3. RIEMANNIAN GEOMETRY AND TENSOR ANALYSIS.

3.1. Euclidean n-dimensional spaces.

A Euclidean n-space, E_n, is such that given n variables $x^1, \dots, x^n$, any set of particular values of these variable is regarded as a point in a manifold of n dimensions, and if x^i, $x^i + dx^i$ $(i = 1, \dots, n)$ are co-ordinates of two neighbouring points and these co-ordinates are rectangular cartesian, then the distance between the two points is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \dots + (dx^n)^2. \tag{3.1}$$

For example, an E_3 is such that

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{3.2}$$

If curvilinear co-ordinates are used the expressions (3.1), (3.2) are less simple. In spherical polar co-ordinates (3.2) becomes

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{3.3}$$

If in E_3 we define general curvilinear co-ordinates by

$$u^1 = u^1(x, y, z), \quad u^2 = u^2(x, y, z), \quad u^3 = u^3(x, y, z),$$

which are one-to-one transformations with non-vanishing Jacobian, then (u^1, u^2, u^3) can be used as co-ordinates in E_3. The inverse transformation is

$$x = x(u^1, u^2, u^3), \quad y = y(u^1, u^2, u^3), \quad z = z(u^1, u^2, u^3).$$

If two neighbouring points have rectangular cartesian co-ordinates (x, y, z) and $(x+dx, y+dy, z+dz)$ respectively, and have curvilinear co-ordinates (u^1, u^2, u^3) and $(u^1+du^1, u^2+du^2, u^3+du^3)$, then we have

$$dx = \frac{\partial x}{\partial u^1} du^1 + \frac{\partial x}{\partial u^2} du^2 + \frac{\partial x}{\partial u^3} du^3, \text{ etc.}$$

so that, from (3.2)

$$ds^2 = A\,du^{1\,2} + B\,du^{2\,2} + C\,du^{3\,2} + 2D\,du^2 du^3 + 2E\,du^3 du^1 + 2F\,du^1 du^2, \tag{3.4}$$

where, in general, A, B, C, D, E, F are functions of (u^1, u^2, u^3). Hence, in terms of curvilinear co-ordinates, an E_n is such that

$$ds^2 = g_{ij}\, du^i du^j, \tag{3.5}$$

where the coefficients g_{ij} are, in general, functions of the curvilinear co-ordinates u^i.

3.2 Generalized n-dimensional spaces.

Since the expression (3.5) refers to an E_n, it is possible to transform the curvilinear co-ordinates u^i to cartesian coordinates x^i so that

$$ds^2 = dx^i dx^i = (dx^1)^2 + \cdots + (dx^n)^2,$$

and, in order for this to be so, the g_{ij} must satisfy certain conditions. We may extend the theory of Euclidean spaces by considering spaces in which the 'distance' ds is given by (3.5) but in which the g_{ij} do not satisfy the Euclidean conditions. Such a space is called a Riemannian space of n-dimensions and is denoted by V_n. An E_n is a particular type of V_n. The expression (3.5) is called the metric of the V_n.

Consider the surface of a sphere of radius a. The co-ordinates of a point on the surface are θ and ϕ and this surface is a two-dimensional space. The distance between two neighbouring points on the surface is obtained by putting $r = a$ in the expression (3.3), i.e.

$$ds^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{3.6}$$

In this case $g_{11} = a^2$, $g_{12} = g_{21} = 0$, $g_{22} = a^2\sin^2\theta$. It is not possible to find co-ordinates (x, y) such that (3.6) takes the form $ds^2 = dx^2 + dy^2$, so this surface is a V_2, but not an E_2.

If a surface is developable, such as a cylinder or a cone, then it is Euclidean.

We can generalize still further by considering spaces in which ds^2 is not expressible as a quadratic differential form and also by considering completely general spaces in which ds, i.e. the distance between two neighbouring points, is not defined. Such a space is called an L_n and we now deal with tensors in these spaces.

3.3 Tensor algebra.

Consider an L_n with co-ordinates x^i $(i = 1, \ldots, n)$, and consider another co-ordinate system $x^{i'}$, where

$$x^{i'} = x^{i'}(x^i) \tag{3.7}$$

If P has co-ordinates $x^i, x^{i'}$ in the two systems and a neighbouring point Q has co-ordinates $x^i + dx^i$, $x^{i'} + dx^{i'}$, then

$$dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i, \tag{3.8}$$

where it is assumed that the co-ordinate transformation (3.7) and its inverse

$$x^i = x^i(x^{i'}) \tag{3.9}$$

are differentiable and have non-zero Jacobian. Note that co-ordinate transformations may be considered as a relabeling of a point in an L_n or as a mapping of a point in one L_n onto a point in another L_n.

If the transformations are linear, the quantities $\frac{\partial x^{i'}}{\partial x^i}$ and $\frac{\partial x^i}{\partial x^{i'}}$ are constants. However, in general, these quantities are functions of the x^i and $x^{i'}$, respectively, and the values of expressions involving these quantities vary from point to point.

<u>Definitions</u>.

(i) A set of n components A^i transforming according to the expression

$$A^{i'} = \frac{\partial x^{i'}}{\partial x^i} A^i \tag{3.10}$$

is a <u>contravariant vector</u>, or a contravariant tensor of rank one, or a $(1,0)$ tensor.

(ii) A set of n components A_i transforming according to the expression

$$A_{i'} = \frac{\partial x^i}{\partial x^{i'}} A_i \tag{3.11}$$

is a <u>covariant vector</u>, or a covariant tensor of rank one, or a $(0,1)$ tensor.

(iii) A set of n^r components $T^{i_1 i_2 \ldots i_r}$ transforming according to the law

$$T^{i_1' i_2' \ldots i_r'} = \frac{\partial x^{i_1'}}{\partial x^{i_1}} \cdots \frac{\partial x^{i_r'}}{\partial x^{i_r}} T^{i_1 i_2 \ldots i_r} \tag{3.12}$$

is a <u>contravariant tensor of rank r</u>, or a $(r,0)$ tensor.

(iv) A set of n^r components $T_{i_1 i_2 \ldots i_r}$ transforming according to the law

$$T_{i_1' i_2' \ldots i_r'} = \frac{\partial x^{i_1}}{\partial x^{i_1'}} \cdots \frac{\partial x^{i_r}}{\partial x^{i_r'}} T_{i_1 \ldots i_r} \tag{3.13}$$

is a <u>covariant tensor of rank r</u>, or a $(0,r)$ tensor.

(v) A set of n^r components $T^{i_1 \dots i_a}_{j_1 \dots j_b}$ $(a+b=r)$ transforming according to the law

$$T^{i_1' \dots i_a'}_{j_1' \dots j_b'} = \frac{\partial x^{i_1'}}{\partial x^{i_1}} \cdots \frac{\partial x^{i_a'}}{\partial x^{i_a}} \frac{\partial x^{j_1}}{\partial x^{j_1'}} \cdots \frac{\partial x^{j_b}}{\partial x^{j_b'}} T^{i_1 \dots i_a}_{j_1 \dots j_b} \qquad (3.14)$$

is a <u>mixed tensor of rank r</u>, or a (a,b) tensor.

(vi) A quantity ϕ which remains unchanged during the co-ordinate transformation, i.e.

$$\phi' = \phi \qquad (3.15)$$

is called a <u>scalar</u> or <u>invariant</u> in L_n.

<u>Examples of tensors.</u>

(a) From (3.8) the coordinate differentials dx^i satisfy the condition (3.10) are so are the components of a contravariant vector. Note that the co-ordinates x^i are not the components of a vector, since the equation $x^{i'} = \frac{\partial x^{i'}}{\partial x^i} x^i$ holds only when the transformations are linear and homogeneous. In this case the x^i are the components of a restricted contravariant vector.

(b) If ϕ is a scalar, then the quantities $\frac{\partial \phi}{\partial x^i}$ satisfy the equations

$$\frac{\partial \phi}{\partial x^{i'}} = \frac{\partial \phi}{\partial x^i} \frac{\partial x^i}{\partial x^{i'}},$$

so that $\frac{\partial \phi}{\partial x^i}$ transforms like a covariant vector. It is called the <u>gradient</u> of ϕ and is written $\phi_{,i}$ thus showing its covariant character.

(c) The Kronecker delta, δ^i_j, is defined by

$$\delta^i_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \qquad (3.16)$$

Now $\delta^i_j = \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{i'}}{\partial x^j} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} \delta^{i'}_{j'}$, so δ^i_j is a mixed tensor of rank two, i.e. a $(1,1)$ tensor, as indicated by the suffixes.

If the components of a tensor A^{ijk} satisfy $A^{ijk} = A^{jik}$, then A^{ijk} is said to be <u>symmetric</u> in the suffixes i and j. If $A^{ijk} = -A^{ikj}$, the tensor is said to be <u>antisymmetric</u> (or <u>skewsymmetric</u>) in the suffixes j and k. It is easily shown that the symmetry properties of a tensor are preserved under transformations as long as the pair of suffixes are either both subscripts or both superscripts. In general, the symmetry properties are not retained if one index is a subscript and one is a superscript. The Kronecker delta is an exception.

A second-rank tensor T^{ij} may be written in the form

$$T^{ij} = \tfrac{1}{2}\left[T^{ij} + T^{ji}\right] + \tfrac{1}{2}\left[T^{ij} - T^{ji}\right], \qquad (3.17)$$

which shows that any such tensor may be regarded as the sum of a part which is symmetric and a part which is antisymmetric in a given pair of suffixes. We introduce the notation

$$T_{k(ij)k\ldots} = \tfrac{1}{2}\left[T_{kijk\ldots} + T_{kjik\ldots}\right], \qquad (3.18)$$
$$T_{k[ij]k\ldots} = \tfrac{1}{2}\left[T_{kijk\ldots} - T_{kjik\ldots}\right],$$

for the symmetric and antisymmetric parts of a tensor $T_{kijk\ldots}$ with respect to a pair of suffixes. The symmetric and antisymmetric parts with respect to non-adjacent suffixes are denoted by

$$T_{(k|i|j)k\ldots} = \tfrac{1}{2}\left[T_{kijk} + T_{jikk}\right], \text{ etc.}$$

3.4 Operations with tensors.

(i) Addition and subtraction.

If $A^{i\ldots j}_{k\ldots l}$ and $B^{i\ldots j}_{k\ldots l}$ are two tensors of the same type, i.e. both (m,n) tensor, then their sum or difference

$$A^{i\ldots j}_{k\ldots l} \pm B^{i\ldots j}_{k\ldots l} = C^{i\ldots j}_{k\ldots l}$$

is also a tensor of the same type. For example, consider

$$\begin{aligned} C^{i'j'} &= A^{i'j'} \pm B^{i'j'} \\ &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}\left(A^{ij} \pm B^{ij}\right) \\ &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}C^{ij}. \end{aligned}$$

Note that the tensors must be added or subtracted at the same point, otherwise $\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}$ will not have the same value for A^{ij} as for B^{ij}, and so is not a common factor.

(ii) Outer multiplication.

Two tensors, say $A^{ij}{}_k$ and $B_{lm}{}^{np}$, can be multiplied together to give a new tensor $C^{ijnp}_{klm} = A^{ij}{}_k B_{lm}{}^{np}$, whose rank is the sum of the ranks of the two tensors and which has the same number of subscripts and superscripts. It is easily shown that C^{ijnp}_{klm} transforms as such a tensor since

$$\begin{aligned} C^{i'j'n'p'}_{k'l'm'} = A^{i'j'}{}_{k'} B_{l'm'}{}^{n'p'} &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^k}{\partial x^{k'}}A^{ij}{}_k \frac{\partial x^l}{\partial x^{l'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{n'}}{\partial x^n}\frac{\partial x^{p'}}{\partial x^p}B_{lm}{}^{np} \\ &= \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^l}{\partial x^{l'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{n'}}{\partial x^n}\frac{\partial x^{p'}}{\partial x^p}A^{ij}{}_k B_{lm}{}^{np} \end{aligned}$$

$$\text{i.e. } C^{i'j'n'p'}_{k'l'm'} = \left(\qquad \text{"} \qquad\right) C^{ijnp}_{klm}$$

(iii) Inner multiplication.

If the above tensors are multiplied together but with summation over a pair of suffixes, the result is a tensor of lower rank. For example, $A^{ij}{}_{k}B_{im}{}^{np} = C^{jnp}_{km}$ is a tensor of rank 5.

