas containing A can be constructed. It makes sense only to consider Maximal Atlases.

Defn 1.3

If (M, A) is a ~~Maximal At~~ Differentiable Manifold ~~and~~ A is a maximal Atlas for M.

(1.2) Vectors and Tensors.

Many important physical laws, eg Maxwells eqns are most conveniently expressed in terms of vectors and tensors. We therefore wish to define vectors and tensors precisely in the context of the structures we are establishing. Unfortunately the usual concept of a vector as an arrow joining two points needs to be revised. For example the surface of the Earth can be considered as a manifold S^2, and homeomorphisms onto R^2 are usually called map projections. Any comprehensive world atlas usually contains maps drawn using different projections. It can easily be seen that the 'vectors' joining say Paris to Peking pass through different areas according to the projection used (i.e 'vector' depends on projection used). The differences become smaller from 'Paris' to 'London' and it becomes clear that a vector can only be defined ~~locally~~ unambiguously as a local concept.

[It is however, to note that vectors in Euclidean space can be associated with differential operators i.e if $\underset{\sim}{v} = (a,b,c)$ at $P \in E^3$ then $\underset{\sim}{v}$ can be identified with

$$\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\right)\Big|_P$$

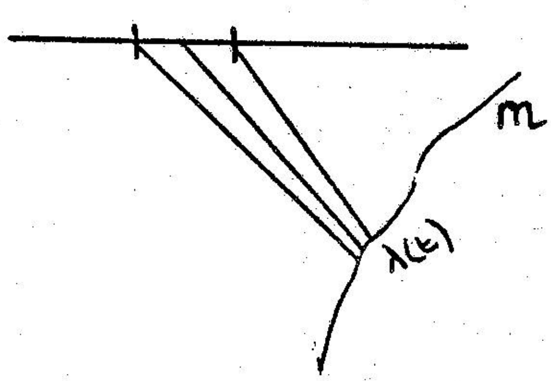
This is used in the defn of a vector]

In order to define direction the idea of a smooth curve is required and this leads to the following rather indirect definition of a vector.

$$\begin{tikzcd} & R^n & \\ m \arrow[ur, "\phi"] & & R \arrow[ll, "\lambda"] \arrow[ul, "\phi\circ\lambda"'] \end{tikzcd}$$

Differentiability has been defined for maps $f: m \to R$ in terms of the differentiability of $f \circ \phi^{-1}$ where ϕ is a chart.
Conversely, the map $\lambda: R \to m$ is differentiable (where defined) if $\phi \circ \lambda : R \to R^n$ is diff'ble (where defined) for all charts ϕ in atlas.

λ is a <u>smooth curve</u> if $\phi \circ \lambda$ is C^∞ for all ϕ in atlas. So a smooth curve λ is a smooth map from an open subset of R into m $\quad \lambda : R \to m \quad t \to \lambda(t)$.



Suppose P is a point $p = \lambda(t_0)$ on a smooth curve $\lambda(t)$, and let C be set of maps $f: m \to R$ which are differentiable at p. For each f in C the map $f \circ \lambda : R \to R$ is certainly diff'ble at $t = t_0$ and we have

<u>Defn 1.4</u> The <u>tangent vector to $\lambda(t)$ at p</u> is a map $(\partial/\partial t)_{\lambda, t_0} : C \to R$ given by

$$\left(\frac{\partial f}{\partial t}\right)_{\lambda, t_0} = \left(\frac{\partial}{\partial t}\right)_{\lambda, t_0} f = \left[\frac{df(\lambda(t))}{dt}\right]_{t = t_0}$$

It will be shown shortly that $\left(\frac{\partial}{\partial t}\right)_{\lambda, t_0}$ really is a vector in the algebraic sence (i.e that the tangent vector belongs to a vector space).

First, we show that $\left(\frac{\partial}{\partial t}\right)_{\lambda, t_0}$ is a vector ~~is~~ independant of curve chosen [i.e Its being a <u>vector</u> is independant of the curve λ]
Let $\phi : m \to R^n$; $\phi(y) = (x^1(y), \ldots, x^n(y))$ be a chart on a neighbourhood of p.
Let $x^i : y \to x^i(y)$ be projection map $m \to R$.
Then

$$f(\lambda(t)) = f \circ \lambda(t) = f \circ \phi^{-1} \circ \phi \circ \lambda(t)$$

[$\phi^{-1} \circ \phi$ is the unit map] $\quad$ where $f \circ \phi^{-1} : R^n \to R$
$\phi \circ \lambda : R \to R^n$

$$\begin{tikzcd} & \mathbb{R}^n \\ M \arrow[ur, "\phi"] \arrow[r, bend right, "f"'] & \mathbb{R} \arrow[u, "\phi\circ\lambda"'] \arrow[l, bend right, "\lambda"'] \end{tikzcd}$$

Using the chain rule

$$\left(\frac{\partial f}{\partial t}\right)_{\lambda, t_0} = \left[\frac{d f(\lambda(t))}{dt}\right]^{(*)}_{t=0} = \left[\underbrace{\frac{\partial (f\circ\phi^{-1})}{\partial x^i}}_{\text{Indep of } \lambda.} \frac{dx^i(\lambda(t))}{dt}\right]_{t=t_0}$$

Hence λ only enters via the second factor on RHS

Also $\left(\frac{\partial f}{\partial t}\right)_{\lambda_1, t_0} = \left(\frac{\partial f}{\partial t}\right)_{\lambda_2, t_0} \Leftrightarrow \frac{dx^i(\lambda_1(t))}{dt} = \frac{dx^i(\lambda_2(t))}{dt}$ at $t=t_0$

