(1.5) Linear Connections.

In setting up differential structure on a manifold one question has been avoided: what is the derivative of a vector field? This might be answered as follows. Let $\lambda(t)$ be a smooth curve through p $(t=0)$ in M. If Y is a vector field on M the derivative of Y along λ at p is

$$(DY)_{\lambda,p} = \lim_{t\to 0} \left\{ t^{-1}\left[Y(\lambda(t)) - Y(\lambda(0)) \right] \right\}$$

At first sight this definition does not make sense, since $Y(\lambda(t))$, $Y(\lambda(0))$ are in $T_{\lambda(t)}$ and T_p resp. and addition of vectors in different vector spaces has not been defined. A solution to this problem has already been suggested by the Lie derivative. One uses the mapping between $T_{\lambda(t)}$, T_p induced by λ to compare the 2 vectors in the same space. The Lie derivative also possesses 2 desirable attributes with which one would wish to endow DY. $\mathcal{L}_X Y$ is linear in Y and $\mathcal{L}_X f = Xf$ for functions f. However one would also require that $D_X Y$ be linear in X so that it depends only on the direction of X. It is this consideration that rules out the Lie deriv. as a possible candidate. As can be seen from the coordinate representation $\mathcal{L}_X Y$ involves the derivs of the components of X, and therefore cannot be linear in X. Since a derivative of Y, $D_X Y$ which is linear in X will certainly be required, some additional structure, affine structure, needs to be imposed on the manifold. This motivates the following definitions.

Def^n 1.8. A linear connection ∇ at a pt p of M is a rule which ~~associates~~ assigns to each vector field X at p a differential operator ∇_X which maps a vector field Y into a vector field $\nabla_X Y$ which satisfies

(1) $\nabla_X Y$ is linear in Y. i.e. if Y, Z are vector fields α, β real constants

$$\nabla_X(\alpha Y + \beta Z) = \alpha \nabla_X Y + \beta \nabla_X Z$$

(2) If f is any fn on M

$$\nabla_X(fY) = f\nabla_X Y + X(f)Y$$

(3) $\nabla_X Y$ is linear in X. i.e if Y, Z are vector fields f, g fns on M

$$\nabla_{(fX+gY)} Z = f\nabla_X Z + g\nabla_Y Z.$$

$(\nabla_X Y)_p$ is called the <u>covariant derivative</u> (with respect to ∇) of <u>Y in the dir^n of X at p</u>. More generally the covariant derivative of Y is the (1,1) tensor field which when contracted with a vector field X produces $\nabla_X Y$. $[X(\nabla Y) = (\nabla Y)X = \nabla_X Y]$.

Now let $(e_a), (\omega^a)$ be dual local vector & 1 form bases in a ngbd U of p. For convenience ∇e_a written ∇_a. According to the rules

$\nabla_a e_b$ is a vector and hence can be written

$$\nabla_a e_b = \Gamma^c{}_{ab} e_c \quad \text{linear combination}$$

Then $\Gamma^c{}_{ab}$ are called the <u>components of the connection</u>. with respect to the chosen bases. If the manifold has a global basis (e_a) i.e R^n then any set of fns $\Gamma^a{}_{bc}$ define a connection, and so a manifold may possess one or more linear connections. However most manifolds do not have a global basis (e_a) and the assumption of the existance of a connection imposes additional structure, <u>affine structure</u> on the manifold.

An equivalent definition of the components of the connection is

$$\Gamma^c{}_{ab} = \omega^c\,(\nabla_a e_b)$$

Since

$$\Gamma^c{}_{ab} = \omega^c(\nabla_a e_b) = \omega^c(\Gamma^d{}_{ab} e_d)$$
$$= \Gamma^d{}_{ab}\,\omega^c(e_d) = \Gamma^d{}_{ab}\,\delta^c_d = \Gamma^c{}_{ab}.$$

Also, consider the covariant derivative of e_c which is a (1,1) tensor & can therefore be written

$$\nabla e_c = \Gamma^a_{\ bc}\, \omega^b \otimes e_a .$$

Contracting this with vector e_d (giving vector)

$$\nabla_d e_c = \Gamma^a_{\ bc}\, \omega^b \otimes e_a (e_d)$$

$$= \Gamma^a_{\ bc}\, \omega^b(e_d)\, e_a = \Gamma^a_{\ dc}\, e_a$$

Similarly

$$\underline{\nabla(fY) = df \otimes Y + f\nabla Y}$$

since, contracting this eq^n with X we get

$$\nabla_X (fY) = [df \otimes Y]X + f\nabla_X Y$$

$$= df(X)Y + f\nabla_X Y$$

$$= X(f)Y + f\nabla_X Y$$

$$= (\nabla_X f)Y + f\nabla_X Y \quad - \text{ equivalent to Leibnitz rule.}$$

i.e $\nabla_X(fY) = \underline{X(f)Y + f\nabla_X Y}$ which is rule (2) of def 1.8

So that we can write $\nabla Y = \nabla(Y^c e_c) = dY^c \otimes e_c + Y^c \Gamma^a_{\ bc}\omega^b \otimes e_a$ (*)

It is useful to write these results out for clarity

(i) $\nabla_d e_c = \Gamma^a_{\ dc} e_a$
(ii) $\nabla e_c = \Gamma^a_{\ bc}\, \omega^b \otimes e_a$
(iii) $e_d \nabla e_c = \nabla_d e_c = \Gamma^a_{\ bc}\, \omega^b(e_d) e_a = \Gamma^a_{\ dc} e_a$.

In many cases (e_a), (ω^a) will be taken to be local coord ~~systems~~ induced bases, $(\partial/\partial x^i)$, (dx^i) so that $df = f_{,i}\, dx^i$ $\quad f_{,i} = \partial f/\partial x^i$ as usual

Then from (*) with $dY^c_{\ i} = \left(\frac{\partial Y^c}{\partial x_j}\right)_j dx^j$

$$\nabla Y = Y^i_{\ ,j}\, dx^j \otimes \left(\frac{\partial}{\partial x^i}\right) + Y^k \Gamma^i_{\ jk}\, dx^j \otimes \left(\frac{\partial}{\partial x^i}\right)$$

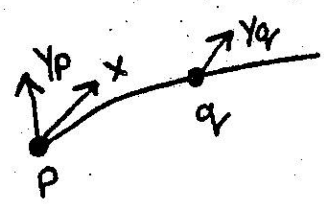
$$= Y^i_{\ ;j}\, dx^j \otimes \left(\frac{\partial}{\partial x^i}\right)$$

where components of coord. derivative ∇Y are

$$Y^i_{\ ;j} = Y^i_{\ ,j} + \Gamma^i_{\ jk} Y^k .$$

Def^n 1.9.

Let $\lambda(t)$ be the integral curve of a vector field X through a point p of M. The vector field Y is said to be parallely transported along λ if $\nabla_X Y = \mu Y$. (some authors require $\mu = 0$)



$$\frac{Y_q - Y_p}{t} \rightarrow \nabla_X Y.$$

Example in R^2.