The inner product of two vectors, A^i and B_j, is a scalar because

$$A^{i'}B_{i'} = \frac{\partial x^{i'}}{\partial x^i}A^i\frac{\partial x^j}{\partial x^{i'}}B_j = \delta_i{}^j A^i B_j = A^i B_i = A^j B_j,$$

so that $A^iB_i = \phi$, the scalar product of the two vectors.

(iv) Contraction.

A mixed tensor such as T^{ij}_k may be contracted by setting a subscript and a superscript as the same letter, e.g. $T^{i}_{i}{}^{j}$. This gives a quantity which as n components, instead of n^3 components, and transforms as a vector, i.e.

$$T^{j'} \equiv T^{i'}_{i'}{}^{j'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^k}{\partial x^{i'}}T^{ij}_k$$
$$= \delta_i^k\frac{\partial x^{j'}}{\partial x^j}T^{ij}_k$$
$$= \frac{\partial x^{j'}}{\partial x^j}T^{i}_{i}{}^{j} = \frac{\partial x^{j'}}{\partial x^j}T^j$$

It follows that a contracted tensor is of rank two less than that of the original tensor, i.e. a (m,n) tensor contracts to a $(m-1,n-1)$ tensor. Note that the tensor resulting from inner multiplication is the contraction of the corresponding outer multiple.

(v) Quotient Theorem.

This allows us to test whether or not a set of quantities form the components of a tensor.

Theorem. A set of symbols, whose product (inner or outer) with an arbitrary tensor is a tensor, forms a tensor.

Let A^i_j be the set of symbols and B^k an arbitrary tensor, and suppose that

$$A^i_j B^j = C^i,$$

where C^i is a tensor. We have to show that A^i_j is a tensor.

Transforming to a new frame, we have

$$A^{i'}_{j'}B^{j'} = C^{i'}.$$

Since B^j, C^i are tensors, we have

$$B^{j'} = \frac{\partial x^{j'}}{\partial x^j}B^j, \quad C^{i'} = \frac{\partial x^{i'}}{\partial x^i}C^i,$$

and thus

$$A^{i'}_{\ j'}\frac{\partial x^{j'}}{\partial x^j}B^j = \frac{\partial x^{i'}}{\partial x^i}C^i$$
$$= \frac{\partial x^{i'}}{\partial x^i}A^i_{\ j}B^j$$
$$\therefore \left(\frac{\partial x^{j'}}{\partial x^j}A^{i'}_{\ j'} - \frac{\partial x^{i'}}{\partial x^i}A^i_{\ j}\right)B^j = 0$$

Since B^j is arbitrary we can choose its components to be $(1,0,0,\dots)$, and then $(0,1,0,\dots)$, and so on. This shows that the quantity in the bracket is identically zero, i.e.

$$\frac{\partial x^{j'}}{\partial x^j}A^{i'}_{\ j'} = \frac{\partial x^{i'}}{\partial x^i}A^i_{\ j}.$$

Inner multiplication by $\frac{\partial x^j}{\partial x^{k'}}$ gives

$$A^{i'}_{\ k'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{k'}}A^i_{\ j},$$

so that $A^i_{\ j}$ is a tensor of the type indicated by its suffixes.

(vi) <u>Conjugate tensors</u>.

Consider a symmetric covariant tensor g_{ij} with determinant $g \equiv |g_{ij}| \neq 0$. Let G^{ij} be the cofactor of the element g_{ij} in the determinant. We shall now show that $g^{ij} = G^{ij}/g$ is a symmetric covariant tensor.

From the theory of determinants we have

$$g_{ij}G^{kj} = g\delta_i^{\ k}, \quad g_{ij}G^{ik} = g\delta_j^{\ k} \qquad (3.19)$$

so that, dividing by g, we find

$$g_{ij}g^{kj} = \delta_i^{\ k}, \quad g_{ij}g^{ik} = \delta_j^{\ k} \qquad (3.20)$$

Choose an arbitrary contravariant vector A^i, then $B_i = g_{ij}A^j$ is also an arbitrary vector, since $g \neq 0$ implies that the components of A^i can always be calculated from this last equation. Now we have

$$g^{ij}B_i = g^{ij}g_{ik}A^k = \delta^j_{\ k}A^k = A^j,$$

using equation (3.20). Hence, by the quotient theorem, g^{ij} is a contravariant tensor, and it follows that it is symmetric like g_{ij}. The tensors g_{ij}, g^{ij} are said to be conjugate to one another.

(vii). <u>Tensor equations</u>.

It follows from the transformation laws that if all components of a given tensor vanish in one co-ordinate system, then all components vanish in all co-ordinate systems. Hence, if a physical law is expressed in tensor form by saying that one tensor equals another, then, since the difference of the tensors is zero, it follows that the tensors are equal in any other co-ordinate system so that the validity of the law is independent of the co-ordinate system employed. Also if a tensor equation is established in a special co-ordinate system, then it is valid in general.

3.5. Tensor densities.

Given a transformation of co-ordinates

$$x^{i'} = x^{i'}(x^i),$$

we denote by J the Jacobian determinant of the transformation, i.e. $J = \left|\frac{\partial x^i}{\partial x^{i'}}\right|$.

We define a relative tensor, $\mathfrak{R}^{i\ldots j}_{k\ldots l}$, as a quantity which transforms according to the rule

$$\mathfrak{R}^{i'\ldots j'}_{k'\ldots l'} = J^W \frac{\partial x^{i'}}{\partial x^i} \cdots \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^k}{\partial x^{k'}} \cdots \frac{\partial x^l}{\partial x^{l'}} \mathfrak{R}^{i\ldots j}_{k\ldots l}, \tag{3.21}$$

where W is a positive or negative integer. $\mathfrak{R}$ is said to be a relative tensor of weight W having covariant and contravariant characteristics as indicated by the suffixes.

If $W = 0$, the relative tensor is an ordinary tensor. If $W = 1$, the relative tensor is called a tensor density.

The following results are easily proved:

(i) The addition or subtraction of two relative tensor of the same type and weight results in new relative tensors of the same type and weight.

(ii) The outer product of two relative tensors is a relative tensor whose weight is the sum of the weights of its factors.

(iii) A relative tensor may be contracted, reducing its rank by two, but leaving its weight unaltered.

Note that area and volume elements are tensor densities of zero rank, i.e. scalar densities. For example, in E_3, using cartesian co-ordinates, the volume element dV is

$$dV = dx\,dy\,dz = dx^1 dx^2 dx^3$$

and using spherical polars it is

$$dV' = r^2 \sin\theta\, dr\, d\theta\, d\phi = (x^{1'})^2 \sin x^{2'}\, dx^{1'} dx^{2'} dx^{3'}.$$

In this transformation from $x^{i'}$ to x^i, $J = \left|\frac{\partial x^i}{\partial x^{i'}}\right| = r^2 \sin^2\theta = (x^{1'})^2 \sin u^{2'}$, so the volume elements obey

$$dx^1 dx^2 dx^3 = J\, dx^{1'} dx^{2'} dx^{3'}$$

and so the volume elements are scalar densities.

Consider a contravariant tensor density of rank n, $\varepsilon^{ij\ldots k}$, which is antisymmetric with respect to every suffix. Its components are determined as follows:

$$\begin{aligned} \varepsilon^{ij\ldots k} &= 0, \text{ if any two suffixes are the same,} \\ &= \varepsilon^{12\ldots n}, \text{ if } i,j,\ldots,k \text{ is an even permutation of } 1,2,\ldots,n. \\ &= -\varepsilon^{12\ldots n}, \text{ if } i,j,\ldots,k \text{ is an odd permutation of } 1,2,\ldots,n. \end{aligned}$$

If in some co-ordinate system we take $\varepsilon^{12\ldots n} = +1$, then in the primed system

$$\begin{aligned} \varepsilon^{1'2'\ldots n'} &= J \frac{\partial x^{1'}}{\partial x^i} \frac{\partial x^{2'}}{\partial x^j} \cdots \frac{\partial x^{n'}}{\partial x^k} \varepsilon^{ij\ldots k} \\ &= JK \end{aligned}$$

where K is the determinant with $i'j$-element $\frac{\partial x^{i'}}{\partial x^j}$, i.e. $K = \left|\frac{\partial x^{i'}}{\partial x^j}\right|$. Now $JK = \left|\frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{i'}}{\partial x^j}\right| = \left|\frac{\partial x^i}{\partial x^j}\right| = |\delta^i_j| = 1$, so $\varepsilon^{1'2'\ldots n'} = 1$, and hence it follows that $\varepsilon^{ij\ldots k}$ has the same components $0, \pm 1$ in all frames.

The tensor density $\varepsilon^{ij\ldots k}$ is known as the <u>Levi-Civita tensor density</u>, or the <u>permutation symbol</u>.

The determinants of second-rank tensors are relative invariants. For example, if A_{ij} is a covariant tensor, then

$$\begin{aligned} A_{i'j'} &= \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} A_{ij} \\ \text{i.e. } |A_{i'j'}| &= \left|\frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} A_{ij}\right| \\ &= \left|\frac{\partial x^i}{\partial x^{i'}}\right| \left|\frac{\partial x^j}{\partial x^{j'}}\right| |A_{ij}| \\ \text{i.e. } |A_{i'j'}| &= J^2 |A_{ij}|, \end{aligned} \qquad (3.22)$$

so that $|A_{ij}|$ is a relative invariant of weight 2.

Similarly, the determinant $|A^{ij}|$ of a contravariant tensor is a relative invariant of weight -2, while the determinant $|A^i{}_j|$ of a mixed tensor is an invariant.

3.6 <u>Parallel displacement and covariant differentiation.</u>

In order to have further operations with tensors, the general space L_n must have some structure. In particular, a satisfactory definition of differentiation of tensors is not possible in the basic L_n.

Now the partial derivative of a tensor (other than a scalar) is not a tensor. For example, if A_i is a vector, then

$$A_{i'} = \frac{\partial x^i}{\partial x^{i'}} A_i ,$$

so that

$$\begin{aligned} \frac{\partial A_{i'}}{\partial x^{j'}} &\equiv A_{i',j'} \\ &= \left(\frac{\partial x^i}{\partial x^{i'}} A_i\right)_{,j'} \\ &= \frac{\partial x^i}{\partial x^{i'}} A_{i,j'} + \frac{\partial^2 x^i}{\partial x^{i'} \partial x^{j'}} A_i \end{aligned}$$

$$\text{i.e.} \quad A_{i',j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} A_{i,j} + \frac{\partial^2 x^i}{\partial x^{i'} \partial x^{j'}} A_i , \qquad (3.23)$$

thus showing that $A_{i,j}$ is not a tensor, unless we a restricted to linear transformations.

Let P and Q be neighbouring points with co-ordinates x^i and $x^i + dx^i$ and let A_i and $A_i + dA_i$ [i.e. $A_i(x^i + dx^i) = A_i(x^i) + dA_i$] be the vectors of a covariant vector field associated with P and Q. Since these tensors are associated with different points, their transformation laws will be different, so their difference, dA_i, will not be a vector. However,

$$dA_i = \frac{\partial A_i}{\partial x^j} dx^j , \qquad (3.24)$$

and dx^j is a vector, so $\frac{\partial A_i}{\partial x^j}$ cannot be a tensor, as before. In order to obtain a derivative which is a tensor we must find a procedure which involves comparison of two vectors defined at the same point.

In an E_n, a vector A_i may undergo parallel displacement from a point P to another point Q so that it suffers no change in magnitude and direction. If rectangular cartesian co-ordinates are employed then the displaced vector will have the same components as the original vector, but if curvilinear co-ordinates are used, the components of the displaced vector and the original vector will not have the same components, since the axes at P and Q will not, in general, be in the same directions. In the more general L_n, we say that parallel displacement (in some sense) takes place giving us a displaced vector with components $A_i + \delta A_i$

which can be compared with the field vector $A_i + dA_i$ at the same point Q. Since these are both vectors defined at the same point, their difference $dA_i - \delta A_i$ is a vector. Generalizing (3.24) we write

$$dA_i - \delta A_i = A_{i;j}\, dx^j, \qquad (3.25)$$

where, by the quotient rule, $A_{i;j}$ is a second rank tensor known as the <u>covariant derivative</u> of A_i.

We have defined the tensor derivative of a vector in terms of parallel displacement, but we have yet to define parallel displacement in L_n. The definition that we adopt must correspond to the usual definition in E_n since E_n is a special case of L_n. In E_n, when cartesian co-ordinates are used, the components of a vector A_i remain unchanged and also the scalar product $A_i B^i$ of two vectors must remain unchanged (since it is a scalar) so that the angle between the vectors is constant. In L_n we assume that when A_i undergoes an infinitesimal parallel displacement, its scalar product with an arbitrary vector B^i remains invariant.