$\Leftrightarrow$ the curves λ_1, λ_2 are tangent at $t=t_0$

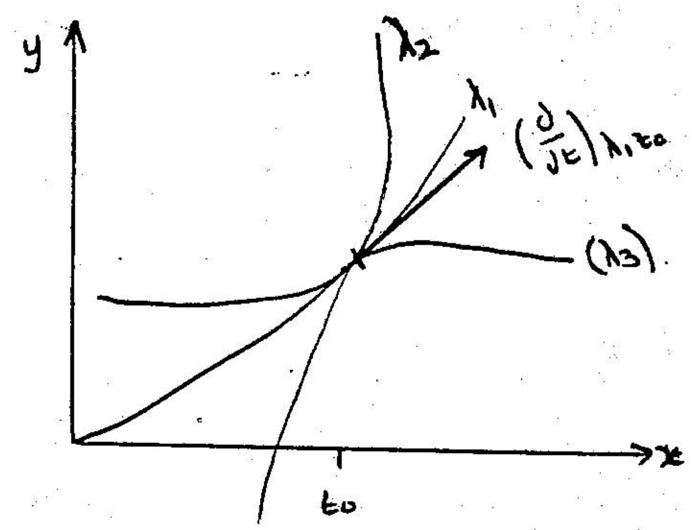
Hence two curves λ_1, λ_2 are <u>tangent</u> at $t=t_0$ if

$$\frac{dx^i(\lambda_1(t))}{dt} = \frac{dx^i(\lambda_2(t))}{dt} \quad \text{at } t=t_0$$

Example $M = \mathbb{R}^2$

$$\lambda_1(t) \to (t, at^2)$$
$$\lambda_2(t) \to \left(t, \tfrac{1}{2}\frac{at^4}{t_0^2} + \tfrac{1}{2}at_0^2\right)$$

Note $\{\lambda_1(t_0) = \lambda_2(t_0)\}$



$$\frac{dx^i(\lambda_1(t))}{dt}\Big|_{t=t_0} = (1, 2at_0)$$
$$\frac{dx^i}{dt}(\lambda_2(t))\Big|_{t=t_0} = (1, 2at_0)$$

So that λ_1 and λ_2 are tangent at $t=t_0$.

Further

$$\left(\frac{\partial f}{\partial t}\right)_{\lambda_1, t_0} = \frac{\partial f}{\partial x} + 2at_0\frac{\partial f}{\partial y}$$

$(*)$ $\left[\frac{d}{dt} f(\lambda(t))\right]_{t=t_0} = \Big|_{t_0} \frac{d}{dt}\left(f\circ\phi^{-1}\circ\phi\circ\lambda(t)\right)$

$$= \Big|_{t_0} \frac{d}{dt}\left(f\circ\phi^{-1}(\phi\circ\lambda(t))\right)$$
$$= \Big|_{t_0} \frac{d}{dt}\left(f\circ\phi^{-1}(\underbrace{x^1(\lambda(t)), \dots, x^n(\lambda(t))}_{\in\mathbb{R}^n})\right)$$

(by chain rule) $= \Big|_{t_0} \frac{\partial f\circ\phi^{-1}}{\partial x^i}\frac{dx^i(\lambda(t))}{dt}$

Proof that tangent vector is a vector.

We write $\frac{dx^i(\lambda(t))}{dt} = \left(\frac{dx^i}{dt}\right)_\lambda$ (Notation)

It is also convenient to identify each point p with its image under the chart ϕ, writing $[\phi(p) = (x^1, \dots x^n)]$

$$\left(\frac{\partial(f\circ\phi^{-1})}{\partial x^i}\right)_{\phi(p)} = \left(\frac{\partial f}{\partial x^i}\right)_p \quad \text{(Notation)}$$

Then $\left(\frac{\partial f}{\partial t}\right)_{\lambda, t_0} = \left(\frac{\partial f}{\partial x^i}\right)_{\lambda(t_0)} \left(\frac{dx^i}{dt}\right)_\lambda$ or

$$\underline{\left(\frac{\partial}{\partial t}\right)_{\lambda, t_0} = \left(\frac{dx^i}{dt}\right)_\lambda \left(\frac{\partial}{\partial x^i}\right)_{\lambda(t_0)}}$$

The operators $\left(\frac{\partial}{\partial x^i}\right)_{\lambda(t_0)} : C \to R$ are called the <u>coordinate derivatives at p.</u>

[in above example they were $\partial/\partial x$, $\partial/\partial y$]

<u>Remarks</u> (i) Every tangent vector is a linear combination of n coord. derivs

(ii) Conversely, every linear combination of n coord. derivs. is a tangent vector.

$X^i\left(\frac{\partial}{\partial x^i}\right)_{t_0}$ is the tangent vector to the curve

$$\phi^{-1}\{x^i(p) + (t-t_0)X^i\}$$

through p at $t = t_0$.

Write $X_p(f) = X^i\left(\frac{\partial}{\partial x^i}\right)_p f$ for all f in C (Notation).

Now, addition of curves in M has not been defined, but addition of the curves $x^i(p) + (t-t_0)X^i$, $x^i(p) + (t-t_0)Y^i$ is defined to be $x^i(p) + (t-t_0)(X^i+Y^i)$. This leads to the obvious definition of addition of tangent vectors at a point, if X_p and Y_p are tangent vectors at p, α, β real nos.

$$(\alpha X_p + \beta Y_p) f = \alpha X_p(f) + \beta Y_p(f).$$

for all f in C. Thus the set of all Tangent vectors at p forms a real vector space, called the <u>tangent space T_p at P over M</u>, abbreviated to T_p when M understood. [This is all that could be asked of a vector – i.e is belongs to a vector space and is thus a vector in the 'normal sence'. It has all the properties of a vector but in curved spacetime it is not intuitively obvious].

It should be noted that the coordinate derivs not only span T_p, but also form a basis for T_p. For if $Y_p = Y^i \left(\frac{\partial}{\partial x^i}\right)_p = 0$ then $Y_p(x^j) = Y^i \delta^i_j = Y^j = 0$. Thus T_p has dimension n. Finally, the set of all pairs $(p, T_p(M))$ is called the <u>Tangent Bundle $T(M)$ over M</u>.