Let (x, y) be obvious global coord system, $e_x = \frac{\partial}{\partial x}$ $e_y = \frac{\partial}{\partial y}$ $\omega^x = dx$ $\omega^y = dy$.

If α, β are constants $X = \alpha e_x + \beta e_y$ is a globally defined vector field. With above coord system, in the usual sense of the word X_p, X_q, the vectors at 2 different points are parallel. This is easy to see since;

If we have connection such that $\Gamma^i{}_{jk} = 0$

Then $X^i{}_{;j} = X^i{}_{,j}$

i.e covariant deriv reduces to the partial deriv and for $X = \alpha e_x + \beta e_y$ and $X^i{}_{,j} = 0$

$\therefore$ concept of parallelism is consistant with def^n 1.9 since $X^i{}_{;j} = 0$.

Less trivial is the case of polar coords (r, θ) where $e_1 = \partial/\partial r$ $e_2 = \frac{1}{r}\partial/\partial\theta$ $\omega^1 = dr$ $\omega^2 = r\,d\theta$.

According to Euclidean geometry

$$\nabla_1 e_a = \nabla_{e_1} e_a = 0$$

$$\nabla_2 e_1 = \frac{e_2}{r}$$

$$\nabla_2 e_2 = -\frac{e_1}{r}$$

And so from $\nabla_a e_b = \Gamma^c{}_{ab} e_c$

$$\Gamma^2{}_{21} = -\Gamma^1{}_{22} = \frac{1}{r}$$

(15)

def^n of comps of df

$\therefore$ the gradient fn $\quad df = e_a(f)\,\omega^a = f_{,r}\,dr + \left(\frac{f_{,\theta}}{r}\right) r d\theta$

with components $(f_{,r},\ r^{-1} f_{,\theta})$ as usual.

Let X, Y be 2 vector fields and $\quad X = X^a e_a$
$\qquad Y = Y^a e_a$.

Since.

$$\nabla(fY) = df \otimes Y + f\nabla Y$$

$$\nabla(Y) = \nabla(Y^a e_a) = dY^a \otimes e_a + Y^a \nabla e_a$$

this result on pg is used

$$= dY^a \otimes e_a + Y^c \nabla e_c$$
$$= dY^a \otimes e_a + Y^c \Gamma^a_{bc}\, \omega^b \otimes e_a$$

$$\therefore \quad \nabla Y = \left(Y^a_{,b} + \Gamma^a_{bc} Y^c\right) \omega^b \otimes e_a \quad \left[dY^a = Y^a_{,b}\,\omega^b\right]$$

So

$$\nabla_X Y = \left(Y^a_{,b} + \Gamma^a_{bc} Y^c\right) \omega^b(X)\, e_a$$
$$= \left(Y^a_{,b} + \Gamma^a_{bc} Y^c\right) \omega^b(X^d e_d)\, e_a$$
$$= \left(Y^a_{,b} + \Gamma^a_{bc} Y^c\right) X^b e_a$$
$$= X(Y^a) e_a + \Gamma^a_{bc} X^b Y^c e_a.$$

These results are quite general. writing out $\nabla_X Y$ in component form gives

$$\nabla_X Y = \left(X(Y^r) - X^\theta Y^\theta / r,\ X(Y^\theta) + X^\theta Y^r / r\right)$$

or in obvious adjustment of notation

from above

$$(X\cdot\nabla)Y = \left((X\cdot\nabla)Y^r - X^\theta Y^\theta/r,\ (X\cdot\nabla)Y^\theta + X^\theta Y^r/r\right)$$

i.e connection in this case associated gradient operator

which is a standard result in vector analysis

Similarly $\nabla\cdot Y$ is a scalar which one might expect to get from ∇Y by contracting ω^b on e_a to give

$$\nabla\cdot Y = Y^a_{\ a} + \Gamma^a_{ac} Y^c \qquad \left\{ \omega^b \otimes e_a \Rightarrow \omega^b(e_a) \Rightarrow \delta^b_a \right\}$$

$$= Y^r_{,r} + r^{-1} Y^\theta_{,\theta} + r^{-1} Y^r = r^{-1}\left[(rY)^r_{,r} + Y^\theta_{,\theta}\right]$$

again a standard result. Thus when applied to R^n the cov. derivative may be regarded as generalisation of usual invariant operators in vector analysis expressed in arbitrary curvilinear coords.

So far nothing has been said about the tensorial properties of the connection or its components Γ^a_{bc}. Suppose that (e_a), (ω^a), $(\bar{e}_a)$, $(\bar{\omega}^a)$ are 2 pairs of dual bases for vectors & 1forms related by

$$\bar{\omega}^a = A^a{}_b\,\omega^b \qquad \bar{e}_b = B^a{}_b\,e_a \qquad \text{where } B = A^{-1}.$$

we consider a change of basis.

$$\begin{aligned}
\bar{\Gamma}^a{}_{bc} &= \bar{\omega}^a(\bar{\nabla}_b\bar{e}_c) \\
&= A^a{}_f\,\omega^f\left(\nabla_{(B^g{}_b e_g)}B^h{}_c e_h\right) \\
&= A^a{}_f B^g{}_b\,\omega^f(\nabla_g B^h{}_c e_h) \\
&= A^a{}_f B^g{}_b\left\{B^h{}_c\,\omega^f(\nabla_g e_h) + e_g(B^h{}_c)\,\omega^f(e_h)\right\}
\end{aligned}$$

[since $\nabla_X(fY) = f\nabla_X Y + X(f)Y$

$$\nabla_g(B^h{}_c e_h) = B^h{}_c\nabla_g e_h + e_g(B^h{}_c)\,e_h.$$

$$\therefore\ \omega^f(\nabla_g B^h{}_c e_h) = B^h{}_c\,\omega^f(\nabla_g e_h) + e_g(B^h{}_c)\,\omega^f(e_h)]$$

$$= \underbrace{A^a{}_f B^g{}_b B^h{}_c\,\Gamma^f{}_{gh}}_{\text{tensor transform law.}} + A^a{}_h\,e_b(B^h{}_c)$$

$\Gamma^f{}_{gh} = \omega^f(\nabla_g e_h)$.

Because 2nd term does not in general, vanish, it is clear that Γ^a_{bc} does not transform like components of a tensor.

So far the covariant derivative has only been applied to vectors. It can also be applied to 1-forms and $\therefore$ to general tensors by imposing the ~~follwg~~ following rules.

(i) If T (r,s) tensor, ∇T is a $(r, s+1)$ tensor

(T is (r,s) $\nabla_X T$ is (r,s) ∇T is $(r, s+1)$)

(ii) ∇ is linear and commutes with contractions

(iii) for any function f, $\nabla f = df$.

(iv) If S, T are arbitary tensor fields, Leibnitz's rule is to hold in the form

$$\nabla(S \otimes T) = \nabla S \otimes T + S \otimes \nabla T$$

(iv), $\nabla(S \otimes T) = \nabla S \otimes T + S \otimes \nabla T$ is ambiguous; there is more than one way to apply it. However there is an accepted convention, which then makes it well defined.