To define parallel displacement in L_n, first consider a vector A_i in E_n. Let x^i be curvilinear co-ordinates and let $x^{i'}$ be rectangular cartesian co-ordinates in E_n. Then

$$A_i = \frac{\partial x^{i'}}{\partial x^i} A_{i'} \quad \text{and} \quad A_{i'} = \frac{\partial x^i}{\partial x^{i'}} A_i. \qquad (3.26)$$

If A_i is parallelly displaced from P to Q, the cartesian components $A_{i'}$ do not change, i.e. $\delta A_{i'} = 0$. From (3.26) we have

$$\delta A_i = \delta\left(\frac{\partial x^{i'}}{\partial x^i} A_{i'}\right) = \delta\left(\frac{\partial x^{i'}}{\partial x^i}\right) A_{i'}$$
$$= \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}\, dx^j A_{i'} \qquad (3.27)$$

Substituting for $A_{i'}$, we find

$$\delta A_i = \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial x^{i'}} A_k\, dx^j,$$
$$\text{i.e. } \delta A_i = \Gamma^k_{ij} A_k\, dx^j, \qquad (3.28)$$

where

$$\Gamma^k_{ij} = \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial x^{i'}} \qquad (3.29)$$

Conforming to this, we define δA_i in L_n by (3.28), where the n^3 quantities Γ^k_{ij} are determined arbitrarily at every point of L_n, subject to being continuous functions of x^i and possessing continuous partial derivatives of the necessary order. The quantities Γ^k_{ij} are the components of an <u>affinity</u> which specifies an <u>affine connection</u> between points of L_n.

We now have

$$dA_i - \delta A_i = A_{i;j}\,dx^j = A_{i,j}\,dx^j - \Gamma^k_{ij} A_k\,dx^j,$$

so the covariant derivative is given by

$$A_{i;j} = A_{i,j} - \Gamma^k_{ij} A_k \qquad (3.30)$$

and this is a tensor.

Note that $A_{i;j}$ and $A_{i,j}$ are equal if the components of Γ^k_{ij} vanish over some region of L_n. This will be true only in the reference frame employed; transformation to another frame will, in general, result in non-zero Γ^k_{ij}. Only covariant derivatives can appear in tensor equations which are to be true in all frames.

3.7 <u>Transformation and properties of the affinity.</u>

Although the components of the affinity may be chosen arbitrarily in any other frame are fixed by its transformation law, which we shall now find.

Since $A_{i;j}$ is a tensor we have

$$A_{i';j'} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} A_{i;j}$$

and also

$$A_{i'} = \frac{\partial x^i}{\partial x^{i'}} A_i$$

Now

$$A_{i';j'} = A_{i',j'} - \Gamma^{k'}_{i'j'} A_{k'}$$

so that

$$\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} A_{i;j} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} A_{i,j} + \frac{\partial^2 x^i}{\partial x^{i'}\partial x^{j'}} A_i - \Gamma^{k'}_{i'j'} A_k \frac{\partial x^k}{\partial x^{k'}} \qquad (3.31)$$

But

$$A_{i;j} = A_{i,j} - \Gamma^k_{ij} A_k,$$

$$\text{i.e.}\quad \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} A_{i;j} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} A_{i,j} - \Gamma^k_{ij}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} A_k. \qquad (3.32)$$

Comparing (3.31) and (3.32) we have

$$\Gamma^{k'}_{i'j'}\frac{\partial x^k}{\partial x^{k'}} = \frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}} + \Gamma^k_{ij}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}$$

Multiplying both sides by $\frac{\partial x^{\ell'}}{\partial x^k}$ we obtain

$$\Gamma^{k'}_{i'j'} = \frac{\partial x^{k'}}{\partial x^k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\Gamma^k_{ij} + \frac{\partial x^{k'}}{\partial x^k}\frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}} \qquad (3.33)$$

which is the required transformation law.

From equation (3.33) we see that if the components of Γ^k_{ij} are zero in one frame, they are not necessarily zero in any other frame. In fact, in general, there is no frame in which the components vanish over a region of L_n.

The major properties of an affinity are:

(i) If Γ^k_{ij} is symmetric with respect to i and j in one frame, then it is symmetric in every frame.

(ii) The difference of two affinities, defined over a region of L_n, is a tensor since

$$\Gamma^{k'}_{i'j'} - \bar{\Gamma}^{k'}_{i'j'} = \frac{\partial x^{k'}}{\partial x^k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\left(\Gamma^k_{ij} - \bar{\Gamma}^k_{ij}\right),$$

from (3.33), so that $\Gamma^k_{ij} - \bar{\Gamma}^k_{ij}$ is a tensor of rank three.

(iii) The average $\frac{1}{2}(\Gamma^k_{ij} + \bar{\Gamma}^k_{ij})$ of two affinities is an affinity, from (3.33).

(iv) It follows that we can write

$$\Gamma^k_{ij} = \Gamma^k_{(ij)} + \Gamma^k_{[ij]},$$

so that $\Gamma^k_{[ij]}$ is a tensor and $\Gamma^k_{(ij)}$ is an affinity. If $\Gamma^k_{[ij]}$ is zero in one frame then it is zero in all, giving an alternative proof of (i).

(v) If we are restricted to linear transformations, then Γ^k_{ij} is a tensor, because the last term in (3.33) will then be zero.

(vi) For any affinity we can make $\Gamma^k_{(ij)} = 0$ in at least one co-ordinate system at one point.

<u>Proof</u>. Let the given point be O and let it be the origin of the co-ordinate system so that, at O, $x^i = 0$. Transform to a new co-ordinate system, $x^{i'}$, by the transformation

$$x^i = \delta^i_{i'} x^{i'} + \tfrac{1}{2} a^i_{j'k'} x^{j'} x^{k'} + O[(x^{i'})^3], \qquad (3.34)$$

where the $a^i_{j'k'}$ are constants. Now

$$\frac{\partial x^i}{\partial x^{i'}} = \delta^i_{i'} + a^i_{j'i'} x^{j'} + \dots$$

$$\frac{\partial^2 x^i}{\partial x^{i'}\partial x^{j'}} = a^i_{j'i'} + \dots$$

At O, $x^i = 0$, so

$$\frac{\partial x^i}{\partial x^{i'}} = \delta^i_{i'}, \quad \frac{\partial^2 x^i}{\partial x^{i'}\partial x^{j'}} = a^i_{j'i'}.$$

$$\text{i.e.}\quad \Gamma^{k'}_{(i'j')} = \delta^{k'}_k \delta^i_{i'} \delta^j_{j'} \Gamma^k_{(ij)} + \delta^{k'}_k a^k_{(j'i')},$$

where we note that, from (3.34), $a^{i'}_{j'k'}$ is symmetric. By choosing $a^{k}_{j'i'}$ in the correct way, $\Gamma^{k'}_{(i'j')} = 0$ (at O only), i.e.

$$a^{k}_{(j'i')} = -\delta^{i}_{i'}\delta^{j}_{j'}\Gamma^{k}_{(ij)}.$$

Such a co-ordinate system is called a <u>geodesic co-ordinate system</u> at the point O, which is its <u>pole</u>. In such a system, if Γ^{k}_{ij} is symmetric, the covariant and partial derivatives are identical at O, which is useful since, if a tensor equation is valid in a geodesic system, it is valid in all frames.

3.8. <u>Covariant derivatives of tensors</u>.

For a covariant vector A_i we have

$$A_{i;j} = A_{i,j} - \Gamma^{k}_{ij}A_k. \qquad (3.35)$$

For other types of tensor we find:

(i) <u>Scalar, ϕ</u>. When ϕ undergoes parallel displacement it remains unaltered, i.e. $\delta\phi = 0$ and so

$$\phi_{;i} = \phi_{,i}, \qquad (3.36)$$

which is a vector, as required.

(ii) <u>Contravariant vector B^i</u>. Now A_iB^i is a scalar, so

$$(A_iB^i)_{;j} = (A_iB^i)_{,j}$$

$$\text{i.e. } A_{i;j}B^i + A_iB^i{}_{;j} = A_{i,j}B^i + A_iB^i{}_{,j}$$

$$\text{i.e. } (A_{i;j} - A_{i,j})B^i = -(B^i{}_{;j} - B^i{}_{,j})A_i$$

$$-\Gamma^{k}_{ij}A_kB^i = -(B^i{}_{;j} - B^i{}_{,j})A_i$$

$$\therefore B^i{}_{;j}A^j = (B^i{}_{,j} + \Gamma^{i}_{kj}B^k)A_i$$

Choose A_i to be an arbitrary vector, so

$$B^i{}_{;j} = B^i{}_{,j} + \Gamma^{i}_{kj}B^k, \qquad (3.37)$$

which is the covariant derivative of B^i.

(iii) <u>Tensor D^i_{jk}</u>. This transforms like $A^iB_jC_k$ and we find that

$$D^i_{jk;l} = D^i_{jk,l} + \Gamma^{i}_{ml}D^m_{jk} - \Gamma^{m}_{jl}D^i_{mk} - \Gamma^{m}_{kl}D^i_{jm}. \qquad (3.38)$$

The extension to tensors of all ranks is obvious.

The usual rules apply to covariant differentiation, i.e.

(a) If $T^i_{jk} = A^i_{jk} + B^i_{jk}$, then $T^i_{jk;l} = A^i_{jk;l} + B^i_{jk;l}$. (3.39)

(b) If $T^i = A^i_{jk}B^{jk}$, then $T^i{}_{;l} = A^i_{jk;l}B^{jk} + A^i_{jk}B^{jk}{}_{;l}$ (3.40)

Note that the Kronecker delta is a constant under covariant differentiation, since

$$\delta^i_{j;k} = \delta^i_{j,k} + \Gamma^i_{\ell k}\delta^\ell_j - \Gamma^\ell_{jk}\delta^i_\ell,$$

but $\delta^i_{j,k} = 0$, so

$$\delta^i_{j;k} = \Gamma^i_{jk} - \Gamma^i_{jk} = 0.$$

3.9 Example.

Consider an E_2 with plane polar co-ordinates (r,θ). Take cartesian co-ordinates (x,y) as the unprimed system and (r,θ) as the primed system, i.e. $(x,y) \equiv (x^1, x^2)$; $(r,\theta) \equiv (x^{1'}, x^{2'})$.

$$x = r\cos\theta, \quad y = r\sin\theta; \quad r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}.$$

For the (x,y) system, $\Gamma^k_{ij} = 0$.

$$\frac{\partial x^{1'}}{\partial x^1} = \cos\theta, \quad \frac{\partial x^{1'}}{\partial x^2} = \sin\theta, \quad \frac{\partial x^{2'}}{\partial x^1} = -\frac{1}{r}\sin\theta, \quad \frac{\partial x^{2'}}{\partial x^2} = \frac{1}{r}\cos\theta$$

$$\frac{\partial^2 x^i}{\partial x^{1'}\partial x^{1'}} = 0, \quad \frac{\partial^2 x^1}{\partial x^{1'}\partial x^{2'}} = -\sin\theta, \quad \frac{\partial^2 x^2}{\partial x^{1'}\partial x^{2'}} = \cos\theta, \quad \frac{\partial^2 x^i}{\partial x^{2'}\partial x^{2'}} = -x^i.$$

Since $\Gamma^k_{ij} = 0$, $\Gamma^{k'}_{i'j'} = \frac{\partial x^{k'}}{\partial x^k}\frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}}$ and is symmetric. Then

$$\Gamma^{i'}_{1'1'} = 0, \quad \Gamma^{1'}_{1'2'} = \cos\theta(-\sin\theta) + \sin\theta(\cos\theta) = 0,$$

$$\Gamma^{1'}_{2'2'} = \cos\theta(-r\cos\theta) + \sin\theta(-r\sin\theta) = -r$$

$$\Gamma^{2'}_{1'2'} = -\frac{1}{r}\sin\theta(-\sin\theta) + \frac{1}{r}\cos\theta(\cos\theta) = \frac{1}{r}$$

$$\Gamma^{2'}_{2'2'} = -\frac{1}{r}\sin\theta(-r\cos\theta) + \frac{1}{r}\cos\theta(-r\sin\theta) = 0.$$

i.e. $\Gamma^{1'}_{2'2'} = -r$ and $\Gamma^{2'}_{1'2'} = \Gamma^{2'}_{2'1'} = \frac{1}{r}$ are the only non-zero Γ's.