[A simple example of these concepts is phase space for 1 particle Kinetic Theory of gases. At each point in space, the momentum space is defined to be the set of all ~~points and~~ possible momenta at that point. Thus is clearly an example of a tangent space. Phase space is the set of all points and their momentum space, i.e is hence a tangent bundle]

The tangent vectors at each point p in M form a vector space T_p. The standard results of linear algebra may be used to construct further useful vector spaces. The dual space to T_p, the <u>cotangent vector space T^*_p at p</u> is the linear space of linear maps $T_p \to R$

$$T^*_p = \{ \psi_p : \psi_p : T_p \to R \text{ is a linear map} \}$$

The image of $X_p \in T_p$ under ψ_p will be written $\psi_p(X_p)$. Elements of T_p^* are called <u>1-forms.</u>

Now let $f \in C$; the set of differentiable maps $M \to R$. For each X_p in T_p, $X_p(f)$ is a real number. Hence, fixing f it is clear that $X_p \to X_p(f)$ is a linear map: $T_p \to R$ and hence a member of T_p^*. Denote this map by df_p, the <u>differential of f at p</u>

$$df_p : T_p \to R$$

$$\text{by } df_p(X_p) = X_p(f)$$

Now, suppose that ϕ is a chart on U. It has already been shown that ϕ induces a basis for T_p, which was denoted by $(\partial/\partial x^i)_p$, the coord derivs at p. This basis induces a basis for T^*_p, the dual basis, denoted by $dx^i{}_p$; according to definitions,

$$dx^i{}_p\left(\frac{\partial}{\partial x^j}\right)_p = \left(\frac{\partial}{\partial x^j}\left(x^i\right)\right)_p = \delta^i{}_j .$$

It is instructive to calculate the components of df_p with respect to this basis. Setting

$$df_p = f_i\, dx^i{}_p \qquad X_p = X^i\left(\frac{\partial}{\partial x^i}\right)_p$$

$$df_p(X_p) = f_i\, dx^i{}_p\left(X^j\left(\frac{\partial}{\partial x^j}\right)_p\right)$$

$$= f_i X^j\, dx^i{}_p\left(\frac{\partial}{\partial x^j}\right)_p$$

$$= f_i X^j \delta^j{}_i = f_i X^i$$

$$= X_p(f) = X^i\left(\frac{\partial f}{\partial x^i}\right)_p$$

[Note: upstairs indices $x^i \Rightarrow$ vectors; downstairs indices $f_i \Rightarrow$ 1-forms; can only sum if 1 index updowns, t'other upstairs]

This holds for all X^i and so $f_i = \left(\frac{\partial f}{\partial x^i}\right)_p$ i.e. the differential of f at p is nothing more than the gradient of f at p, or alternatively the gradient of f is a 1-form [i.e. df has same components as ∇f] Thus a 1-form can be thought of as a (local) hypersurface element.

These results apparently depend on the chart or coord system used to define bases in T_p, T^*_p. Suppose chart is changed, i.e. let ϕ, $\bar{\phi}$ be 2 charts on U with projection maps $x^i, \bar{x}^i : U \to R$ resp. Since $\bar{x}^i$ is a differentiable map : $U \to R$, setting $f = \bar{x}^i$ gives

$$d\bar{x}^i{}_p = \left(\frac{\partial \bar{x}^i}{\partial x^j}\right)_p dx^j{}_p \quad \text{— contravariant Law.}$$

Now $d\bar{x}^i{}_p$ clearly basis element for T^*_p induced by $\bar{\phi}$, hence above gives rule for change of ~~b~~ coord-induced bases on T^*_p.

The rule for change of basis on T_p now follows as

$$\left(\frac{\partial}{\partial \bar{x}^j}\right)_p = \left(\frac{\partial x^i}{\partial \bar{x}^j}\right)_p\left(\frac{\partial}{\partial x^i}\right)_p \quad \text{covariant law}$$

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[Since $\delta^i_j = d\bar{x}^i_p \left(\frac{\partial}{\partial \bar{x}^j}\right)_p = d\bar{x}^i_p \left(\frac{\partial x^i}{\partial \bar{x}^j}\right)_p \left(\frac{\partial}{\partial x^i}\right)_p$

$$= \left(\frac{\partial x^i}{\partial \bar{x}^j}\right)_p \left(\frac{\partial \bar{x}^i}{\partial x^i}\right)_p = \left(\frac{\partial \bar{x}^i}{\partial \bar{x}^j}\right)_p = \delta^i_j \text{]}$$

The rules for transforming components of vectors and 1-forms are then seen to be

Let $\quad X_p = \bar{X}^i \left(\frac{\partial}{\partial \bar{x}^i}\right)_p = X^j \left(\frac{\partial}{\partial x^j}\right)_p = X^j \left(\frac{\partial \bar{x}^i}{\partial x^j}\right)_p \left(\frac{\partial}{\partial \bar{x}^i}\right)_p$

$$\therefore \quad \bar{X}^i = \left(\frac{\partial \bar{x}^i}{\partial x^j}\right)_p X^j$$

$$y_p = y_j \, dx_p^j = \bar{y}_i \, d\bar{x}_p^i = y_j \left(\frac{\partial x^j}{\partial \bar{x}^i}\right)_p d\bar{x}_p^i$$

$$\therefore \quad \bar{y}_i = \left(\frac{\partial x^j}{\partial \bar{x}^i}\right)_p y_j$$

A warning on terminonology. Quantities which transform like $(\partial/\partial x^i)_p$ are called components of ~~contra~~covariant vectors while quantities like dx^i_p are called components of contravariant vectors.
Notice that

X^i, dx^i transform contravariantly

(components of vectors) (basis elements of 1-form)

while y_i, $\left(\frac{\partial}{\partial x^i}\right)$ transform covariantly

(comps of 1-forms) (basis of vectors.)

If the determinant of $\left(\frac{\partial \bar{x}^i}{\partial x^j}\right)_p$ is positive for all $p, \bar{x}^i, x^i$ then M is <u>orientable</u>.
[R^n is orientable, a möbius strip is not].