In particular $\nabla(\omega^a \otimes e_b) = \nabla \omega^a \otimes e_b + \omega^a \otimes \nabla e_b$

$$= \nabla \omega^a \otimes e_b + \omega^a \otimes \Gamma^d_{fb} \omega^f \otimes e_d$$

Contract over a and b

$$0 = \nabla(\omega^a(e_b)) = \nabla \omega^a(e_b) + \omega^a(\Gamma^d_{fb} \omega^f \otimes e_d)$$

$$= \nabla \omega^a(e_b) + \Gamma^d_{fb} \omega^f \omega^a(e_d) \qquad [\text{since } \nabla e_b = \Gamma^d_{fb} \omega^f \otimes e_d \text{, } b \text{ free suffix}]$$

$$= \nabla \omega^a(e_b) + \Gamma^a_{fb} \omega^f$$

$$= \nabla \omega^a(e_b) + \Gamma^a_{cb} \omega^c \qquad \text{(change suffices from } f \text{ to } c\text{)}$$

Consistant with this eq^n we may write

$$\nabla \omega^a = -\Gamma^a_{bc} \omega^b \otimes \omega^c$$

Since $\nabla \omega^a(e_b) \otimes \omega^b = -\Gamma^a_{cb} \omega^c \otimes \omega^b = -\Gamma^a_{bc} \omega^b \otimes \omega^c$
and $\nabla \omega^a(e_b) \otimes \omega^b = \nabla \omega^a$
where in calculating $\nabla \omega^a(e_d)$ it is the last term on

[Note: convention: (see notes)
$\nabla\omega = \psi_{i;j}\,\omega^j \otimes \omega^i$
$\nabla_X\omega = \nabla\omega_{\text{contracted with } X} = \psi_{i;j}\,\omega^j(X)\,\omega^i$ [not $\omega^j \omega^i(X)$]]

on right that is contracted with e_d

$$\nabla\omega^a = -\Gamma^a_{\ bc}\,\omega^b \otimes \omega^c$$

By convention (purely) $\nabla\omega^a(e_d) = -\Gamma^a_{\ bc}\,\omega^b\,\omega^c(e_d)$.

So we have $\nabla Y = \left(Y^i_{\ ,j} + \Gamma^i_{\ jk} Y^k\right) dx^j \otimes \frac{\partial}{\partial x^i}$

$$\nabla\psi = \left(\psi_{i,j} - \Gamma^k_{\ ji}\psi_k\right) dx^j \otimes dx^i \quad \text{– Ex 1.14}$$

where $Y = Y^i \left(\frac{\partial}{\partial x^i}\right)$ $\psi = \psi_i\,dx^i$ wrt to coord induced bases.

Using these rules the covariant derivatives of any tensor may be constructed. Suppose T is $(1,1)$ tensor
$T = T_a^{\ b}\,\omega^a \otimes e_b$ then

$$\nabla T = dT_a^{\ b} \otimes \omega^a \otimes e_b + T_a^{\ b}\,\nabla\omega^a \otimes e_b + T_a^{\ b}\,\omega^a \otimes \nabla e_b$$

$$= dT_a^{\ b} \otimes \omega^a \otimes e_b - T_a^{\ b}\Gamma^a_{\ cd}\,\omega^c \otimes \omega^d \otimes e_b + T_a^{\ b}\,\omega^a \otimes \Gamma^d_{\ cb}\,\omega^c \otimes e_b.$$

and; By convention (see note)

$$\nabla_X T = dT_a^{\ b}(X)\,\omega^a \otimes e_b - \Gamma^a_{\ cd}T_a^{\ b}\,\omega^c(X)\,\omega^d \otimes e_b + T_a^{\ b}\,\omega^a(X)\,\Gamma^d_{\ cb}\,\omega^c \otimes e_b$$

$$= X(T_a^{\ b})\,\omega^a \otimes e_b - \Gamma^a_{\ cd}T_a^{\ b}X^c\,\omega^d \otimes e_b + \Gamma^d_{\ cb}T_a^{\ b}X^a\,\omega^c \otimes e_b.$$

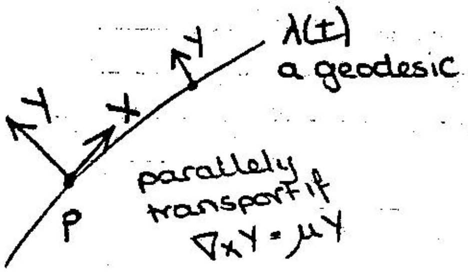
If the bases are coord induced, the component form of these eq^{n}s is

$$T_i^{\ j}{}_{;k} = T_i^{\ j}{}_{,k} - \Gamma^m_{\ ki}T_m^{\ j} + \Gamma^j_{\ km}T_i^{\ m}$$

plus the same eq^n multiplied by X^k. This result generalises to higher orders in obvious way.

Let p be a point in R^n, and X a vector defined at p. X defines a class of smooth curves at p, those which are tangent to X. From this class one can pick out one curve whose tangent vector is everywhere parallel to X, the straight line through p parallel to X. Now def^n 1.9 has already introduced the idea of parallel transport along a smooth curve in an affine differentiable manifold and so the idea of a smooth curve whose tangent vectors at diff points are parallel generalises to M. To avoid confusion such curves are called geodesics, rather than straight lines.

<u>Def^n 1.10.</u> A <u>Geodesic</u> is a smooth curve whose tangent vector is everywhere parallel.



Suppose that $\lambda(t)$ is a geodesic with tangent vector X. Let x^i be local coords, so that λ is $x^i = x^i(t)$. Denoting $\frac{d}{dt}$ by dot '$\bullet$' along curve, it is clear X has components $X^i = \dot{x}^i(t)$ with respect to coord induced basis for vectors.

$$\therefore \quad x^i = x^i(t) \quad , \quad X^i = \dot{x}^i(t)$$

X has to satisfy the eq^n $\nabla_X X = \mu(x^i(t))\, X$ along the curve. The local coord form of this eq^n is

$$X(X^i) + \Gamma^i{}_{JK} X^J X^K = \mu X^i$$

Let $X = X^i e_i$ from $\nabla_X(fY) = X[df \otimes Y + f\nabla_X Y]$

$$\Rightarrow \quad \nabla_{X^K e_K}(X^i e_i) = X[dX^i \otimes e_i] + X^i \nabla_{X^K e_K} e_i$$

$$\Rightarrow \nabla_X X = X(X^i)\, e_i + X^i X^K \nabla_K e_i$$

$$= X(X^i)\, e_i + X^i X^K \Gamma^m_{Ki}\, e_m \qquad \text{def}^n \text{ of } \Gamma$$

$$= X(X^i) e_i + X^J X^K \Gamma^i_{JK}\, e_i \quad = \mu X^i e_i$$

$$\therefore \quad X(X^i) + X^J X^K \Gamma^i{}_{JK} = \mu X^i$$

or, alternatively $[X = \frac{d}{dt}]$

$$\ddot{x}^i + \Gamma^i{}_{JK}\, \dot{x}^J \dot{x}^K = \mu \dot{x}^i$$

Before discussing this equation further some simplification can be made. The defn of the geodesic did not explicitely involve the parameter t along the curve. Thus a change of parameter $t \to s(t)$ is always possible. Denote $\frac{d}{ds}$ by "$'$" dash then

$$\frac{d}{dt} = \dot{s}\frac{d}{ds} \qquad \frac{d^2}{dt^2} = \ddot{s}\frac{d}{ds} + \dot{s}^2\frac{d^2}{ds^2}$$

and eqn of geodesic becomes

$$x^{i''} + \Gamma^i_{jk}\, x^{j'} x^{k'} = x^{i'}(\mu\dot{s} - \ddot{s})/\dot{s}^2$$