3.10. Curl of a covariant vector

Consider a covariant vector A_i. Then $A_{i;j} = A_{i,j} - \Gamma^k_{ij}A_k$. Twice the antisymmetric part of this expression is

$$2A_{[i;j]} = 2A_{[i,j]} - 2\Gamma^k_{[ij]}A_k \qquad (3.41)$$

The first term is the covariant curl, the second term is the ordinary curl $(A_{i,j} - A_{j,i})$ and the third term is a tensor. Hence, the ordinary curl is a tensor. If Γ^k_{ij} is symmetric, the covariant curl and ordinary curl are identical.

3.11. Covariant differentiation of relative tensors.

By using the concept of parallel displacement and requiring that the covariant derivatives of $\varepsilon^{ij\dots k}$ and $\varepsilon_{ij\dots k}$ be identically zero (since they have the same components in all frames and at all points), it can be shown that the covariant derivative of a relative tensor of weight W is

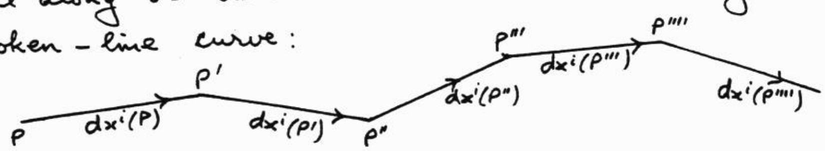
$$\mathfrak{R}^{k}{}_{i;j} = \mathfrak{R}^{k}{}_{i,j} + \Gamma^{k}_{\ell j}\mathfrak{R}^{\ell}{}_{i} - \Gamma^{\ell}_{ij}\mathfrak{R}^{k}{}_{\ell} - W\Gamma^{\ell}_{\ell j}\mathfrak{R}^{k}{}_{i}, \qquad (3.42)$$

and, in particular, for a relative invariant $\mathfrak{R}$,

$$\mathfrak{R}_{;i} = \mathfrak{R}_{,i} - W\Gamma^{\ell}_{\ell i}\mathfrak{R}. \qquad (3.43)$$

3.12. <u>Geodesics.</u>

Starting with some infinitesimal displacement vector, dx^i, at a point P, we may parallel-transport this vector along its own direction to the point P'. This gives a new infinitesimal vector at P' that we can displace along its own direction to P''. Continuing this process we obtain a broken-line curve:



Hence, one can parallel-transport the vector dx^i from any point to some other point. As the size of the displacement tends to zero, the broken line becomes a continuous curve. This curve starts from P with a well-defined direction and continues to another point at a finite distance. Such a curve is called a geodesic.

Now introduce a parameter to designate points along the curve. Let s be chosen to transform as a scalar and to have an invariant value at P and P'. Then $\frac{dx^i}{ds}$ is a vector. The condition for a geodesic is that the components of the vector $\left(\frac{dx^i}{ds}\right)_P$ displaced to a point P' are identical with those of the vector $\left(\frac{dx^i}{ds}\right)_{P'}$, i.e.

$$\left(\frac{dx^i}{ds}\right)_P - \Gamma^i_{jk}\frac{dx^j}{ds}\Delta x^k = \left(\frac{dx^i}{ds}\right)_{P'}, \qquad (3.44)$$

where Δx^k corresponds to dx^k used in equation (3.28), for example. Dividing this equation by Δs, the displacement from P to P', and taking the limit as $\Delta s \to 0$:

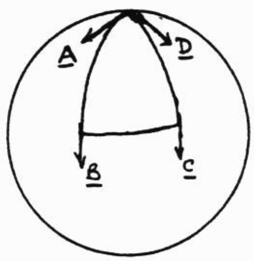
$$\lim_{\Delta s \to 0}\left\{\frac{(dx^i/ds)_{P'} - (dx^i/ds)_P}{\Delta s} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{\Delta x^k}{\Delta s}\right\} = 0$$

$$\text{i.e.} \quad \frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \qquad (3.45)$$

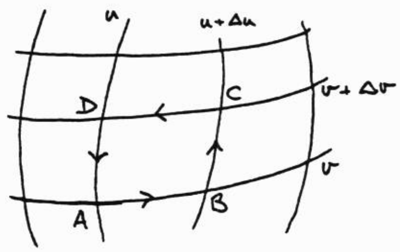
This is the equation for a geodesic in terms of a scalar parameter s for a curve. Note that, in E_n using cartesian co-ordinates, (3.45) becomes $\frac{d^2x^i}{ds^2} = 0$, which is the equation for straight lines. Hence, in E_n geodesics are straight lines.

3.13. The Riemann – Christoffel curvature tensor.

If a vector undergoes parallel transport about a closed path in E_n, then the resultant vector is identical with the original one. However, in general, this is not true in a curved space. For example, consider the surface of a sphere on which there is a spherical triangle composed of geodesic curves, i.e. two lines of longitude and the equator. When $\underline{A}$ is parallel-transported it finally becomes $\underline{D} \neq \underline{A}$.



Consider the change in the components of a vector under parallel displacement about an infinestimal closed path defined by four curves of the two parameter set
$$x^i = f^i(u,v).$$



Let the path pass through the four points

$A(u,v)$; $B(u+\Delta u, v)$; $C(u+\Delta u, v+\Delta v)$; $D(u, v+\Delta v)$,

so that ABCDA is a complete circuit. Now the absolute change in a vector A_i due to parallel displacement through dx^j is $\delta A_i = dA_i - A_{i;j}dx^j$. But the sum of dA_i round the complete circuit is zero, so we ignore it in what follows. As we go round the circuit we find:

(i) from A to B, the absolute change is $-A_{i;j}\frac{\partial x^j}{\partial u}\Delta u$, calculated for v,

(ii) from B to C, the absolute change is $-A_{i;j}\frac{\partial x^j}{\partial v}\Delta v$, calculated for $u+\Delta u$,

(iii) from C to D, the absolute change is $+A_{i;j}\frac{\partial x^j}{\partial u}\Delta u$, calculated for $v+\Delta v$,

(iv) from D to A, the absolute change is $+A_{i;j}\frac{\partial x^j}{\partial v}\Delta v$, calculated for u.

From (ii) and (iv) the net result is the difference of the changes $A_{i;j}\frac{\partial x^j}{\partial v}\Delta v$ at $u+\Delta u$ and at u, respectively. We require not $\frac{\partial}{\partial u}(A_{i;j}\frac{\partial x^j}{\partial v}\Delta v)\Delta u$, but the absolute difference
$$-\left(A_{i;j}\frac{\partial x^j}{\partial v}\Delta v\right)_{;k}\frac{\partial x^k}{\partial u}\Delta u$$

$$\text{i.e.} \quad -A_{i;jk}\frac{\partial x^j}{\partial v}\frac{\partial x^k}{\partial u}\Delta u\Delta v, \qquad (3.46)$$

since $dx^j = \frac{\partial x^j}{\partial v}\Delta v$ is the same for (ii) and (iv).

Similarly, (i) and (iii) give

$$+A_{i;jk}\frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}\Delta u\Delta v,$$

so that the total absolute change round the circuit is

$$\Delta A_i = (A_{i;jk} - A_{i;kj})\frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}\Delta u\Delta v \qquad (3.47)$$

In a Euclidean space, the covariant and partial derivatives are identical and (3.47) would be zero, showing that there is no absolute change in the vector on parallel displacement round a circuit. However, in general, we have $A_{i;jk} \neq A_{i;kj}$, so that A_i will be changed when we reach the initial point. Now

$$A_{i;jk} = (A_{i,j} - \Gamma^l_{ij}A_l)_{,k} + \Gamma^m_{ik}(A_{m,j} - \Gamma^l_{mj}A_l) - \Gamma^m_{jk}(A_{i,m} - \Gamma^l_{im}A_l),$$
$$A_{i;kj} = (A_{i,k} - \Gamma^l_{ik}A_l)_{,j} + \Gamma^m_{ij}(A_{m,k} - \Gamma^l_{mk}A_l) - \Gamma^m_{kj}(A_{i,m} - \Gamma^l_{im}A_l),$$

so that

$$A_{i;jk} - A_{i;kj} = (\Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk})A_l - (\Gamma^m_{jk} - \Gamma^m_{kj})A_{i;m} \qquad (3.48)$$

Since $\Gamma^m_{jk} - \Gamma^m_{kj}$ is a tensor, it follows that the quantity in the first bracket is a tensor. We write

$$A_{i;jk} - A_{i;kj} = R^l{}_{ijk}A_l - (\Gamma^m_{jk} - \Gamma^m_{kj})A_{i;m} \qquad (3.49)$$

where

$$R^l{}_{ijk} = \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^l_{mj}\Gamma^m_{ik} - \Gamma^l_{mk}\Gamma^m_{ij} \qquad (3.50)$$

is a fourth-rank tensor known as the <u>Riemann-Christoffel tensor</u>, or <u>curvature tensor</u>. Note that $R^l{}_{ijk}$ is anti-symmetric in j and k.

If the affinity is symmetric then

$$A_{i;jk} - A_{i;kj} = R^l{}_{ijk}A_l \qquad (3.51)$$

In rectangular cartesian co-ordinates $A_{i;jk} - A_{i;kj} = A_{i,jk} - A_{i,kj} = 0$ so $R^l{}_{ijk} = 0$ for Euclidean space, and, since it is a tensor, it will be zero whatever co-ordinate system is used in this space. The vanishing of $R^l{}_{ijk}$ guarantees that an arbitrary vector will not change under parallel displacement. It follows that $R^l{}_{ijk} = 0$ is a necessary and sufficient condition for the space to be flat (Euclidean) and, on the other hand, it provides a measure of the curvature of non-flat spaces.

From (3.47) we see that the absolute change in A_i round a closed circuit is

$$\Delta A_i = R^l{}_{ijk}A_l\frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}\Delta u\Delta v. \qquad (3.52)$$

Since this expression contains only first order terms in $\Delta u, \Delta v$, it is more precise to write

$$\lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \frac{\Delta A_i}{\Delta u \Delta v} = R^{k}{}_{ijk} A_k \frac{\partial x^j}{\partial u} \frac{\partial x^k}{\partial v} \qquad (3.53)$$

3.14 Properties of the curvature tensor.

(i) If $R^{k}{}_{ijk}$ is contracted with respect to k and j, we obtain the Ricci tensor $R_{ik} \equiv R^{k}{}_{ikk} = -R^{k}{}_{ikk}$, where

$$R_{ik} = \Gamma^{k}_{ik,k} - \Gamma^{k}_{ik,k} + \Gamma^{k}_{mk}\Gamma^{m}_{ik} - \Gamma^{k}_{mk}\Gamma^{m}_{ik}. \qquad (3.54)$$

Contraction of $R^{k}{}_{ijk}$ with respect to k and i produces the tensor S_{jk} given by

$$S_{jk} \equiv R^{k}{}_{kjk} = \Gamma^{k}_{kk,j} - \Gamma^{k}_{kj,k}. \qquad (3.55)$$

(ii) Using (3.50) and cyclically permuting the lower suffixes, we obtain

$$R^{k}{}_{jki} = \Gamma^{k}_{ji,k} - \Gamma^{k}_{jk,i} + \Gamma^{k}_{mk}\Gamma^{m}_{ji} - \Gamma^{k}_{mi}\Gamma^{m}_{jk},$$

$$R^{k}{}_{kij} = \Gamma^{k}_{kj,i} - \Gamma^{k}_{ki,j} + \Gamma^{k}_{mi}\Gamma^{m}_{kj} - \Gamma^{k}_{mj}\Gamma^{m}_{ki},$$

and adding these to (3.50) yields

$$R^{k}{}_{ijk} + R^{k}{}_{jki} + R^{k}{}_{kij} = A^{k}{}_{ik,j} + A^{k}{}_{ji,k} + A^{k}{}_{kj,i} + \Gamma^{k}_{mj}A^{m}{}_{ik} + \Gamma^{k}_{mk}A^{m}{}_{ji} + \Gamma^{k}_{mi}A^{m}{}_{kj}, \qquad (3.56)$$

where $A^{k}{}_{ik} = \Gamma^{k}_{ik} - \Gamma^{k}_{ki} = 2\Gamma^{k}_{[ik]}$. Note that if Γ^{k}_{ij} is symmetric, then

$$R^{k}{}_{ijk} + R^{k}{}_{jki} + R^{k}{}_{kij} = 0. \qquad (3.57)$$

(iii) Choose a geodesic co-ordinate system at some point. Then, at this point Γ^{k}_{ij}, but not necessarily its derivatives, vanish, so that (3.50) becomes

$$R^{k}{}_{ijk} = \Gamma^{k}_{ik,j} - \Gamma^{k}_{ij,k},$$

$$\text{i.e. } R^{k}{}_{ijk;l} = R^{k}{}_{ijk,l} = \Gamma^{k}_{ik,jl} - \Gamma^{k}_{ij,kl}.$$

Writing down two similar expressions obtained by cyclically permuting j, k, l, we find

$$R^{k}{}_{ijk;l} + R^{k}{}_{ikl;j} + R^{k}{}_{ilj;k} = 0 \qquad (3.58)$$

Since this is a tensor equation and valid in the geodesic frame, it must be valid in all frames. Also, the chosen point can be any point of L_n, so it valid at all points. The expression (3.58) is the Bianchi identity.