The transformations rules can be generalised. Although coordinduced bases for T_p, T_p^* will often be used there also exist bases which are not induced by coord system. Let $(e_1, \dots e_n)$ be any basis ~~and~~ for T_p and $(\omega^1, \dots \omega^n)$ dual basis for T_p^* ($\omega^j(e_i) = \delta^j_i$). Under change of basis

$\bar{\omega}^i = A^i{}_j \omega^j$; $e_j = (A^{-1})^i{}_j e_i$ components change according to $(X = X^i e_i)$ $\quad \bar{X}^i = A^i{}_j X^j$, $\quad \bar{y}_j = (A^{-1})^i{}_j y_i$

<u>Tensors</u>. It is not difficult to define more general tensors (in rough notes (*) gives case of (2,1) tensor).
Let

$\Pi_p{}^s{}_r$ be the cartesian product

$$\Pi_p{}^s{}_r = \underbrace{T_p^* \times \dots \times T_p^*}_{r \text{ factors}} \times \underbrace{T_p \times \dots \times T_p}_{s \text{ factors.}}$$

Thus an element of $\Pi_p{}^s{}_r$ is an ordered set of ~~s~~ r forms and s vectors.

<u>Def^n 1.5.</u> A <u>tensor T of type (r,s) at p</u> is a map: $\Pi_p{}^s{}_r \to R$ which is linear in each argument (i.e multilinear).

The Image of $(\psi^1, \dots, \psi^r, X_1, \dots, X_s)$ is written $T(\psi^1, \dots \psi^r, X_1, \dots X_s)$
[i.e multilinearity $\Rightarrow$ $T(\psi^1, X_1 + X_1', X_2) = T(\psi^1, X_1, X_2) + T(\psi^1, X_1', X_2)$]

The multilinearity ~~of~~ implies that addition of tensors of the same type and multiplication by a scalar can be defined in the obvious way. e.g. if S, T are both of type (1,1), α scalar, then $(S+T)$, αT are also of type (1,1) ~~and~~ where

$$\begin{aligned} (S+T)(\psi, X) &= S(\psi, X) + T(\psi, X) \\ (\alpha S)(\psi, X) &= \alpha\,(S(\psi, X)) \end{aligned} \quad \forall \; \begin{matrix} \psi \in T_p^* \\ X \in T_p \end{matrix}$$

Thus space of all (r,s) tensors at p is a real vector space denoted by

$$T_p{}^r{}_s = \underbrace{T_p \otimes \dots \otimes T_p}_{r \text{ factors}} \otimes \underbrace{T_p^* \otimes \dots \otimes T_p^*}_{s \text{ factors}}$$

(note change in order).
Suppose that $Y_i \in T_p$, $\tau^i \in T^*_p$. Let $Y_1 \otimes \dots \otimes Y_r \otimes \tau^1 \otimes \dots \otimes \tau^s$ denote the element of $T_p{}^r{}_s$ whose effect on $(\psi^1, \dots \psi^r, X_1, \dots X_s)$ is given by

$$\begin{aligned} Y_1 \otimes \dots \otimes Y_r \otimes \tau^1 \otimes \dots \otimes \tau^s & (\psi^1, \dots \psi^r, X_1, \dots X_s) \\ &= \psi^1(Y_1) \dots \psi^r(Y_r)\, \tau^1(X_1) \dots \tau^s(X_s) \end{aligned}$$

$\otimes$ is called <u>the tensor (direct) product</u>.
By an obvious extension if $T \in T_p{}^s{}_r$, $S \in T_p{}^t{}_q$

$T \otimes S$ is an element of $T_p{}^{s+t}{}_{r+q}$ defined by –

$$(T\otimes S)(\psi^1,\dots,\psi^r,\tau^1,\dots\tau^q,X_1\dots X_s,Y_1,\dots Y_t)$$
$$= T(\psi^1\dots\psi^r,X_1\dots X_s)\,S(\tau^1,\dots\tau^q,Y_1,\dots Y_t).$$

With this product the tensor spaces at p form form an algebra. Note that
$$T_p{}^1{}_0 = T_p^* \;;\; T_p{}^0{}_1 = T_p.$$

Now let $(e_i),(\omega^i)$ be dual bases of T_p, T_p^*. It is clear that $e_1\otimes\dots\otimes e_r\otimes\omega^1\otimes\dots\otimes\omega^s$ generates a basis for $T_p{}^r{}_s$. An arbitary element can be expanded as
$$T = T^{i_1\dots i_r}{}_{j_1\dots j_s}\, e_1\otimes\dots e_r\otimes\omega^1\otimes\dots\otimes\omega^s.$$
and

$T^{i_1\dots i_r}{}_{j_1\dots j_s}$ are <u>components of T</u> wrt basis induced by $(e_i),(\omega^i)$. These can also be defined by
$$\underline{T^{i_1\dots i_r}{}_{j_1\dots j_s} = T(\omega^{i_1}\dots\omega^{i_r},e_{j_1},\dots e_{j_s})}$$
[Ex 1.3 and (*) through]

Given an (r,s) tensor in component form $T^{i_1\dots i_r}{}_{j_1\dots j_s}$, the <u>contraction</u> of the pth vector index with the qth form index is defined to be the $(r-1,s-1)$ tensor with components $T^{i_1\dots i_{p-1}\,k\,i_{p+1}\dots i_r}{}_{j_1\dots j_{q-1}\,k\,j_{q+1}\dots j_s}$

[cannot contract over 2 contravariant | covariant indices as this is dependant on the basis]
We show that a contraction produces a $(r-1,s-1)$ tensor. Contracting over ~~1st~~ i_1 and j_1.
$$T^{k i_2\dots i_r}{}_{k j_2\dots j_s} = T(\omega^k,\omega^{i_2},\dots\omega^{i_r},e_k,e_{j_2},\dots)$$
$$= T^{k k_2\dots k_r}{}_{k l_2\dots l_s}(e_k\otimes e_{k_2}\otimes\dots\otimes\omega^k\otimes\omega^{l_2}\dots)(\omega^k,\omega^{i_2},\dots,e_k e_{j_2},\dots)$$
$$= T^{k k_2\dots k_r}{}_{k l_2\dots l_s}\Big\{\underbrace{\omega^k(e_k)}_{=1}\,\omega^{i_2}(e_{k_2})\dots\underbrace{\omega^k(e_k)}_{=1}\,\omega^{j_2}(e_{l_2})\dots\Big\}$$
$$= T^{k k_2\dots k_r}{}_{k l_2\dots l_s}\, e_{k_2}\otimes\dots\otimes e_{k_r}\otimes\omega^{l_2}\dots\otimes\omega^{l_s}(\omega_{i_2},\dots\omega_{i_r},e_{j_2},\dots)$$
This is obviously components of an $(r-1,s-1)$ tensor.