Now, by choosing $\dot{s}$ to satisfy $\ddot{s} = \mu\dot{s}$, the RHS vanishes. In this case s is called an <u>affine parameter</u>. This eqn does not specify s uniquely, for if s is an affine parameter, then so is $\alpha s + \beta$ (α, β constant). i.e affine parameters are only specified to linear transformations. Without loss of generality we always assume parameter t along geodesic is affine, so that the eqn of geodesic is

$$\ddot{x}^i + \Gamma^i_{jk}\,\dot{x}^j\dot{x}^k = 0$$

writing in the form $\ddot{x}^i = -\Gamma^i_{jk}\,\dot{x}^j\dot{x}^k$

It is easy to see that a geodesic is uniquely specified by giving $x(t)$ and $\dot{x}(t)$ for one value of t, i.e there is a unique geodesic through each point p of M tangent to a specified direction at p, which will be denoted by $\lambda_X(t)$. All considerations here have been local; the geodesic $\lambda_X(t)$ may not be defined for all t. If however the parameter t can take all values the geodesic is said to be a <u>complete</u> geodesic. If all geodesics in M are complete, M is <u>geodesically complete</u>.

Consider all the geodesics passing through a point p of M. Each geodesic is specified uniquely by its tangent vector X_p at p.

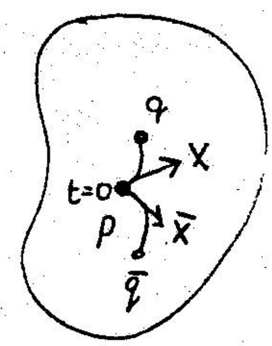
Def^n 1.11. The <u>exponential map</u> at p $\exp: T_p \to M$ maps each tangent vector T_p into the point a unit parameter distance from p along the geodesic through p with tangent vector X_p. Note that unless M is geodesically complete the exponential map may not be defined for all $X_p \in T_p$

However for sufficiently small values of the indep variable, solutions of ordinary diff. eq^{n}s depend on initial values, and it is intuitively clear that there is some sufficiently small neighbourhood of p which is diffeomorphic to some neighbourhood of the origin in T_p. We may define the coords of a point q in this nghbd as X^a, where $q = \exp(X^a e_a)$ and e_a is a basis of T_p. Such coords are called <u>Normal coords</u>

Roughly speacking

$x^i(t_0), \dot{x}^i(t_0)$ determine geodesics

$\therefore$ $x^i(t_0)$ $x^i(t)$ " "
provided $t \simeq t_0$



q determines X (up to arb. constant which chosen so that q is parameter distance 1 up curve) call components of $X \in T_p$ coords of $q \in M$.

i.e. $T_p \to M$ $\quad X = X^a e_a$
$X \to q$ $\quad q$ has coords X^a $\Big\}$ exp map $q = \exp(X)$

Normal coords are very useful in actual calculations because of following property. Let q be pt unit distance along geodesic X with coords x^i_0. Any other pt on geodesic has normal coords $x^i(t) = x^i_0 t$. Then $x^i(t) = x^i_0 t$ is a geodesic Since $\ddot{x}^a(t) = 0$ and $\dot{x}^a = x^a_0$

It follows that $\Gamma^a_{bc}(x^a_0 t) x^b_0 x^c_0 = 0$ $\quad \left[\text{since } \ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0\right]$

Diffing wrt t

$$\Gamma^a_{\ bc,d}(x^a_0 t) x^b_0 x^c_0 x^d_0 = 0$$

In particular setting $t = 0$, it follows that $\Gamma^a_{(bc)}(p) = \Gamma^a_{(bc,d)}(p) = 0$ so that the choice of normal coords at p always allows one to set

$$\Gamma^a_{(bc)} = \Gamma^a_{(bc,d)} = 0 \text{ at } p.$$

(1.6) Curvature and Torsion.

It has already been shown that the components of a connection on M do not define a tensor. There are however two important, geometrically significant tensors that can be built from a connection.

Def^n(1.12) The Torsion tensor T is a $(1,2)$ tensor which can be defined by.

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

Again this is an oblique definition which needs to be examined with care. Clearly it is a $(1,2)$ object since T is a map $T_p \times T_p \to T_p$. It is also necessary to show that it is multilinear.

It is evident that $T(X,Y) = -T(Y,X)$ so it is only necessary to show T is linear in first argument.

T is linear in X; Let f be an arbitary fn on M. Then

$$T(fX,Y) = \nabla_{fX} Y - \nabla_Y fX - [fX,Y]$$

$$= f\nabla_X Y - \cancel{df(Y)X} - f\nabla_Y X - f[X,Y] + \cancel{Y(f)X}$$

$$[fX,Y]g = [fX(Yg) - Y(fXg)]$$

$$= fX(Yg) - fY(Xg) - Xg\,Y(f)$$

$$= f[X,Y]g - XY(f)g$$

$$\therefore\ [fX,Y] = f[X,Y] - Y(f)X. \qquad \text{Also } df(Y) = Y(f)$$

$$= f\nabla_X Y - f\nabla_Y X - f[X,Y]$$

$$= fT(X,Y)$$

Thus T multilinear and T is a tensor;

Now suppose (e_a) is a basis for vectors in some ngbd U in M. The commutation coefs of the basis the components of the commutator $[e_a, e_b]$,

$$[e_a, e_b] = \gamma^c{}_{[ab]}\, e_c. \qquad \{[e_a,e_b] = -[e_b,e_a]\}$$

Now $T(e_a, e_b) = \nabla_a e_b - \nabla_b e_a - [e_a, e_b]$

$$= \Gamma^c{}_{ab} e_c - \Gamma^c{}_{ba} e_c - \gamma^c{}_{ab} e_c$$

$$= 2\Gamma^c{}_{[ab]} e_c - \gamma^c{}_{ab} e_c \qquad \left[T^a{}_{(bc)} = 0 \text{ since } \gamma^a_{ab} = \gamma^a_{..}\right]$$

$$\therefore \; T^c{}_{ab} = T^c{}_{[ab]} = \left(2\Gamma^c{}_{[ab]} - \gamma^c{}_{ab}\right) \qquad [T(X,Y) = -T(Y,X)]$$

where $T^c{}_{ab}$ are <u>components of</u> T.