Example.

Using plane polar co-ordinates in E_2 show that the curvature tensor vanishes.

$r = x^1$, $\theta = x^2$. Non-zero Γ's are $\Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{r}$, $\Gamma^{1}_{22} = -r$.

Curvature tensor components must be of the form $R^{k}{}_{i12}$ since it is antisymmetric in the last two suffixes.

$$R^{k}{}_{i12} = \Gamma^{k}_{i2,1} - \Gamma^{k}_{i1,2} + \Gamma^{k}_{m1}\Gamma^{m}_{i2} - \Gamma^{k}_{m2}\Gamma^{m}_{i1}.$$

If $k = i = 1$ or $k = i = 2$, components are zero since we then have Γ's with two 1's or all 2's. Only other components are

$$\begin{aligned} R^1_{\ 212} &= \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^1_{m1}\Gamma^m_{22} - \Gamma^1_{m2}\Gamma^m_{21} \\ &= \Gamma^1_{22,1} - 0 + 0 - \Gamma^1_{22}\Gamma^2_{21} \\ &= -1 - (-r)(\tfrac{1}{r}) = 0. \\ R^2_{\ 112} &= \Gamma^2_{12,1} - \Gamma^2_{11,2} + \Gamma^2_{m1}\Gamma^m_{12} - \Gamma^2_{12}\Gamma^m_{11} \\ &= \Gamma^2_{12,1} - 0 + \Gamma^2_{21}\Gamma^2_{12} - 0 \\ &= -\tfrac{1}{r^2} + \tfrac{1}{r^2} = 0. \end{aligned}$$

Hence, all components of $R^k_{\ ijk} = 0$.

3.15. Riemannian space.

We now further specify the space L_n by introducing the idea of 'distance' between neighbouring points. This is given by

$$ds^2 = g_{ij}\,dx^i\,dx^j, \qquad (3.59)$$

where the n^2 coefficients g_{ij} are specified at every point of L_n in some co-ordinate system. Such a space is called a Riemannian space and is denoted by V_n. The expression (3.59) is called the metric of V_n and corresponds to the first fundamental form for a two-dimensional surface. The quantity ds is regarded as an invariant for any two neighbouring points, so it follows that g_{ij} is a second-rank tensor which may be taken to be symmetric, without loss of generality. It is called the fundamental tensor, or metric tensor, of V_n.

As in section 3.4 (vi), we can form the conjugate tensor g^{ij}, i.e. the contravariant form of g_{ij}, which exists provided that $|g_{ij}| \equiv g \neq 0$ which is assumed to be true.

If A^j is a contravariant tensor (vector) defined at a point of V_n, then $A_i = g_{ij}A^j$ is a covariant vector at the same point; A_i and A^i are regarded as the covariant and contravariant components of the same vector.

Similarly, we have $A^i = g^{ij}A_j$, so that the suffix can be raised. Any suffix or suffixes can be raised in the same way, e.g.

$$A^{ij}_{\ \ k} = g^{il}A_l{}^j{}_k = g^{il}g_{km}A_l{}^{jm} = g^{il}g_{km}g^{jn}A_{ln}{}^m.$$

If a suffix of g_{ij} is raised we have

$$g^i_j = g^{ik}g_{kj} = \delta^i_j,$$

so that the Kronecker delta is the mixed component of the fundamental tensor.

The inner product of two vectors A^i, B_i is

$$A^i B_i = g^{ij} A_j B_i = A_j B^j = A_i B^i. \tag{3.60}$$

For rectangular cartesian co-ordinates in E_n, the metric is

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2,$$

$$\text{i.e.} \quad ds^2 = g_{ij} dx^i dx^j,$$

where

$$g_{ij} = 1 \ (i = j) \ ; \ g_{ij} = 0 \ (i \neq j)$$

Since $g = 1$, we have

$$g^{ij} = 1 \ (i = j) \ ; \ g^{ij} = 0 \ (i \neq j).$$

If A_i is a vector in E_n, then $A^i = g^{ij} A_j$, so that, in E_n with rectangular cartesian co-ordinates, the covariant and contravariant components are the same.

Note that if the metric of V_n is diagonal, i.e.

$$g_{ij} = 0 \ , \text{ if } i \neq j,$$

then we find

$$g^{ij} = 0 \ (i \neq j) \ ; \ g^{ij} = \frac{1}{g_{ij}} \ (i = j). \tag{3.61}$$

The squared magnitude of a vector is defined as the invariant $(A)^2$ given by

$$(A)^2 = A^i A_i = g_{ij} A^i A^j = g^{ij} A_i A_j. \tag{3.62}$$

Note that A_i, A^i have the same magnitudes. If $A^i A_i = 0$, then A^i is a <u>null vector</u>. Such a vector can only exist if the metric is indefinite.

The angle, θ, between two vectors is, by analogy with E_3, given by

$$AB\cos\theta = A^i B_i$$

$$\text{i.e.} \quad \cos\theta = \frac{A^i B_i}{\sqrt{(A^k A_k)(B^m B_m)}}. \tag{3.63}$$

If $\theta = \frac{\pi}{2}$, then $A_i B^i = 0$, and the two vectors are said to be <u>orthogonal</u>.

3.16 Christoffel symbols.

At any point P of V_n it is possible to find a regular linear transformation of co-ordinates, which may involve complex coefficients, such that

$$g_{ij}dx^i dx^j = (dy^1)^2 + (dy^2)^2 + \dots + (dy^n)^2.$$

The y^i will behave like rectangular cartesian co-ordinates in E_n over a small neighbourhood of P and in this neighbourhood Γ^k_{ij} will be of the form (3.29) and so is symmetric in i and j at P. Since this is true at every point of V_n, we take the affinity of a Riemannian space to be symmetric.

Apart from its symmetry, the affinity is still arbitrary. Now the covariant derivative $A_{i;j}$ is obtained from A^i by lowering the suffix and differentiating and, unless these two operations commute, difficulties will arise. Hence, we define the affinity so that these operations commute, ie.

$$(g_{ij}A^j)_{;k} = g_{ij;k}A^j + g_{ij}A^j{}_{;k}$$
$$= g_{ij}A^j{}_{;k},$$

so that we must have

$$g_{ij;k}A^j = 0.$$

Since this is true for arbitrary A^j, we have

$$g_{ij;k} = 0 \qquad (3.64)$$

This is a set of $\frac{1}{2}n^2(n+1)$ equations and, if the Γ^k_{ij} are symmetric, they also have $\frac{1}{2}n^2(n+1)$ components, so we should be able to find the Γ^k_{ij} in terms of g_{ij} and thus specify the affinity.

Equation (3.64), and the two similar equations obtained by cyclicly permuting i, j, and k, can be written in the form

$$g_{ij,k} - \Gamma^l_{ik}g_{lj} - \Gamma^l_{jk}g_{il} = 0 \qquad (3.65)$$
$$g_{jk,i} - \Gamma^l_{ji}g_{lk} - \Gamma^l_{ki}g_{jl} = 0 \qquad (3.66)$$
$$g_{ki,j} - \Gamma^l_{kj}g_{li} - \Gamma^l_{ij}g_{kl} = 0 \qquad (3.67)$$

Remembering that g_{ij} and Γ^k_{ij} are symmetric, (3.66) + (3.67) − (3.65) leads to

$$g_{kl}\Gamma^l_{ij} = [ij,k], \qquad (3.68)$$

where

$$[ij,k] = \tfrac{1}{2}(g_{jk,i} + g_{ki,j} - g_{ij,k}). \qquad (3.69)$$

The quantity $[ij,k]$ is called the <u>Christoffel Symbol of the First Kind</u>.

It is neither a tensor nor an affinity.

Multiplying both sides of (3.68) by g^{lk}, we obtain

$$\Gamma^k_{ij} = \left\{ {k \atop ij} \right\}, \tag{3.70}$$

where

$$\left\{ {k \atop ij} \right\} = g^{kl}[ij,l] = \tfrac{1}{2} g^{kl}(g_{jl,i} + g_{li,j} - g_{ij,l}). \tag{3.71}$$

The quantity $\left\{ {k \atop ij} \right\}$ is called the <u>Christoffel Symbol of the Second Kind</u>. It is an affinity and is symmetric in i and j.

If the affinity is determined by (3.70) then the equation (3.64) is satisfied. It also follows that

$$g^{ij}{}_{;k} = 0 \tag{3.72}$$

since $g^{ij} g_{kj} = \delta^i_k$.

The affinity in this case is called the metric affinity. Note that in E_n, in rectangular cartesian co-ordinates, the g_{ij} are constants so that the Christoffel symbols are all zero. Hence, covariant and partial derivatives will be the same. In future we shall always use the metric affinity $\left\{ {k \atop ij} \right\}$.

3.17. <u>The Curvature tensor and the Einstein tensor.</u>

The contravariant suffix in $R^l{}_{ijk}$ can be lowered to give the covariant curvature tensor R_{lijk}, i.e.

$$R_{lijk} = g_{lm} R^m{}_{ijk}. \tag{3.73}$$

By writing out the right side of (3.73) explicitly, using (3.54) with $\Gamma^k_{ij} = \left\{ {k \atop ij} \right\}$, and noting the following results:

$$g_{ij}\left\{ {i \atop kl} \right\}_{,m} = \left[g_{ij}\left\{ {i \atop kl} \right\} \right]_{,m} - \left\{ {i \atop kl} \right\} g_{ij,m} ;$$

$$g_{ij}\left\{ {i \atop kl} \right\} = [kl,j] ;$$

$$g_{ij,k} = [ik,j] + [jk,i] ,$$

it follows that

$$R_{lijk} = \tfrac{1}{2}(g_{lk,ij} + g_{ij,lk} - g_{lj,ik} - g_{ik,lj}) + g_{mn}\left[\left\{ {m \atop ij} \right\}\left\{ {n \atop lk} \right\} - \left\{ {m \atop ik} \right\}\left\{ {n \atop lj} \right\} \right] \tag{3.74}$$

From (3.74) we find that

$$R_{lijk} = -R_{iljk} = -R_{likj} = R_{jkli}, \tag{3.75}$$

i.e. R_{lijk} is antisymmetric in the first pair of suffixes, antisymmetric in the last pair of suffixes, and symmetric under interchanges of the first pair with the last pair. Also, from (3.57) we have

$$R_{lijk} + R_{ljki} + R_{lkij} = 0. \tag{3.76}$$

The Ricci tensor $R_{ij} = R^k{}_{ikj}$ is symmetric and is given by

$$R_{ij} = \left\{ {k \atop ij} \right\}_{,k} - \left\{ {k \atop ik} \right\}_{,j} + \left\{ {k \atop mk} \right\}\left\{ {m \atop ij} \right\} - \left\{ {k \atop mj} \right\}\left\{ {m \atop ik} \right\}. \qquad (3.77)$$

If we contract the Bianchi identity (3.58) we obtain

$$R_{ik;l} + R^k{}_{ikl;k} - R_{il;k} = 0,$$

$$\text{i.e. } g^{ik}\left[R_{ik;l} + R^k{}_{ikl;k} - R_{il;k}\right] = 0,$$

$$R_{,l} - R^k{}_{l;k} - R^k{}_{l;k} = 0,$$

$$\text{i.e. } R_{,l} - 2R^k{}_{l;k} = 0,$$

$$\text{i.e. } \left(R^k{}_l - \tfrac{1}{2}\delta^k{}_l R\right)_{;k} = 0,$$

$$\text{i.e. } G^k{}_{l;k} = 0, \qquad (3.78)$$

where $R = g^{ij}R_{ij}$ is the <u>Ricci scalar</u> and $G^k{}_l = R^k{}_l - \frac{1}{2}\delta^k{}_l R$ is the <u>Einstein tensor</u>. Note that

$$G_{ij} = R_{ij} - \tfrac{1}{2}g_{ij}R \quad \text{and} \quad G^{ij} = R^{ij} - \tfrac{1}{2}g^{ij}R$$

On contraction the Einstein tensor gives:

$$G = g^{ij}G_{ij} = G^i_i = R - \tfrac{1}{2}nR$$

$$\text{i.e. } G = -\tfrac{1}{2}(n-2)R \qquad (3.79)$$

Note that equation (3.78) states that the divergence of the Einstein tensor is zero.