The symmetrisation and antisymmetrisation operators acting on ~~a~~ $(r,0)$ tensors are defined by

$$\tilde{S}\,T(y^1,\dots,y^r) = \frac{1}{r!}\sum_{\pi} T(y^{\pi(1)},\dots,y^{\pi(r)})$$

sum over all permutations

$$\tilde{A}\,T(y^1,\dots,y^r) = \frac{1}{r!}\sum_{\pi} \mathrm{Sign}(\pi)\, T(y^{\pi(1)},\dots,y^{\pi(r)})$$

In component form these relations are: (see ex 1.4)

$$T^{(12)} = \tilde{S}T^{12} = \tfrac{1}{2}\left(T^{12}+T^{21}\right)$$

$$T^{[123]} = \tilde{A}T^{123} = \tfrac{1}{6}\left(T^{123}+T^{231}+T^{312}-T^{321}-T^{213}-T^{132}\right)$$

Similar operations are of course defined for subsets of indices, for $(0,s)$ tensors and of the same type in mixed tensors. e.g.

$$T^{(12)34} = \tfrac{1}{2}\left(T^{1234}+T^{2134}\right)$$

$$T^{(12)[34]} = \tfrac{1}{4}\left(T^{1234}-T^{1243}+T^{2134}-T^{2143}\right)$$

$$T^{((12)3)} = T^{(123)}$$

$$T^{((12)(34))} = \tfrac{1}{2}\left(T^{(12)(34)}+T^{(34)(12)}\right) \neq T^{(1234)}$$

If $T = \tilde{S}T$ (i.e. $T^{ijk} = T^{(ijk)}$) ~~or~~ then T is <u>totally symmetric</u>. If $T = \tilde{A}T$ (i.e. $T^{ij} = T^{[ij]}$) then T is <u>totally antisymmetric</u> or <u>skew</u>.

Any totally skew $(0,s)$ tensor is called a s-form which is consistant with the def^n of a 1-form. By convention fns $M \to R$ are 0-forms.

A 2-form can be constructed from a $(0,2)$ tensor $A_{ij}\,\omega^i \otimes \omega^j$ as

$$\begin{aligned} A &= \tilde{A}\left(A_{ij}\,\omega^i\otimes\omega^j\right) \\ &= \tfrac{1}{2}\left(A_{ij}-A_{ji}\right)\omega^i\otimes\omega^j \\ &= A_{ij}\,\tfrac{1}{2}\left(\omega^i\otimes\omega^j-\omega^j\otimes\omega^i\right) \\ &= A_{ij}\,\omega^i\wedge\omega^j \\ &= A_{[ij]}\,\omega^i\wedge\omega^j \quad \text{- Ex 1.4.} \end{aligned}$$

where $\wedge$ the <u>exterior product</u> of 2 forms (not nec 1 forms) is defined by

$$A\wedge B = \tilde{A}(A\otimes B).$$

Similarly a p form can be written $A = A_{[i_1\dots i_p]}\,\omega^{i_1}\wedge\dots\wedge\omega^{i_p}$

(1.3) Maps of Manifolds.

It will often be necessary to consider maps between 2 manifolds M, N of dim m, n resp. If ϕ, ψ denote charts from the corresponding atlases then the map $h : M \to N$ will be said to be differentiable if the map $\psi \circ h \circ \phi^{-1}$ ($: R^m \to R^n$) is differentiable. Note that h will not have an inverse in general, and even if it does h^{-1} need not be differentiable. Let

C_M be set of differential fns of M
C_N be set of diffble fns on N

$$\begin{tikzcd}
R^m \arrow[rr, "\psi_\circ h_\circ \phi^{-1}"] & & R^n \\
M \arrow[u, "\phi"] \arrow[rr, "h"] \arrow[dr, "h^* f"'] & & N \arrow[u, "\psi"'] \arrow[dl, "f"] \\
& R &
\end{tikzcd}$$

Suppose $f \in C_N$
If h is diffble then $f \circ h$ is a diffble map $: M \to R$
$\therefore f \circ h \in C_M$

$\therefore$ $h : M \to N$ induces a natural map
$h^* : C_N \to C_M$ given by

$$h^* f(p) = f(h(p)) \quad \forall p \in M, \ f \in C_N$$

[Example. $M = R^3 \quad N = R^2$
$h : (x,y,z) \to (x,y)$ by $h(x,y,z) = (x,y)$.
Let $f \in C_N$ i.e $f(x,y)$ exists.

Then $h^* f : C_N \to C_M$ by $h^* f\ (x,y,z)$
$= f(h(x,y,z)) = f(x,y)$]

There exists a second map induced by h: Suppose $\lambda(t)$ is a smooth curve through p in M. The tangent vector at p is $(\partial/\partial t)_{\lambda, p}$. The image of $\lambda(t)$ under h will be a smooth curve through $h(p)$ which will have a tangent vector also.

$$\begin{tikzcd}
T_p(M) \arrow[rr, "h_*"] \arrow[dr, "X h^* f"'] & & T_{h(p)}(N) \arrow[dl, "h_* X"] \\
& R &
\end{tikzcd}$$

Clearly h induces a linear map $h_* : T_p(M) \to T_{h(p)}(N)$ which can be characterised by the relation

$$\underset{\in T_p(M)}{X}\underset{\in M}{{}_p}(\underset{\in C_M}{h^* f}) = \underset{\in T_{h(p)}(N)}{(h_* X)}\underset{\in N}{{}_{h(p)}}(\underset{\in C_N}{f}) \qquad \forall f \text{ in } C_N, \text{ all } X_p \text{ in } T_p(M)$$