Note
1) Exercise shows that for coord induced basis $\gamma^i_{jk} = 0$ (by def^n $\gamma^a_{bc} e_a = [e_b, e_c]$ — for coord induced basis $[\partial/\partial x^b, \partial/\partial x^c] = 0$ commute)
2) In what follows essentially concerned with torsion free Γ — $\Gamma^a{}_{[bc]} = 0$

If the basis is coord-induced then the components are $\underline{T^k{}_{ij} = 2\Gamma^k{}_{[ij]}}$ (this follows from ex 1.16) Now it is clear that although a linear connection on M determines the geodesics, a complete knowledge of the geodesics on M is only sufficient to determine the symmetric part of the connection $\Gamma^a{}_{(bc)}$. The torsion tensor can therefore be regarded as that part of the connection which does not influence the geodesics.

$$\Gamma^i{}_{jk} = \underbrace{\Gamma^i{}_{(jk)}}_{\text{geodesics}} + \underbrace{\Gamma^i{}_{[jk]}}_{\text{torsion}}$$

The second tensor determined by the connection is of considerably more importance for gravitation theory.

<u>Def^n 1.13.</u> Given a linear connection on M the <u>Riemann curvature tensor</u> is a $(1,3)$ tensor defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for arbitrary vector fields X, Y, Z.

This is another oblique def^n. R is clearly a map $T_p \times T_p \times T_p \to T_p$ for each p in M. So R is obviously a $(1,3)$ object [acting on 3 vectors, produces vector]. We must show T is multilinear. But $R(X,Y)Z = -R(Y,X)Z$ so it is sufficient to show linearity in X and Z.

Linearity in X; Let f be arbitrary fn;

$$R(fX, Y)Z = \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z$$

$$= f\nabla_X \nabla_Y Z - \nabla_Y (f \nabla_X Z) - \nabla_{(f[X,Y] - Y(f)X)} Z$$

We use linearity of ∇ [ie $\nabla_{\alpha X + \beta Y} = \alpha\nabla_X + \beta\nabla_Y$]

$$\nabla_Y(f\nabla_X Z) = \nabla_Y f \otimes \nabla_X Z + f\nabla_Y\nabla_X Z$$
$$= Y(f)\nabla_Y\nabla_X Z + f\nabla_Y\nabla_X Z$$

$$= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - \nabla_{(f[X,Y] - Y(f)X)} Z$$

$$= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - \cancel{Y(f)\nabla_X Z}^{\,0}$$
$$- f\nabla_{[X,Y]}Z + \cancel{Y(f)\nabla_X Z}^{\,0}$$

$$= f\{\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\}$$

$$\therefore R(fX,Y)Z = f\,R(X,Y)Z.$$

Linearity in Z. [Ex 1.16].

(i) Note that $\nabla_X(fY) = \nabla_X f Y + f\nabla_X Y$
(ii) $\nabla_X f = X(f)$

$$R(X,Y)fZ = \nabla_X\nabla_Y fZ - \nabla_Y\nabla_X fZ - \nabla_{[X,Y]}fZ$$

$$= \nabla_X\{\underset{\uparrow \text{fn}}{Y(f)}Z + f\nabla_Y Z\} - \nabla_Y\{X(f)Z + f\nabla_X Z\}$$
$$- \{(\nabla_{[X,Y]}f)Z + f\nabla_{[X,Y]}Z\}$$

$$= \cancel{X(Y(f))Z} + \cancel{Y(f)\nabla_X Z} + \cancel{X(f)\nabla_Y Z} + f(\nabla_X\nabla_Y Z$$
$$- \cancel{Y(X(f))Z} - \cancel{X(f)\nabla_Y Z} - \cancel{Y(f)\nabla_X Z} - f\nabla_Y\nabla_X Z$$
$$- \cancel{[X,Y]f Z} - f\nabla_{[X,Y]}Z \qquad \left[[X,Y]f = X(Y(f)) - Y(X(f))\right.$$

$$= f\{\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\}$$

$$= f\,R(X,Y)Z.$$

Hence R is indeed a tensor. If (e_a) is a basis for vectors the components of R may be defined by

$$R(e_a, e_b)e_c = R^d{}_{cab}\, e_d \qquad \text{(Note order)}$$

Then
$$R^d{}_{cab}\, e_d = \nabla_a\nabla_b e_c - \nabla_b\nabla_a e_c - \nabla_{[e_a,e_b]}e_c$$

$$= \nabla_a(\Gamma^d_{bc}e_d) - \nabla_b(\Gamma^d_{ac}e_d) - \nabla_{\gamma^d_{ab}e_d}e_c$$

$$= \nabla_a(\Gamma^d_{bc}e_d) - \nabla_b(\Gamma^d_{ac}e_d) - \gamma^d_{ab}\nabla_{e_d}e_c.$$

Liebnitz

$$= e_a(\Gamma^d_{bc})e_d + \Gamma^d_{bc}\Gamma^f_{ad}e_f - e_b(\Gamma^d_{ac})e_d$$

$$- \Gamma^d_{ac}\Gamma^f_{bd}e_f - \gamma^f_{ab}\Gamma^d_{fc}e_d$$

Changing suffices f to d to get e_d common we have

$$R^d{}_{cab} = e_a(\Gamma^d_{bc}) - e_b(\Gamma^d_{ac}) + \Gamma^f_{bc}\Gamma^d_{af} - \Gamma^f_{ac}\Gamma^d_{bf} - \gamma^f_{ab}\Gamma^d_{fc}$$

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Results of ex 1.18 and 1.19.
In coord Induced basis $(\partial/\partial x^i)$

$$(1.17)\qquad R^{l}_{\ kij} = \Gamma^{l}_{jk,i} - \Gamma^{l}_{ik,j} + \Gamma^{m}_{jk}\Gamma^{l}_{im} - \Gamma^{m}_{ik}\Gamma^{l}_{jm}$$

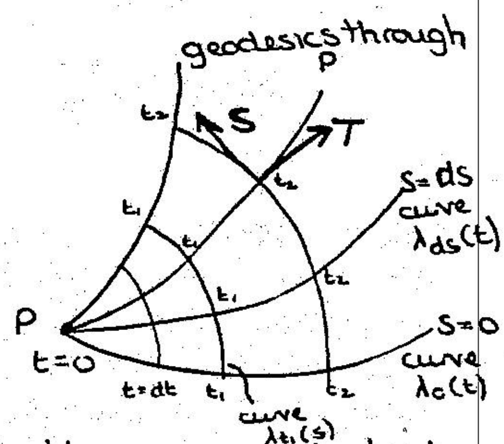
In coords "$\nabla X - X^i_{;k}$"
Define $\nabla\nabla X - X^i_{;kj}$

$$(1.18)\qquad R^{i}_{\ mkj}X^m = X^i_{;jk} - X^i_{;kj} \qquad \underline{\text{Ricci Identity}}$$

i.e. when $R \neq 0$, covariant derivs do not commute. With obvious notational changes the same identity can be shown to hold for components with respect to arbitrary bases.