3.18 <u>Some useful results.</u>

If we contract the covariant derivative $A^i{}_{;j}$ we obtain the scalar $A^i{}_{;i}$, known as the divergence of A^i.

$$A^i{}_{;i} = A^i{}_{,i} + \left\{ {i \atop ji} \right\} A^j \qquad (3.80)$$

From (3.71) we find that

$$\left\{ {i \atop ji} \right\} = \tfrac{1}{2}g^{il}\left(g_{il,j} + g_{lj,i} - g_{ji,l}\right)$$

$$\text{i.e. } \left\{ {i \atop ji} \right\} = \tfrac{1}{2}g^{il}g_{il,j} \qquad (3.81)$$

Now, if g is the determinant $|g_{ij}|$ and Δ^{ij} is the cofactor of g_{ij} in this determinant, then

$$\frac{\partial g}{\partial g_{ij}} = \Delta^{ij} = g g^{ij} \qquad (3.82)$$

$$\text{i.e. } dg = g g^{ij} dg_{ij} = -g g_{ij} dg^{ij} \qquad (3.83)$$

since $d(g_{ij}g^{ij}) = 0$. Hence, we have

$$\frac{\partial g}{\partial x^k} = g g^{ij} g_{ij,k} = -g g_{ij} g^{ij}{}_{,k} \qquad (3.84)$$

Hence, (18.2) becomes

$$\left\{ {i \atop ji} \right\} = \tfrac{1}{2}g^{il}g_{il,j} = -\tfrac{1}{2}g_{il}g^{il}{}_{,j}$$

$$= \frac{1}{2|g|}\frac{\partial |g|}{\partial x^j}$$

$$\text{i.e. } \left\{ {i \atop j\,i} \right\} = \frac{1}{\sqrt{|g|}} \frac{\partial \sqrt{|g|}}{\partial x^j} \tag{3.85}$$

$$= \left(\log_e \sqrt{|g|}\right)_{,j} \tag{3.86}$$

Equation (3.80) can thus be written in the form

$$A^i{}_{;i} = A^i{}_{,i} + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}\left(\sqrt{|g|}\right) A^i$$

$$\text{i.e. } A^i{}_{;i} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} A^i\right)_{,i} \tag{3.87}$$

If A_i is the gradient of a scalar, i.e. $A_i = \frac{\partial \phi}{\partial x^i} \equiv \phi_{,i}$, then $A^i = g^{ij} \frac{\partial \phi}{\partial x^j}$. It follows that

$$A^i{}_{;i} = \operatorname{div} \operatorname{grad} \phi = \nabla^2 \phi,$$

$$\text{i.e. } \nabla^2 \phi = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|}\, g^{ij} \frac{\partial \phi}{\partial x^j}\right) \tag{3.88}$$

In E_n this has the form

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^i \partial x^i} \tag{3.89}$$

<u>Example</u>. Express the Laplacian in terms of spherical polar coordinates in E_3.

The metric of E_3 in spherical polars is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\psi^2,$$

so that $g = r^4 \sin^2\theta$. Using (3.88) and the fact that g_{ij} and g^{ij} are diagonal, we obtain

$$\nabla^2\phi = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta\, g^{11} \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(r^2 \sin\theta\, g^{22} \frac{\partial \phi}{\partial \theta}\right) + \frac{\partial}{\partial \psi}\left(r^2 \sin\theta\, g^{33} \frac{\partial \phi}{\partial \psi}\right)\right]$$

$$= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{\partial}{\partial \psi}\left(\operatorname{cosec}\theta \frac{\partial \phi}{\partial \psi}\right)\right]$$

$$= \frac{1}{r^2 \sin\theta}\left[r^2 \sin\theta \frac{\partial^2 \phi}{\partial r^2} + 2r\sin\theta \frac{\partial \phi}{\partial r} + \sin\theta \frac{\partial^2 \phi}{\partial \theta^2} + \cos\theta \frac{\partial \phi}{\partial \theta} + \operatorname{cosec}\theta \frac{\partial^2 \phi}{\partial \psi^2}\right]$$

$$\text{i.e. } \nabla^2\phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r^2}\cot\theta \frac{\partial \phi}{\partial \theta} + \frac{1}{r^2}\operatorname{cosec}^2\theta \frac{\partial^2 \phi}{\partial \psi^2}$$

Consider the divergence of a second rank tensor A^{ij}.

$$A^{ij}{}_{;j} = A^{ij}{}_{,j} + \left\{ {j \atop m\,j} \right\} A^{im} + \left\{ {i \atop m\,j} \right\} A^{mj}$$

$$= \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} A^{ij}\right)_{,j} + \left\{ {i \atop m\,j} \right\} A^{mj} \tag{3.90}$$

If A^{ij} is antisymmetric, the last term is zero and so

$$A^{[ij]}{}_{;j} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} A^{[ij]}\right)_{,j} \tag{3.91}$$

For a symmetric tensor, we have (3.90). For a mixed tensor

$$A_i{}^j{}_{;j} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} A_i{}^j\right)_{,j} - \left\{ {j \atop i\,m} \right\} A_j{}^m, \tag{3.92}$$

and if the corresponding A^{ij} is symmetric, then the last term can be rewritten thus

$$A_i{}^j{}_{;j} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} A_i{}^j\right)_{,j} - \tfrac{1}{2} g_{jk,i} A^{jk}. \tag{3.93}$$

In section 3.5 we saw that the determinant of a second-rank covariant tensor is a relative invariant of weight 2. Hence the transformation for g is

$$g' = J^2 g \tag{3.94}$$

where $J = \left|\frac{\partial x^i}{\partial x^{i'}}\right|$. Hence $\sqrt{|g'|} = J\sqrt{|g|}$ and the n-dimensional volume element $dx^1 \dots dx^n \equiv d^n x$, which transforms thus

$$d^n x' = J^{-1} d^n x \tag{3.95}$$

satisfies

$$\sqrt{|g'|}\, d^n x' = \sqrt{|g|}\, d^n x. \tag{3.96}$$

Hence $\sqrt{|g|}\, d^n x$ is an invariant (scalar) known as the <u>invariant volume element</u>

3.19 <u>Geodesics</u>

If P and Q are two neighbouring points on a curve C in a V_n and if s is a parameter defined on C such that s and $s + ds$ are the values at P and Q, then the interval ds is given by

$$ds^2 = g_{ij}\, dx^i\, dx^j.$$

This can be rewritten in the form

$$g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 1 \tag{3.97}$$

and the quantity $\frac{dx^i}{ds}$, which is thus a unit vector, is defined to be the unit tangent vector to the curve at P.

A geodesic is defined as a curve such that each element of it is a parallel displacement of the preceding element; i.e. the tangents at all its points are parallel.

After parallel displacement from P to Q, the new unit tangent has components

$$\frac{dx^i}{ds} + \delta\left(\frac{dx^i}{ds}\right) = \frac{dx^i}{ds} - \Gamma^i_{jk} \frac{dx^j}{ds} dx^k, \tag{3.98}$$

whereas the unit tangent at Q has components

$$\frac{dx^i}{ds} + d\left(\frac{dx^i}{ds}\right) = \frac{dx^i}{ds} + \frac{d^2x^i}{ds^2} ds. \tag{3.99}$$

The vectors defined by (3.98) and (3.99) are identical if and only if

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \tag{3.100}$$

which is thus the equations for a geodesic.

The class of curves for which $ds = 0$, which exist only if the metric is indefinite, are called <u>null curves</u>. In finding the equation of a null geodesic we can have equation (3.100) since this requires ds in the denominators. In this case we define some other parameter λ on the curve assuming that $\frac{dx^i}{d\lambda}$ exists at each point of the curve, where $\frac{dx^i}{d\lambda}$, the zero tangent is a null vector. In this case the equation of the null geodesic is

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0 \quad , \quad ds = 0. \tag{3.101}$$

In equations (3.100) and (3.101) we shall take $\Gamma^i_{jk} = \left\{ {i \atop jk} \right\}$.

This definition of a geodesic corresponds in E_3 to a straight line, the shortest distance between two points, and suggests that, in the general case, a geodesic can be defined as a curve of extremal length. We require the condition for the integral $\int ds$ to have a stationary value, i.e.

$$\delta\int ds = 0,$$

where δ means a variation corresponding to a change in co-ordinates and $\frac{dx^i}{dt}$ as we go from one possible path to another

$$\text{i.e. } \delta\int\sqrt{g_{ij}\,dx^i dx^j} = 0$$

$$\text{i.e. } \delta\int\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\,dt = 0,$$

where t is some parameter, which may be s (except for null geodesics).

Put $L^2 = g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}$, then $\delta\int L\,dt = 0$. Now

$$\delta\int L\,dt = \int\left[\frac{\partial L}{\partial x^i}\delta x^i + \frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)}\delta\left(\frac{dx^i}{dt}\right)\right]dt,$$

and, using $\delta\left(\frac{dx^i}{dt}\right) = \frac{d}{dt}(\delta x^i)$, the second term may be rewritten as

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)}\delta x^i\right\} - \frac{d}{dt}\left\{\frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)}\right\}\delta x^i$$

The first term vanishes on integration since $\delta x^i = 0$ the end points. Hence, we have

$$\delta\int L\,dt = \int\left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left\{\frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)}\right\}\right]\delta x^i\,dt = 0, \tag{3.102}$$

and, since this vanishes for arbitrary displacements δx^i, the integrand must vanish identically, i.e.

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)}\right\} - \frac{\partial L}{\partial x^i} = 0 \tag{3.103}$$

Now $\frac{\partial L}{\partial x^i} = \frac{1}{2}\left(g_{lk}\frac{dx^l}{dt}\frac{dx^k}{dt}\right)^{-1/2}\frac{\partial g_{mn}}{\partial x^i}\frac{dx^m}{dt}\frac{dx^n}{dt}$ and

$$\frac{\partial L}{\partial\left(\frac{dx^i}{dt}\right)} = \left(g_{lk}\frac{dx^l}{dt}\frac{dx^k}{dt}\right)^{-1/2} g_{ij}\frac{dx^j}{dt},$$

so equation (3.103) can be written as

$$\frac{d}{dt}\left\{ \frac{g_{ij}\frac{dx^j}{dt}}{\frac{ds}{dt}} \right\} - \frac{g_{mn,i}\frac{dx^m}{dt}\frac{dx^n}{dt}}{2\frac{ds}{dt}} = 0.$$

$$\text{i.e.}\quad g_{ij,k}\frac{dx^k}{dt}\frac{dx^j}{dt} + g_{ij}\frac{d^2x^j}{dt^2} - g_{ij}\frac{dx^j}{dt}\frac{\frac{d^2s}{dt^2}}{\frac{ds}{dt}} - \tfrac{1}{2}g_{mn,i}\frac{dx^m}{dt}\frac{dx^n}{dt} = 0$$

Multiply by g^{ki}, then

$$\frac{d^2x^k}{dt^2} + \tfrac{1}{2}g^{ik}\left(g_{ij,k} + g_{ik,j} - g_{kj,i}\right)\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{dx^k}{dt}\cdot\frac{\frac{d^2s}{dt^2}}{\frac{ds}{dt}} \qquad (3.104)$$

Now choose the parameter t to be the arc length of the curve, i.e. set

$$\frac{ds}{dt} = \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}} = 1.$$

Then equation (3.104) simplifies to

$$\frac{d^2x^k}{ds^2} + \left\{ {k \atop jk} \right\}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \qquad (3.105)$$

which are the equations of the geodesics. Hence geodesics may be defined either as curves such that each element is a parallel displacement of the previous element, or as curves of extremal length. Either definition gives rise to equation (3.105) or (3.100).

3.20. Absolute derivative.

In section 3.6 we saw that the quantity

$$dA_i = \frac{\partial A_i}{\partial x^j}dx^j$$

is not a tensor, but the quantity

$$dA_i - \delta A_i = A_{i;j}dx^j$$

is a tensor. We write

$$DA_i = dA_i - \delta A_i = A_{i;j}dx^j, \qquad (3.106)$$

and DA_i is called the <u>absolute differential</u>, or <u>intrinsic differential</u>, of A_i.