This process is now be repeated. The contraction of a vector with a one-form is a scalar which will be required to be invariant under h. Thus a map $h_*: T_p(M) \to T_{h(p)}(N)$ induces a map $T^*_{h(p)}(N) \to T^*_p(M)$ (note reversed order) which is denoted by h^*, such that for any $X \in T_p(M)$ $\psi \in T^*_{h(p)}(N)$

$$\underbrace{h^*(\psi)}_{\in T^*_p(M)}(\underset{\in T_p(M)}{X}) = \underset{\in T^*_{h(p)}(N)}{\psi}(\underset{\in T_{h(p)}(N)}{h_*(X)}).$$

At first sight this multiplicity of linear maps may seem perplexing. However it is permissable to identify a real fn with a 0-form. Then if

h is a map from M to N $\quad h: M \to N$
h^* is map of forms or contravariant tensors from $N \to M$
h_* is map of vectors and contravariant tensors from $M \to N$

Example: Let M be R^3, N be R^2 and $h: R^3 \to R^2$ by $h(x,y,z) = (x,y)$
clearly h has no inverse. Let $f \in C_N$.

Then $h^* f$ is differentiable fn on M defined by

$$h^* f(x,y,z) = f(h(x,y,z)) = f(x,y) \qquad h^* f \in C_M$$

If X is vector at the point (x,y,z) with components (X,Y,Z) then

$$\underset{\in T(M)}{X}(\underset{\in C_M}{h^* f}) = \left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right) f(x,y)$$
$$= X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} = h_* X(f)$$

So that $h_* X$ is a vector at the point (x,y) with components (X,Y).

Conversely, if A is the 1 form $A\,dx + B\,dy$ at (x,y) $h^* A$ is the form $A\,dx + B\,dy + 0\,dz$ at (x,y,z)

Now $h_* X = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}$

$$\therefore \psi(h_* X) = (A\,dx + B\,dy)\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}\right) = AX + BY = (h^*\psi)(X)$$
$$= (A\,dX + B\,dY + 0\,dz)\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right) \quad \forall (x,y,z)$$

$\therefore h_*(A\,dx + B\,dy) = A\,dx + B\,dy + 0\,dz$

(1.4). Exterior and Lie derivatives:

The operations on vectors and tensors which have already been defined have involved only vectors and tensors defined at a point. Of course it is very easy to define a tensor field over a manifold M of type (r,s) as a mapping which associates to each point p of M an element of $T_p{}^s{}_r$. While such a def^n is very simple it gives no clue as to how one could compare tensors defined at different points, an essential prerequisite of defining derivatives of tensors. Indeed, additional structure must be added to the manifold to make this possible, and this occupy some of later sections. There exist, however, 2 related derivatives which do not involve this additional structure, and these are discussed now.

In the last section it was shown that the concept of a 1-form could be extended to a p-form using the exterior product. There is a certain derivative, the exterior derivative d which maps p-forms to p+1 forms. Indeed an example of this operation has already be shown. A smooth map: $f: M \to R$ is a 0-form and (in normal coords bases)

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \quad \text{is a 1-form.}$$

Assume (dx^i) is a coord induced basis for T^*_p.

Def^n (1.6). The exterior product maps p-forms at $p \in M$ into p+1 forms according to the rules

(i) If f is a 0-form $\quad df = f_{,i}\,dx^i \quad f_{,i} = \left(\frac{\partial f}{\partial x_i}\right)_p$

(ii) If ω is r-form, ψ a p-form then

$$d(\omega \wedge \psi) = d\omega \wedge \psi + (-1)^r \omega \wedge d\psi$$

(iii) $\quad d(d\omega) = 0$

(iv) $\quad d(\omega + \psi) = d\omega + d\psi$.

This rather oblique defn does not make it clear that the rules define $d\omega$ (ω an r-form), further that it is a $(r+1)$ form and that it is independant of the coord basis used in the defn. It has already been shown that df is coord invariant. Set $f = x^i$ so that the 1-form dx^i satisfies $d(dx^i) = 0$ by (iii)

Suppose ω is a 1 form : $\omega = \underset{\text{0 form}}{\omega_i}\ \underset{\text{1-form}}{dx^i}$

$$\therefore\ d\omega = d(\omega_i dx^i) = d\omega_i \wedge dx^i + (-1)^0 \cancel{\omega \wedge d(dx^i)}^{\,0}$$

since $d(dx^i) = 0$

$$= d\omega_i \wedge dx^i$$

$$= \omega_{i,j}\, dx^j \wedge dx^i \qquad \text{since } \omega_i \text{ is 0-form using (i)}$$

$$= \omega_{[i,j]}\, dx^j \wedge dx^i \qquad \text{– since antisymmetric in } i, j.$$

$\therefore$ $d\omega$ is a 2-form.

We must show coord independance. Let $d\bar{x}^i$ be a different coord basis, so that

$$\omega = \bar{\omega}_i\, d\bar{x}^i \quad \text{where } \bar{\omega}_i = \left(\frac{\partial x^k}{\partial \bar{x}^i}\right)\omega_k$$

– contravariant law.

We must show that $d\omega = \bar{\omega}_{[i,j]}\, d\bar{x}^j \wedge d\bar{x}^i$
$= \omega_{[i,j]}\, dx^j \wedge dx^i$.

Now $\bar{\omega}_{[i,j]} = \dfrac{\partial \bar{\omega}_i}{\partial \bar{x}^j} - \dfrac{\partial \bar{\omega}_j}{\partial \bar{x}^i}$ and $\dfrac{\partial}{\partial \bar{x}^j}(\bar{\omega}_i) = \dfrac{\partial}{\partial \bar{x}^j}\left(\dfrac{\partial x^k}{\partial \bar{x}^i}\omega_k\right)$

$$= \frac{\partial^2 x^k}{\partial \bar{x}^j \partial \bar{x}^i}\omega_k + \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial \omega_k}{\partial \bar{x}^j}$$

$$\therefore\ \bar{\omega}_{[i,j]} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial \omega_k}{\partial \bar{x}^j} + \cancel{\frac{\partial^2 x^k}{\partial \bar{x}^j \partial \bar{x}^i}\omega_k}$$
$$- \frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial \omega_k}{\partial \bar{x}^i} - \cancel{\frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^j}\omega_k}$$

$$= \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial \omega_k}{\partial \bar{x}^j} - \frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial \omega_k}{\partial \bar{x}^i}$$