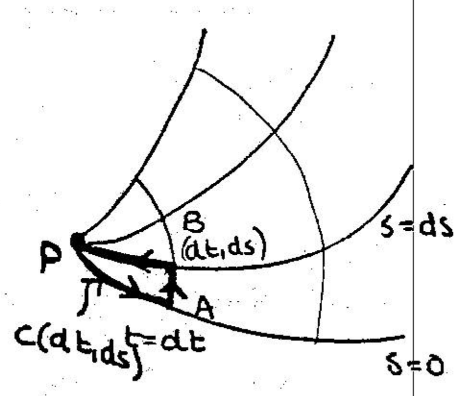
The geometrical interpretation of (1.18) is as follows.

Let $\lambda_s(t)$ be a 1-parameter family of ~~smooth~~ geodesics passing through the point p ($t=0$) in M. As s varies, the points of constant t define a smooth curve in M denoted $\lambda_t(s)$. Note that for $t=0$ curve $\lambda_t(s)$ degenerates to a pt. Let T, S be tangent vectors to curve $\lambda_s(t)$, $\lambda_t(s)$ respectively.



From the geometric Interpretation of the commutator it follows that $[T, S] = 0$. Now let X_p be any vector in T_p. We can propogate X_p parallely along the curves $\lambda_s(t)$ thereby defining a vector field X by the conditions $\nabla_T X = 0$, $X(p) = X_p$.

Now let $C(dt, ds)$ be the closed curve formed by starting at p, going along $\lambda_0(t)$ to $\lambda_0(dt)$, then along $\lambda_{dt}(s)$ to $\lambda_{dt}(ds)$ and then along $\lambda_{ds}(t)$ back to p as in figure. Let $X(dt, ds)$ be the vector in T_p obtained by parallely transporting X around $X(dt, ds)$. The difference $X(dt, ds) - X_p$ is a well defined vector which can be computed in the limit of small dt, ds as follows.



It is clear that the paths from p to $(dt, 0)$ and from (dt, ds) back to p contribute nothing to this difference since these segments have tangent vector T and X is parallely transported along T.

Define the 'acceleration' of Z to be $\nabla_X \nabla_X Z$

To calculate this first note that since commutator vanishes $[X, Z] = 0$

$$\nabla_X Z = \nabla_Z X \quad (\text{since torsion free } \nabla_X Z - \nabla_Z X - [X,Z] = T(X,Z) = 0).$$

and since $\nabla_X X = 0$

$$R(X,Z)X = \nabla_X \nabla_Z X - \cancel{\nabla_Z \nabla_X X}^{\,0} - \cancel{\nabla_{[X,Z]} X}^{\,0}$$

$$= \nabla_X \nabla_X Z$$

Hence $\underline{R(X,Z)X = \nabla_X \nabla_X Z}$ is the eq^n of <u>geodesic deviation</u>.

The significance of the name can be seen if a 1-parameter family of integral curves of X is considered. The points of the geodesics having the same affine parameter t form a smooth curve $\lambda_t(s)$ where s labels the geodesics. Then the Tangent Vector field to $\lambda_t(s)$, Z is a connecting Vector. Now let ε be a small real no. At each point t on one geodesic $s=0$, the exponential map sends εZ ~~at~~ into the point $\lambda_\varepsilon(t)$ on a 'neighbouring' geodesic. This is the significance of the term connecting, and the above eq^n then shows how neighbouring geodesics deviate. As was to be expected from R^n example, the 'velocity' of the connecting vector depends on the initial conditions, it is the 'acceleration' which is related to the curvature.

It is obvious from the definition that $R^a{}_{b(cd)} = 0$

Now it is straight forward but lengthy excercise in tensor algebra to show that

(dont try.)

$$R^a{}_{[bcd]} = 2e_b\left(T^a{}_{[cd]}\right) - 4T^e{}_{[bc}T_{d]e}$$

Thus when the ~~torsion~~ torsion vanishes ($T^i{}_{jk} = 0$) $R^a{}_{[bcd]} = 0$. Now it is clear that if the geodesic deviation can be measured for all geodesic and ~~...~~ connecting vectors, $R^a{}_{(bc)d}$ is determined. It follows that when $T = 0$, this is sufficient to determine R entirely, since $R^a{}_{bcd} = \frac{2}{3}\left[R^a{}_{(bc)d} - R^a{}_{(bd)c}\right]$

$$\therefore R^a{}_{bcd} = \underset{\substack{\text{acceleration} \\ \text{(geod. dev)}}}{R^a{}_{(bc)d}} + \underset{\text{torsion.}}{R^a{}_{[bc]d}}$$

Recall: $R^a{}_{bcd} = e_a(\Gamma^a_{bc}) - e_b(\Gamma^a_{ac}) + \Gamma^f_{bc}\Gamma^a_{af} - \Gamma^f_{ac}\Gamma^a_{bf} - \gamma^f_{ab}\Gamma^a_{fc}$ where $\gamma^f{}_{[ab]}$

curve, parallel transport from p to q is path independant. It follows that the (e_a) at T_p generate basis vector fields throughout R^n by parallel transport i.e $\nabla_a e_b = 0$ and if (ω^a) are the dual basis fields of 1-forms $\nabla_a \omega^b = 0$.
Let Y^i be an arbitary coord system, and set
arbitary 1-form → $e^a = e^a{}_i \, dY^i$. It follows that $0 = \nabla \omega^b = e_{i;k} \, dY^i \otimes dY^k$
so that

$$e^b{}_{i;k} = 0$$

$$\therefore \; 0 = e^b{}_{[i;k]} = e^b{}_{[i,k]} - \cancel{\Gamma^j{}_{[ki]}} \, e^b{}_j$$

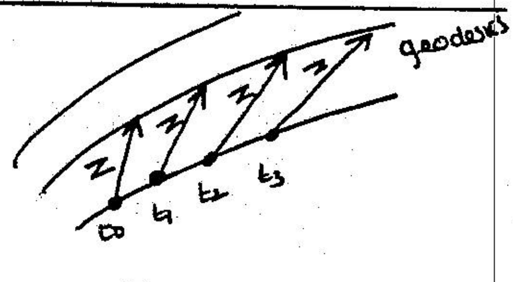
0 since Torsion tensor zero $(T \sim \Gamma^i{}_{[jk]})$

$$= e^b{}_{[i,k]}.$$

It follows that $e^b{}_i$ is the gradient of a function x^b say, so that $\omega^b = dx^b$. It follows that $e_b = \partial/\partial x^b$ and bases (e_a) and (ω^b) are coordinate induced. But by construction the components of the connection with respect to (e_a), (ω^b) vanish, which proves result.

If the torsion vanishes there is another geometrical interpretation of R which will prove of considerable significance later. First consider 2 coplanar geodesics in R^n (with Euclidean connection) i.e straight lines, with affine parameter t. Let Z_t be straight line joining 2 points of the same t. Now as t varies the direction and length of Z_t will change linearly in it. If the geodesics are not coplanar Z_t will also be rotated through an angle which varies linearly in t. In other words Z acquires a constant 'velocity', or the 'acceleration' vanishes, when t is thought of as a time coordinate. These concepts can be carried over to a general Affine manifold, leading to the idea of geodesic deviation (see note).