If t is any scalar, then the absolute derivative of A_i with respect to t is defined by

$$\frac{DA_i}{Dt} = A_{i;j}\frac{dx^j}{dt}. \qquad (3.107)$$

This is a vector and obeys the usual rules of differentiation. Similarly, the absolute derivative of any tensor is given by

$$\frac{D}{Dt}T^{k\ldots i}_{j\ldots k} = T^{k\ldots i}_{j\ldots k;l}\frac{dx^l}{dt} \qquad (3.108)$$

Note that the equations of geodesics (3.105) can be written in the form

$$\frac{D}{Ds}t^k = 0, \qquad (3.109)$$

where $t^k \equiv \frac{dx^k}{ds}$ is the unit tangent vector.

3.21 The permutation tensor and dual tensors.

The permutation symbol $\varepsilon^{ij\ldots k}$ is a tensor density, i.e. a relative tensor of weight 1, so it follows that its covariant counterpart $\varepsilon_{ij\ldots k}$ is a relative tensor of weight -1. (Note that the suffixes on the permutation symbol are raised and lowered by using the metric tensor of the locally flat Euclidean space, i.e. the tangent space, expressed in Cartesian co-ordinates). Since $\sqrt{|g|}$ is a relative invariant of weight 1, it follows that the quantity

$$\eta_{ij\ldots k} = \pm\sqrt{|g|}\,\varepsilon_{ij\ldots k} \qquad (3.110)$$

is a tensor whose values are given by

$$\left.\begin{aligned}\eta_{ij\ldots k} &= +\sqrt{|g|}\,\varepsilon_{12\ldots n}, \text{ if } i,j,\ldots,k \text{ is an even perm of } 1,2,\ldots,n\\ \eta_{ij\ldots k} &= -\sqrt{|g|}\,\varepsilon_{12\ldots n}, \text{ if } i,j,k \text{ is an odd perm of } 1,2,\ldots,n\\ \eta_{ij\ldots k} &= 0, \text{ if any two suffixes are the same.}\end{aligned}\right\} \qquad (3.111)$$

The ambiguity in sign in equation (3.110) is due to the fact that we define

$$\eta^{ij\ldots k} = \frac{1}{\sqrt{|g|}}\,\varepsilon^{ij\ldots k} \qquad (3.112)$$

and the positive (negative) sign in equation (3.110) arises according as g is positive or negative.

The tensor $\eta^{ij\ldots k}$ is called the permutation tensor. Note that both $\eta^{ij\ldots k}$ and $\eta_{ij\ldots k}$ have zero covariant derivative.

Note that if $T^{ij\ldots k}$ is a tensor antisymmetric in all its suffixes and of rank less than the dimension of the space, then we define the dual tensor $^*T^{ij\ldots}$ by.

$$^*T^{i_1 i_2\ldots i_m} = \frac{1}{m!}\,\eta^{i_1 i_2\ldots i_m\ldots i_n}\,T_{i_{m+1}\ldots i_n} \qquad (3.113)$$

For example in a V_4, the dual of an antisymmetric tensor T_{ij} is

$$^*T_{ij} = \tfrac{1}{2}\eta_{ijkl}T^{kl}, \qquad (3.114)$$

$$^*T^{ij} = \tfrac{1}{2}\eta^{ijkl}T_{kl}. \qquad (3.115)$$

Note that the dual of the dual is given by

$$\begin{aligned}{}^{**}T_{ij} &= \tfrac{1}{2}\eta_{ijkl}\,{}^*T^{kl}\\ &= \tfrac{1}{4}\eta_{ijkl}\,\eta^{klmn}T_{mn}\\ &= \pm T_{mn}, \text{ depending on the signature.}\end{aligned}$$

$$\text{i.e.}\quad {}^{**}T_{ij} = (\mathrm{sgn})\,T_{mn} \qquad (3.116)$$

where (sgn) is the sign of g.

Also in V_4, the dual of a vector A_i is a third-rank tensor

$^*A^{jkl}$ and vice-versa.

The tensor $\delta^{l\dots i}_{j\dots k}$ is defined by

$$\delta^{l\dots i}_{j\dots k} = \det \begin{bmatrix} \delta^l_j & \cdots & \delta^i_j \\ \vdots & & \vdots \\ \delta^l_k & & \delta^i_k \end{bmatrix} \qquad (3.117)$$

is identically zero if there are more than n suffixes (upper or lower). In a V_4 with negative g we find

$$\delta^{li}_{jk} = -\tfrac{1}{2}\eta^{limn}\eta_{jkmn}. \qquad (3.118)$$

3.22 <u>Orthogonal ennuples.</u>

In a V_n we call n mutually orthogonal non-null vector fields an orthogonal ennuple. If the n vectors are denoted by h^i_α, the latin suffix is the tensor suffix, which is raised and lowered by the metric tensor g_{ij}, and the greek suffix is the numbering suffix which denotes the n vectors and is raised and lowered by the metric tensor $\eta_{\alpha\beta}$ of the locally flat (Euclidean) space in Cartesian co-ordinates. The ennuple satisfies the following relations

$$\left.\begin{aligned} h_\alpha{}^i h^\alpha{}_j &= \delta^i_j , \quad h_{\alpha i} h^\alpha{}_j = g_{ij} \\ h_\alpha{}^i h^\beta{}_i &= \delta_\alpha^\beta , \quad h_\alpha{}^i h_{\beta i} = \eta_{\alpha\beta} \end{aligned}\right\} \qquad (3.119)$$

The ennuple components for a diagonal metric are just the square roots of the metric tensor. For example, in an E_2 with polar co-ordinates the metric is

$$ds^2 = dr^2 + r^2 d\theta^2$$

and the ennuple components are

$$h_{(1)i} = (1,0) \quad , \quad h_{(1)}{}^i = (1,0),$$
$$h_{(2)i} = (0,r) \quad , \quad h_{(2)}{}^i = (0,\tfrac{1}{r}),$$

and the ennuple suffixes (1),(2) are raised by the E_2 cartesian metric tensor, i.e. by 1's.

If we take a space which does not have positive definite signature, e.g.

$$ds^2 = e^{2\mu}dx^2 + e^{2\nu}dy^2 - e^{2\lambda}dz^2,$$

then the ennuple is

$$h_{(1)}{}^i = (e^{-\mu},0,0) \;,\; h_{(2)}{}^i = (0,e^{-\nu},0) \;,\; h_{(3)}{}^i = (0,0,e^{-\lambda})$$
$$h_{(1)i} = (e^{\mu},0,0) \;,\; h_{(2)i} = (0,e^{\nu},0) \;,\; h_{(3)i} = (0,0,e^{\lambda})$$

but, since the local flat space is

$$ds^2 = dx^2 + dy^2 - dz^2,$$

the ennuple suffixes (1), (2), (3) are raised and lowered by +1, +1, −1, respectively.

Given a vector A_i, the four scalar A_α defined by the expression

$$A_\alpha = h_\alpha{}^i A_i, \qquad (3.120)$$

are the components of A_i referred to the local cartesian co-ordinate system and are called the local components of A_i. Similarly, the local components of a mixed tensor $B_i{}^j$ are

$$B_\alpha{}^\beta = h_\alpha{}^i h^\beta{}_j B_i{}^j. \qquad (3.121)$$

The local components are sometimes called the physical components; the reason for this is illustrated by the following example: Consider an E_2 and the velocity vector u^i defined by

$$u^i = \frac{dx^i}{dt}$$

where t is time, a scalar. In cartesians

$$u^i = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}\right) = (\dot{x}, \dot{y}).$$

Changing to polar co-ordinates, $x^{i'} = (r, \theta)$, then

$$u^{i'} = \frac{dx^{i'}}{dt} = (\dot{r}, \dot{\theta}).$$

However, these are not the usual components of velocity in polar co-ordinates — the usual ones are the physical components found by taking

$$h^\alpha{}_{i'} u^{i'} = (1.u^{1'}, r u^{2'}) = (\dot{r}, r\dot{\theta}).$$

3.23. <u>Riemannian curvature and spaces of constant curvature.</u>

Let λ^i, μ^i be two contravariant vector fields at a point P in V_n. Then, in a neighbourhood of P, the geodesics through P having tangent vectors of the form

$$\xi^i = \alpha\lambda^i + \beta\mu^i,$$

where α and β are real parameters, form a portion S of a two-dimensional surface. The Gaussian curvature K of S at P is called the <u>Riemannian curvature</u> of V_n at P in the two-dimensional direction determined by λ^i and μ^i. It can be shown that this curvature is given by the expression

$$K(g_{hj}g_{ik} - g_{hk}g_{ij})\lambda^h\mu^i\lambda^j\mu^k = R_{hijk}\lambda^h\mu^i\lambda^j\mu^k. \qquad (3.121)$$

<u>Schur's Theorem.</u>

If the Riemannian curvature of a V_n is the same for every pair of vectors at every point, then the curvature is a constant at every point of V_n.

<u>Proof.</u> From equation (3.121) we see that if K is independent of the choice of λ^i and μ^i, then

$$R_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij}). \qquad (3.122)$$

Taking the covariant derivative and cyclicly permuting suffixes, we find

$$R_{hijk;l} + R_{hikl;j} + R_{hilj;k} = K_{,l}(g_{hj}g_{ik} - g_{hk}g_{ij}) + K_{,j}(g_{hk}g_{il} - g_{hl}g_{ik}) + K_{,k}(g_{hl}g_{ij} - g_{hj}g_{il})$$

The left-hand side is zero from the Bianchi identities, and inner multiplication of the right-hand side by $g^{hj}g^{ik}$ leads to

$$(n-1)(n-2)K_{,l} = 0,$$

so that, assuming $n \geq 3$, K is a constant, thus proving the theorem.

A V_n satisfying equation (3.122) is called a <u>space of constant curvature</u>.

A V_n is called an <u>Einstein space</u> if its Ricci tensor is proportional to its metric tensor, i.e.

$$R_{ij} = \lambda g_{ij}, \qquad (3.123)$$

where λ is a scalar. In fact, λ must be a constant because, contracting (3.123), we have $R = n\lambda$, so that the Einstein tensor G_{ij} is given by

$$G_{ij} \equiv R_{ij} - \tfrac{1}{2}Rg_{ij} = (1-\tfrac{n}{2})\lambda g_{ij},$$

and since $G^{ij}{}_{;j} \equiv 0$, it follows that $\lambda_{,i} = 0$. Contracting equation (3.122) with g^{hj}, we obtain

$$R_{ik} = K(ng_{ik} - g_{ik}) = K(n-1)g_{ik}, \qquad (3.124)$$

so that a space of constant curvature is a special case of an Einstein space. Note that $R = n(n-1)K$, so that the Ricci scalar of a space of constant curvature is also a constant.

3.24 <u>Conformal spaces.</u>

If two spaces $V_n, \bar{V}_n$ are such that their metric tensors are related by an expression of the form

$$\bar{g}_{ij} = e^{2\sigma} g_{ij}\,, \qquad (3.125)$$

where σ is a scalar function of the co-ordinates, it follows that the magnitudes of the displacement vectors at points of V_n and $\bar{V}_n$ with the same co-ordinates are proportional, and the angles between two corresponding directions at corresponding points are equal. The spaces V_n, $\bar{V}_n$ are said to be in conformal correspondence to each other, and the mapping given by equation (3.125) maps V_n conformally onto $\bar{V}_n$.

From equation (3.125) we have

$$\bar{g}^{ij} = e^{-2\sigma} g^{ij}, \qquad (3.126)$$

and the Christoffel symbols in V_n and $\bar{V}_n$ are related by

$$\overline{\left\{{k \atop ij}\right\}} = \left\{{k \atop ij}\right\} + \delta^k_i \sigma_{,j} + \delta^k_j \sigma_{,i} - g_{ij} g^{kk} \sigma_{,k}\,. \qquad (3.127)$$

If A_{ij} denotes the quantity

$$A_{ij} \equiv \sigma_{;ij} - \sigma_{,i}\sigma_{,j}\,, \qquad (3.128)$$

where the semi-colon denotes covariant differentiation using $\left\{{k \atop ij}\right\}$ and not $\overline{\left\{{k \atop ij}\right\}}$, then we find that the curvature tensors of V_n and $\bar{V}_n$ are related by

$$e^{-2\sigma}\bar{R}_{kijk} = R_{kijk} + g_{kk}A_{ij} + g_{ij}A_{kk} - g_{kj}A_{ik} - g_{ik}A_{kj} + (g_{kk}g_{ij} - g_{kj}g_{ik})\sigma_{,m}\sigma_{,}^{\;m}, \qquad (3.129)$$

where $\sigma_{,}^{\;m} = g^{mn}\sigma_{,n}$. Remembering that suffixes on barred and unbarred quantities are raised and lowered by $\bar{g}_{ij}$ and g_{ij}, respectively, the expression (3.129) can be rewritten as.

$$\bar{R}^k{}_{ijk} = R^k{}_{ijk} + \delta^k_k A_{ij} + g_{ij}A^k{}_k - \delta^k_j A_{ik} - g_{ik}A^k{}_j + (\delta^k_k g_{ij} - \delta^k_j g_{ik})\sigma_{,m}\sigma^{,m}. \qquad (3.130)$$

The components of the Ricci tensor are given by

$$\bar{R}_{ik} \equiv \bar{R}^k{}_{ikk} = R_{ik} + A_{ik} + A_{ik} - nA_{ik} - g_{ik}A^j{}_j + g_{ik}(1-n)\sigma_{,m}\sigma_{,}^{\;n}$$

$$\text{i.e. } \bar{R}_{ik} = R_{ik} - (n-2)(\sigma_{;ik} - \sigma_{;i}\sigma_{;k}) - (n-2)g_{ik}(\sigma_{;}{}^{j}{}_{j} + \sigma_{j}\sigma^{j})\,; \qquad (3.131)$$

where $\sigma_{;}{}^{j}{}_{j} = g^{kj}\sigma_{;kj}$.