Now, using contravariant law of trans of coord basis

$$d\bar{x}^j = \left(\frac{\partial \bar{x}^j}{\partial x^i}\right) dx^i.$$

$\therefore$ $\bar{\omega}_{[i,j]}\, d\bar{x}^j \wedge d\bar{x}^i$

$$= \left(\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial \omega_k}{\partial \bar{x}^j} - \frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial \omega_k}{\partial \bar{x}^i}\right)\left\{\frac{\partial \bar{x}^j}{\partial x^m}\frac{\partial \bar{x}^i}{\partial x^n}\right\} dx^m \wedge dx^n$$

$$= \left\{\left(\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial x^n}\right)\left(\frac{\partial \omega_k}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^m}\right) - \left(\frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^m}\right)\left(\frac{\partial \omega_k}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial x^n}\right)\right\} dx^m \wedge dx^n$$

By chain rule

$$= \left\{\frac{\partial x^k}{\partial x^n}\frac{\partial \omega_k}{\partial x^m} - \frac{\partial x^k}{\partial x^m}\frac{\partial \omega_k}{\partial x^n}\right\} dx^m \wedge dx^n$$

$$= \left\{\delta_{kn}\frac{\partial \omega_k}{\partial x^m} - \delta_{km}\frac{\partial \omega_k}{\partial x^n}\right\} dx^m \wedge dx^n$$

$$= \left(\frac{\partial \omega_n}{\partial x^m} - \frac{\partial \omega_m}{\partial x^n}\right) dx^m \wedge dx^n$$

$$= \omega_{[n,m]}\, dx^m \wedge dx^n.$$

Thus $d\omega$ is coord invariant

Note: A vector field is a rule which gives an element of T_p, for each p in M. i.e map: $p \to T_p$ $p \in M$.

Now suppose that f is diff'ble fn on M $f: M \to R$, and X, Y are two vector fields over M. For each p, M $X_p f$ is a real no, and so the map $p \to X_p f$ is for fixed f another diff'ble fn on $M: M \to R$, and is denoted by Xf

$$Xf(p) = X_p(f).$$

<u>Defn 1.7.</u> <u>The commutator</u> of 2 vector fields X, Y over M is the vector field $[X, Y]$ defined by

$$[X,Y]_p f = \underset{X_p(\,)}{X_p}\underset{\substack{Yf: M\to R \\ \text{real no.}}}{(Yf)} - Y_p(Xf). \quad \text{for all fns } f$$

This is again an oblique defn. We will show that $[X, Y]_p$ is an element of T_p.

Suppose X, Y have components X^i, Y^i with respect to coord induced basis $(\partial/\partial x_i)_p$ of T_p

Then $\quad X = X^i \frac{\partial}{\partial x^i} \qquad Y = Y^j \frac{\partial}{\partial x^j}$

Then

$$Yf = Y^j \frac{\partial f}{\partial x^j}$$

and $$X_p(Yf) = X^i \frac{\partial}{\partial x^i}\left[Y^j\left(\frac{\partial f}{\partial x^j}\right)\right]_p$$
$$= X^i Y^j{}_{,i} \frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j}$$

Similarly $$Y_p(Xf) = Y^j X^i{}_{,j} \frac{\partial f}{\partial x^i} + X^i Y^j \frac{\partial^2 f}{\partial x^j \partial x^i}$$

Hence;

$$[X,Y]_p f = \left(X^i Y^j{}_{,j} - Y^i X^j{}_{,i}\right)\left(\frac{\partial f}{\partial x^j}\right)_p$$

So that $[X,Y]_p$ is indeed a tangent vector at p with components $(X^i Y^j{}_{,i} - Y^i X^j{}_{,i})$.

Now suppose X is any vector field over M whose ~~coord~~ components with respect to charts are diff'ble fns on M. Then through any point of M it is possible to construct a smooth curve whose tangent vector everywhere is X. To see this let (x^i) be a local coord system in some neighbourhood of p. Let X_q have components $X^i(q)$ wrt coord basis of T_q, where q is in some ngbd of p. Now consider set of ordinary differential eq^{n}s

$$\frac{dx^i}{dt} = X^i(x^1(t), \ldots, x^n(t))$$

By the standard existance and uniqueness theorems there exist precisely one solution of this system such that $x^i(0) = x^i(p)$. The solution obviously defines a smooth curve $\lambda(t)$ whose tangent vector at any point q on the curve is X_q. This curve is called the <u>integral curve of X through p</u>.

Note that its existance has only been demonstrated in a nhbd U of p.

There exists an $\varepsilon > 0$ and a family of diffeomorphisms $\phi_t : U \to M$ (diff'ble maps) defined by X as follows.

If $q \in U$ and $|t| < \varepsilon$; $\phi_t(q)$ is the point reached by moving a parameter distance t along the

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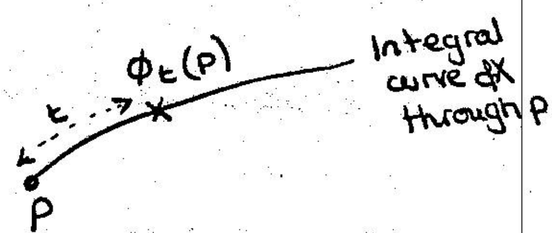
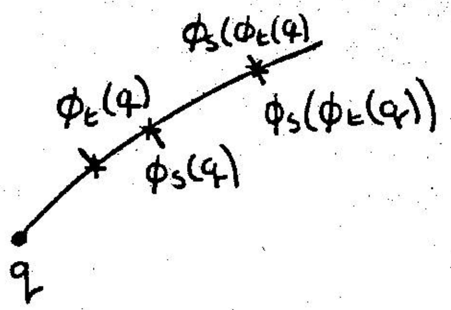
integral curve of X through q



From this def^n it is immediate that $\phi_s(\phi_t(q)) = \phi_t(\phi_s(q)) = \phi_{t+s}(q)$

$\phi_{-t}(q) = (\phi_t)^{-1}(q)$ for sufficiently small s, t, so that ϕ_t form a 1-parameter local group of diffeomorphisms



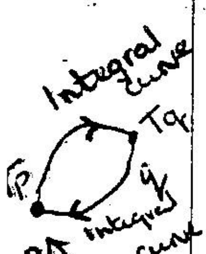
Now let T be a (r,0) or (0,s) tensor field on M. The diffeomorphism ϕ_t induces a linear map between (r,0) or (0,s) tensors on M sending T_p to $\phi_{t*}(T_p)$ at $\phi_t(p)$.