Let M be a manifold with a symmetric linear connection i.e $T = 0$. Let X be a geodesic vector field i.e $\nabla_X X = 0$. Let Z be a vector field defined along some integral curve of X satisfying $[X, Z] = 0$. From obvious geometrical interpretation Z will be called the <u>connecting vector field</u>.



$\nabla_X X = 0$
and
$[X, Z] = 0$ by construction.

[Note: defⁿ of scalar product of 2 vectors $(X \circ Y) = g(X,Y)$].

If $a(X,Y) = g(X,Y) = 0$, X, Y are orthogonal.

If (e_a) is a basis for T_p, the components of g are $g_{ab} = g(e_a, e_b)$

Example.

Consider R^2. If (x,y) are the usual cartesian coords then $\partial/\partial x$, $\partial/\partial y$ are usually required to have length 1 and to be orthogonal. Thus $g_{xx} = g_{yy} = 1$ $g_{xy} = g_{yx} = 0$. The magnitude of the vector with components (X^i) is

$$d(X) = (X^{x^2} + X^{y^2})^{1/2}$$

and the angle between the vectors X, Y, is

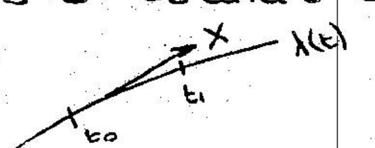
$$a(X,Y) = (X^x Y^x + X^y Y^y) / d(X) d(Y) \quad \text{as usual.}$$

In polar coords (r, θ)

$$g_{rr} = 1 \quad g_{\theta\theta} = r^2 \quad g_{r\theta} = 0 .$$

Let $\lambda(t)$ be a smooth curve with tangent vector X. The length of the curve between points $t = t_0$ and $t = t_1$ is

$$\int_{t_0}^{t_1} d(X)\, dt .$$



g nondegen. iff matrix of coefs g_{ab} is non sing.

If $g(X,Y) = 0$ for all Y in T_p implies that $X = 0$ the metric is said to be non-degenerate. Hence forth we shall always assume g is non degenerate.

Then there is a unique $(2,0)$ tensor with components g^{ab} given by

$$g^{ab} g_{bc} = \delta^a{}_c$$

(this exists since g_{ab} non-singular, i.e. the matrix g^{ab} is the inverse of matrix g_{ab}, and it follows that g^{ab} non singular and induced g' nondegenerate)

Now, since both g_{ab} and g^{ab} are non-degenerate they define isomorphisms $T_p \leftrightarrow T_p{}^*$

$$X \leftrightarrow Y, \quad X^a = g^{ab} Y_b \quad Y_a = g_{ab} X^b .$$

(Thus we have a (1,1) correspondance between vectors & 1-forms. Similarly there are isomorphisms between all tensor spaces $T_p{}^r{}_s$ for which $r+s$ = constant, i.e. for $r+s = 2$

$$T^{ab} \leftrightarrow T^a{}_b \leftrightarrow T_a{}^b \leftrightarrow T_{ab} ; \quad T^{ab} = g^{bc} T^a{}_c , \quad T^a{}_b = g^{ac} g_{bd} T_c{}^d \text{ etc.}$$

There is another identity involving the Riemann tensor, the <u>Bianchi Identities</u>

$$\text{if} \quad T^i{}_{jk} = 0 \qquad R^i{}_{j[kl;m]} = 0$$

A direct proof is lengthy. But, instead, we argue as follows. Since $R^i{}_{j[kl;m]}$ represent the components of a ~~vector~~ tensor, they will vanish with respect to all coord systems if they vanish in any one coord system. Now let p be any pt. in M, and choose a normal coord system so that $\Gamma^i{}_{(jk)} = 0$ (by def^n). Now if $T^i{}_{[jk]} = 0$ $\Gamma^i{}_{[jk]} = 0$ and

$$\Gamma^i{}_{jk} = \Gamma^i{}_{(jk)} + \Gamma^i{}_{[jk]} = 0$$

Hence

$$R^i{}_{jkl} = \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l}$$

and

$$R^i{}_{jkl;m} = \Gamma^i{}_{lj,km} - \Gamma^i{}_{kj,lm}$$

def^n of $R^a{}_{bcd}$ in terms of $\Gamma^m{}_{np}$ and $\Gamma^m{}_{np} = 0$ at a pt [$\Gamma^m{}_{np,q}$ not nec. zero]

Antisymmetrising gives result $R^i{}_{j[kl;m]} = 0$ at a point. Note the usefulness of ~~computing~~ verifying tensor eq^{n}s by computing them in special coordinate systems. A final def^n is needed.

<u>Def^n 1.14.</u> The <u>Ricci Tensor</u> is a (0,2) tensor whose components R_{ab} are found by contracting those of the Rieman tensor

$$R_{ab} = R^c{}_{acb}$$

<u>1.7. Pseudo-Riemannian geometry</u>

The final structure to be introduced on the manifold is metric structure. It will be shown that a metric structure automatically induces affine structure, so that it is the most special structure so far.

<u>Def^n 1.15.</u> A <u>metric tensor g</u> at a point p in M is a symmetric (0,2) tensor. It assigns a ~~positive~~ <u>magnitude</u> $(g(X,X))^{1/2}$ to each vector X in T_p, denoted by $d(X)$, and defines the <u>angle</u> $a(X,Y)$ between any 2 vectors X, Y in T_p such that $(d(X)d(Y) \neq 0)$

$$a(X,Y) = \cos^{-1}\left\{\frac{g(X,Y)}{d(X)\,d(Y)}\right\}$$

It ~~has~~ already been stated that metric structure is more specialised than affine structure. This is because there is a unique torsion free (i.e symmetric) linear connection on M induced by g through the condition that

$$\nabla g = 0$$

$$\therefore \text{ i.e } \nabla_a g_{bc} = 0 \quad \text{i.e } g_{bc;a} = 0$$

Such a connection, the metric or Levi-Civita Connection has the property that if the vector fields Y, Z are parallely transported along an integral curve of the vector field X, then $g(Y,Z)$ is constant along the curve (so in particular magnitudes of vectors invariant under parallel transport). Thus every metric induces a metric connection. It is not true however that every ~~metric~~ connection is a metric connection for some metric.

The proof that a metric connection exists and is unique relies on assuming that it exists and then showing how it is explicitely determined from g. Thus suppose ∇ is a metric connection.

Let X, Y, Z be arbitary vector fields.