The Ricci scalars, R and $\bar{R}$, are related by

$$e^{2\sigma}\bar{R} = R - (n-1)(n-2)\sigma_{,j}\sigma_{,}^{\;j} - 2(n-1)\sigma_{;}{}^{j}{}_{j} \qquad (3.132)\,.$$

The <u>Weyl conformal curvature tensor</u> $C^k{}_{ijk}$ is defined by

$$C^k{}_{ijk} = R^k{}_{ijk} - \frac{1}{n-2}\left(\delta^k_j R_{ik} - \delta^k_k R_{ij} + g_{ik}R^k{}_j - g_{ij}R^k{}_k\right) - \frac{R}{(n-1)(n-2)}\left(\delta^k_k g_{ij} - \delta^k_j g_{ik}\right). \qquad (3.133)$$

From equations (3.130) to (3.133) it follows that two spaces are in conformal correspondence with each other if and only if their Weyl tensors are identical, i.e.

$$\bar{C}^{k}{}_{ijk} = C^{k}{}_{ijk} \tag{3.134}$$

The expression (3.133) does not exist when $n = 2$. However, it can be shown that any V_2 metric, i.e. a quadratic differential form in two variables, can be reduced to the form

$$ds^2 = \lambda(x^1, x^2)\left[(dx^1)^2 \pm (dx^2)^2\right] \tag{3.135}$$

so that any V_2 is conformal to any other V_2. In the case of a V_3, it can be shown that $C^{k}{}_{ijk}$ is identically zero.

If a V_n is conformal to an E_n (which, of course, will have the same signature as the V_n) then, since the Weyl tensor of the E_n is zero, it follows that V_n satisfies

$$C^{k}{}_{ijk} = 0. \tag{3.136}$$

A V_n $(n \geq 4)$ satisfying this condition is said to be <u>conformally flat</u>. As can be seen from equation (3.135), any V_2 is conformally flat. However, although any V_3 satisfies the condition (3.136), this does not imply that a V_3 is necessarily conformally flat. The necessary and sufficient condition for a V_3 to be conformally flat is that the <u>Cotton-York tensor</u>

$$Y^{ij} = 2\eta^{imn}\left(R^{j}{}_{m} - \tfrac{1}{4}\delta^{j}{}_{m}R\right)_{;n} \tag{3.137}$$

vanishes. This condition can also be written in the form

$$R_{ij;k} - R_{ik;j} + \tfrac{1}{4}\left(g_{ik}R_{,j} - g_{ij}R_{,k}\right) = 0 \tag{3.138}$$

$$\text{i.e.} \qquad R_{i[j;k]} + \tfrac{1}{4}\, g_{i[k}R_{,j]} = 0.$$

Thus we have the following results concerning conformally flat spaces:

(i) Any V_2 is conformally flat.

(ii) A V_3 is conformally flat if and only if the Cotton-York tensor vanishes.

(iii) A V_n $(n > 3)$ is conformally flat if and only if the Weyl tensor vanishes.

From equations (3.122) and (3.133) it follows that any space of constant curvature is conformally flat, but the converse is not necessarily true.

3.25 Isometries and Killing vectors.

Let $\xi^i(x^j)$ be a contravariant vector field defined in V_n with metric

$$ds^2 = g_{ij}\,dx^i dx^j.$$

Suppose that each point of the V_n is subjected to an infinitesimal transformation

$$x'^i = x^i + \xi^i \delta t \qquad (3.139)$$

$$\text{i.e.}\quad \delta x^i = \xi^i \delta t$$

This transformation leads to

$$\delta(dx^i) = d(\delta x^i) = \xi^i{}_{,j}\,dx^j \delta t \qquad (3.140)$$

$$\delta(g_{ij}) = g_{ij,k}\,\delta x^k = g_{ij,k}\,\xi^k \delta t \qquad (3.141)$$

In general, the effect of such a transformation would be to distort the metric, but if the field ξ^i is such that the metric remains invariant under the transformation (3.139), then this transformation is called an <u>isometry</u> or <u>motion</u> of the V_n. In this case we have

$$\delta(ds^2) = 0$$

$$\text{i.e.}\quad \delta g_{ij}\,dx^i dx^j + g_{ij}\,\delta(dx^i)dx^j + g_{ij}\,dx^i\,\delta(dx^j) = 0$$

$$\text{i.e.}\quad g_{ij,k}\,\xi^k \delta t\, dx^i dx^j + g_{ij}\,\xi^i{}_{,k}\,dx^k dx^j \delta t + g_{ij}\,dx^i\,\xi^j{}_{,k}\,dx^k \delta t = 0.$$

Rearranging this equation we obtain

$$g_{ij,k}\,\xi^k + g_{ik}\,\xi^k{}_{,j} + g_{jk}\,\xi^k{}_{,i} = 0 \qquad (3.142)$$

Since $g_{ij;k} = 0$, we have $g_{ij,k} = \left\{{l \atop ik}\right\} g_{lj} + \left\{{l \atop jk}\right\} g_{il}$. Using this, equation (3.142) becomes

$$\left\{{l \atop ik}\right\} g_{lj}\,\xi^k + \left\{{l \atop jk}\right\} g_{il}\,\xi^k + g_{ik}\,\xi^k{}_{,j} + g_{jk}\,\xi^k{}_{,i} = 0.$$

$$\text{i.e.}\quad g_{il}\,\xi^l{}_{;j} + g_{jl}\,\xi^l{}_{;i} = 0$$

$$\text{i.e.}\quad \xi_{i;j} + \xi_{j;i} = 0 \qquad (3.143)$$

These equations (or equations (3.142)) are called <u>Killing's equations</u> and any solutions ξ_i are known as <u>Killing vectors</u>. These vectors define the possible infinitesimal isometries of the form (3.139).

In V_n the maximum number of independent solutions of the equations (3.143) is $\frac{1}{2}n(n+1)$ and the V_n will admit this maximum number if and only if V_n is a space of constant curvature. If the V_n admits $r \le \frac{1}{2}n(n+1)$ independent Killing vectors, then the V_n is said to admit an <u>r-parameter group of motions (or isometries)</u>. If V_n admits the maximum number of Killing vectors then it is said to be <u>maximally symmetric</u>.

the m^{th} component of the vector is 1 and all other components are zero, the equation (3.142) yields $g_{ij,m} = 0$, which implies that the metric tensor is independent of x^m in this co-ordinate system. In example 1, two of the Killing vectors are (1,0) and (0,1) and the metric tensor is independent of x and y. In example 2, the third Killing vector shows that the metric tensor is independent of ϕ.

3.26 Lie derivatives.

The directional derivative of a scalar f along a given vector ξ^i is defined as the inner product of ξ^i with the gradient of f, i.e.

$$\nabla_\xi f = f_{,i}\,\xi^i \tag{3.150}$$

In fact, the quantity $\nabla_\xi f.\delta t$ is the change in f under the infinitesimal transformation (3.139).

The Lie derivative of a scalar function with respect to the vector field ξ^i is defined to be the directional derivative, i.e.

$$\mathcal{L}_\xi f = \nabla_\xi f = f_{,i}\,\xi^i \tag{3.151}$$

For a vector field A^i, we define the Lie derivative with respect to ξ^i as the commutator of ξ^i and A^i, i.e.

$$\begin{aligned}\mathcal{L}_\xi A^i &= [\xi^i, A^i] = \nabla_\xi A^i - \nabla_A \xi^i \\ &= A^i{}_{;j}\,\xi^j - \xi^i{}_{;j}\,A^j\end{aligned} \tag{3.152}$$

It is easily seen that this can be rewritten in the form

$$\mathcal{L}_\xi A^i = A^i{}_{,j}\,\xi^j - \xi^i{}_{,j}\,A^j \tag{3.153}$$

Using the fact that $A_i A^i$ is a scalar and also equations (3.150) and (3.152), we find that the Lie derivative of a covariant vector is (assuming the Leibnitz rules for differentiation):

$$\mathcal{L}_\xi A_i = A_{i;j}\,\xi^j + \xi^j{}_{;i}\,A_j, \tag{3.154}$$

$$\text{i.e.}\quad \mathcal{L}_\xi A_i = A_{i,j}\,\xi^j + \xi^j{}_{,i}\,A_j. \tag{3.155}$$

Similarly, for a tensor $T^{ij}{}_k$, the Lie derivative is found to be

$$\mathcal{L}_\xi T^{ij}{}_k = T^{ij}{}_{k;m}\,\xi^m - T^{mj}{}_k\,\xi^i{}_{;m} - T^{im}{}_k\,\xi^j{}_{;m} + T^{ij}{}_m\,\xi^m{}_{;k}, \tag{3.156}$$

and, again, the covariant derivatives can be replaced by partial derivatives.

In particular, the Lie derivative of the metric tensor g_{ij} is

$$\mathcal{L}_\xi g_{ij} = g_{ij;m}\,\xi^m + g_{mj}\,\xi^m{}_{;i} + g_{im}\,\xi^m{}_{;j} \tag{3.156}$$

Using the fact that $g_{ij;m} = 0$, this becomes

$$\mathcal{L}_\xi g_{ij} = \xi_{j;i} + \xi_{i;j} \tag{3.157}$$

If ξ_i is a Killing vector, then equation (3.143) shows that the right-hand side of the above equation is zero, so the Lie derivative of the metric tensor with respect to a Killing vector is zero. This just means that there is no change in the metric tensor in the direction of the Killing vector, which is in accord with the definition of an isometry.

Another definition of the Lie derivative $\Phi^A[x^i(P)]$ (A represents all the tensor suffixes and $x^i(P)$ are the co-ordinates of the point P) is as follows: Make an infinitesimal of the form (3.139), i.e. $P \to P'$ by $x^i(P) = x^i(P') + \xi^i(P')$, where ξ^i is infinitesimal and so can be evaluated either at P or P'. Also make an infinitesimal co-ordinate transformation that makes the numerical values of the co-ordinates of P' the same as those of P in the original co-ordinates, i.e.

$$x'^i(P') = x^i(P)$$

Then we define

$$\mathcal{L}_\xi \Phi^A(P) = \lim_{\xi \to 0} \left[\Phi^A(P) - \Phi'^A(P')\right]. \tag{3.158}$$

It can be shown that this definition is equivalent to those of equations (3.150), (3.152), (3.154), and (3.156).

3.27. The generalized Kronecker delta.

The <u>generalized Kronecker delta</u> $\delta^{i_1 \dots i_n}_{j_1 \dots j_n}$ is defined as

$$\delta^{i_1 \dots i_r}_{j_1 \dots j_r} = \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_2}_{j_1} & \dots & \delta^{i_n}_{j_1} \\ \delta^{i_1}_{j_2} & \delta^{i_2}_{j_2} & \dots & \delta^{i_n}_{j_2} \\ \vdots & \vdots & & \vdots \\ \delta^{i_1}_{j_r} & \delta^{i_2}_{j_r} & \dots & \delta^{i_r}_{j_r} \end{vmatrix} \tag{3.159}$$

When $r = 1$, this is the usual Kronecker delta δ^i_j. When $r = 2$, we have

$$\delta^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}. \tag{3.160}$$

In general, $\delta^{i_1 \dots i_r}_{j_1 \dots j_r}$ is the sum of $r!$ terms, each of which is the product of r Kronecker deltas. Since δ^i_j is a $(1,1)$ tensor, it follows that $\delta^{i_1 \dots i_r}_{j_1 \dots j_r}$ is a (r, r) tensor.