[Recall: Vector field ⇒ Integral curve ⇒ diffeomorphism i.e Given tensor field, we have created a linear map from a ~~point p~~ tensor ∈ tensor field at p to a tensor ∈ tensor field at $\phi_t(p)$. By letting $t \to 0$ we can define a map from ~~a~~ a tensor ∈ tensor field at p to a tensor of the same type at p as follows].

<u>Def^n 1.7.</u> <u>The Lie derivative of T with respect to X at p</u> is

$$(\pounds_X T)_p = \lim_{t\to 0} \frac{1}{t}\left\{T_p - (\phi_{t*}T)_p\right\}$$

$$= -\left[\frac{d(\phi_{t*}T)_p}{dt}\right]_{t=0}$$



$(\phi_{t*}T)_p$ is a tensor at p.

Where $(\phi_{t*}T)_p = \phi_{t*}(T_{\phi^{-t}(p)})$ and is a (r,s) tensor at p

This somewhat complicated def^n leads to some surprisingly simple properties. First it is clear that the limit involves the difference of two tensors of the same type at p. Thus $(\pounds_X T)_p$ is a (r,s) tensor at p. It is also clear that $\pounds_X$ considered as a map in $T_p{}^r{}_s$ is linear. It is not difficult to show that $\pounds_X$ is a derivative, because it satisfies Leibnitz's rule

$$\pounds_X (S \otimes T) = \pounds_X S \otimes T + S \otimes \pounds_X T$$

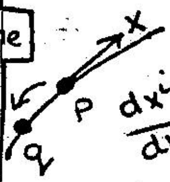
for arbitary fields S, T. However, such a general result will not be needed here.

Now it is possible to compute the effects of $\pounds_X$ on functions, forms, vectors etc in a coord indep. manner. However, it is more advantageous by first doing calculations using a local coord system (x^i) In what follows, q is always the point a parameter distance $-t$ along the integral curve of X through p from p.

[Thus the coords of q are $x^i(q) \mp x^i(p) + \int_0^{-t} X^i(x(s))\,ds$ and $dx^i(q)/dt = -X^i(q)$].

(1) Let $T = f : M \to R$ i.e. Let T be a fn (0-form).

From defn $\pounds_X(f) = -\left[\dfrac{d(\phi_{t*} f(p))}{dt}\right]_{t=0}$



$\dfrac{dx^i(q)}{dt} = -X^i(q)$

$$= -\left[\frac{d\, f(\phi_t(p))}{dt}\right]_{t=0} = -\left[\frac{df(q)}{dt}\right]_{t=0}$$

$$= -\left[\frac{\partial f(q)}{\partial x^i(q)} \frac{dx^i(q)}{dt}\right]_{t=0} = \left[X^i(q) \frac{\partial f(q)}{\partial x_i}\right]_{t=0}$$

$$= [X_q f]_{t=0} \quad \therefore \underline{\underline{\pounds_X(f) = X_p f}}$$

Let Y be a vector: To calculate $\pounds_X Y$ first consider

$$(\phi_t * Y)^i_p = (\phi_t * Y)_p(x^i(p)) = Y^j_q \frac{\partial}{\partial x^j(q)}(x^i(p))$$

$$= \frac{\partial x^i(\phi_t(q))}{dx^j(q)} Y^j_q$$

Now $\dfrac{dx^i}{dt}(\phi_t(q)) = X^i_p$

and so $\left[\dfrac{d}{dt}\left(\dfrac{\partial x^i(\phi_t(q))}{\partial x^j(q)}\right)\right]_{t=0} = \dfrac{\partial}{\partial x^j(q)}\left[\dfrac{d}{dt}(x^i(\phi_t(q))\right]_{t=0} = (X^i{}_{,j})_p$.

Thus $(\pounds_X Y)^i = -\left[\dfrac{d}{dt}(\phi_{t*} Y)^i\right]_{t=0} = Y^i{}_{,j} X^j - X^i{}_{,j} Y^j = [X, Y]$

[Using: (similar to above) $[\;] = -\left[\dfrac{d}{dt}\left(\dfrac{\partial x^i}{\partial x^j}(\phi_t(q)) Y^j\right)\right] \overset{\text{Prod. Rule}}{=} -\left[\left\{\dfrac{d}{dt}\left(\dfrac{\partial x^i}{\partial x^j}(\phi_t(q))\right)\right\} Y^j + \dfrac{\partial x^i}{\partial x^j}\dfrac{\partial Y^j}{\partial x^k}\dfrac{dx^k}{dt}\right]$] $\therefore$ $\underline{\underline{\pounds_X Y = [X, Y]}}$.

If ω is a 1-form then $\omega(Y)$ is a fn. From above rules

$$(\pounds_X \omega) Y = \pounds_X \omega(Y) - \omega(\pounds_X Y) = X(\omega(Y)) - \omega([X, Y])$$

which shows effect of Lie deriv. on 1-forms.

Notes.

1) For fns, 1 forms, define ϕ_t by $p \xrightarrow{t} q$, $\phi_t(p) = q$

2) for vectors $q \xrightarrow{t} p$, $\phi_t(q) = p$

$X = X^i \frac{\partial}{\partial x^i}$, jth comp $= X^i \frac{\partial}{\partial x^i}(x_j) = X^j$

Above $\dfrac{dx^i(q)}{dt} = -X_q$

since $p \xrightarrow{t} q$ this time $\dfrac{dx^i}{dt}(p) = X_p$

$q \xrightarrow{t} p$