(i) $\nabla g = 0$

(ii) Consider $g(X,Y)$ = scalar - which is a 0-form

$$\nabla_X(g(X,Y)) = d(g(X,Y))(X) \quad \text{by def}^n$$

$$= X(g(X,Y)).$$

$$\therefore \quad X(g(Y,Z)) \overset{\text{by (ii)}}{=} \nabla_X(g(Y,Z)) = \overset{0 \text{ by (i)}}{\cancel{\nabla_X g}}(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

$\therefore$ Similarly

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$$

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

Adding first two eq^{n}s and subtracting third gives and using $\nabla_X Y - \nabla_Y X = [X,Y]$

$$g(Z, \nabla_X Y) = \tfrac{1}{2}\Big\{ -Z(g(X,Y)) + Y(g(Z,X)) + X(g(Y,Z)) + g(Z,[X,Y]) + g(Y,[Z,X]) - g(X,[Y,Z]) \Big\} \quad (*)$$

(13)

Hence given a metric, $T^{ab}, T^a{}_b, T_a{}^b$ and T_{ab} are equivalent (although in practice we may have more than one metric and care would then be needed). So in general we shall regard such associated covariant and contravariant tensors as representations of the same geometric object (and in particular $g_{ab}, \delta_a{}^b, g^{ab}$ may be thought of as representations of the same geometric object g).

The signature of g at p is the number of positive eigenvalues at p − number of negative eigenvalues. If g is non degenerate and continuous, the signature will be constant on M. Since g_{ab} is symmetric, by Linear Algebra $\exists$ a coord system basis in which

$$g_{ab} = \mathrm{diag}(\underbrace{+1,+1,\ldots +1}_{r},\underbrace{-1,\ldots,-1}_{s}) \quad \sim \text{canonical form}$$

(and so $\operatorname{sign} g = r-s$)
[In this case basis vectors form an orthonormal set at p].

If in canonical form

$$g_{ab} = \mathrm{diag}\{+1,\ldots,+1\} = n \qquad \text{and } d(X)=0 \Rightarrow X=0$$

then the metric is positive definate. (metric in topological sense)

Of more interest in relativity theory; if in canonical form

$$g_{ab} = \mathrm{diag}(-1,+1,\ldots,+1) = n-2.$$

are the Lorentz or pseudo-Riemannian metric with sig $n-2$. It is no longer true that $d(X)=0$ implies $X=0$.

In special Relativity we have Lorentz metric η_{ij} where

$$\eta_{ij} = \mathrm{diag}(-1,+1,+1,+1) \quad \text{has sign } 2.$$

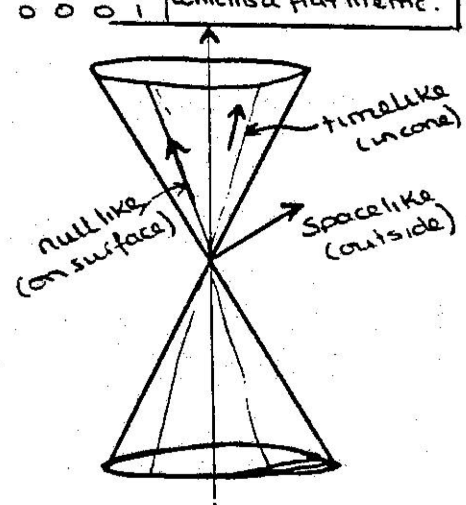
$$\text{In SR} \quad g_{ij} = \eta_{ij} = \begin{array}{c|cccc|} & & \beta\rightarrow & & \\ \alpha\downarrow & -1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

~ Lorentz metric which is a flat metric.

Indeed vectors are classified according to the sign of $g(X,X)$ when Lorentz metric;

$0 < g(X,X)$ spacelike vector
$0 = g(X,X)$ null vector
$0 > g(X,X)$ timelike vector

The null vectors form a double cone in T_p which seperates the timelike from spacelike vectors.



null cone
defined by Lorentz metric

Now if (e_a) is a vector basis, setting $\left.\begin{matrix} Z = e_a \\ X = e_b \\ Y = e_c \end{matrix}\right\}$ (**) gives

$$\Gamma_{abc} = g(e_a, \nabla_b e_c) = g_{ad}\Gamma^d{}_{bc}$$

By def^n $g(e_a, \nabla_b e_c) = g(e_a, \Gamma^d{}_{bc} e_d)$

$$= g(e_a, e_d)\Gamma^d{}_{bc} = g_{ad}\Gamma^d{}_{bc}.$$

So we can express $\Gamma^d{}_{bc}$, the connection coefs in terms of g_{ad} and $g(e_a, \nabla_b e_c)$. Also $\Gamma_{abc} = g_{ad}\Gamma^d{}_{bc}$ ('lowering & raising' indices) and has same geometric interpretation, hence we express Γ_{abc} in terms of $g(e_a, \nabla_b e_c)$

$$\therefore \quad \Gamma_{abc} = g(e_a, \nabla_b e_c)$$

by (*) & (**) $$= \tfrac{1}{2}\{e_c g_{ab} + e_b g_{ca} - e_a g_{bc} + \gamma^d{}_{bc} g_{ad} + \gamma^d{}_{ab} g_{cd} - \gamma^d{}_{ca} g_{bd}\}$$

[note $\gamma^a{}_{bc}$ commutator coefs $[e_a, e_b] = \gamma^a{}_{bc} e_c$]

If the basis is metric induced then the last 3 terms vanish, and

$$\Gamma_{ijk} = \tfrac{1}{2}\left(g_{ij,k} + g_{ki,j} - g_{jk,i}\right)$$

are the coord comps of connection. If M possesses a metric it will henceforth be assumed that any connection used is the metric one, without explicit reference to this fact being made.

If M has a metric structure the Riemann tensor possesses further symmetry's. It has already been shown that

$$R^a{}_{bcd} = R^a{}_{b[cd]} \qquad (1)$$

Always true.

(since $R^a{}_{b(cd)} = 0$)

and we have the Bianchi Identities

$$R^a{}_{b[cd;e]} = 0 \qquad (2)$$

Now let $R_{abcd} = g_{ae} R^e{}_{bcd}$.
Using Riemannian coords it follows that

$$R_{ijkl} = \Gamma_{ilj,k} - \Gamma_{ikj,l}$$

$$= \tfrac{1}{2}\left(g_{il,jk} + g_{kj,il} - g_{lj,ik} - g_{ik,jl} \right)$$

In this coord system $R_{(ij)kl} = 0$

$$R_{ijkl} = R_{klij}$$

Since these are tensor eq^{n}s they must be valid in all basis systems.

Hence

$$\left.\begin{aligned} R_{abcd} &= R_{[ab]cd} \\ R_{abcd} &= R_{cdab} \end{aligned}\right\} (3)$$

only true for metric connection.

Def^n 1.16

The Ricci Scalar is $R = g^{ab} R_{ab} = R^a{}_a$ the contraction of the Ricci tensor.

The Einstein tensor is given by

$$G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab}.$$

Ex 1.20.

$$\nabla G = 0$$
$$\nabla_a G^a{}_b = 0$$
$$\nabla_a R^a{}_b - \tfrac{1}{2} e_b(R) = 0.$$

(contracted Bianchi identity $R_{ij}{}^{ij} = \tfrac{1}{2} R^{ji}$)