Newton's Theory of Gravity

First law: In absence of gravitation $\exists$ privileged class of coord systems called Inertial frame s.t bodies not acted upon by any force move in a straight line with uniform velocity.

Notes 1. Absolute time.

2. Principle of relativity: laws of mechanics are true in any inertial frame, inertial frames are related by Galilean transformations: (plus translations and rotations)

e.g. $x' = x - vt \quad y' = y \quad z' = z \quad t' = t$

(i.e if true in frame S, then true in frame S' moving with rel uniform velocity relative to S).

3. Background space of Newtonian mechanics is $\mathbb{R}^3$.

First law A body under no forces moves along a geodesic of $\mathbb{R}^3_3$.

Second law: Wrt inertial frames

$$\frac{d^2 x^\alpha}{dt^2} = -\phi_{,\alpha} \qquad \alpha = 1,2,3$$

Newtonian Grav. potential

Poisson's law $\nabla^2 \phi = 4\pi G \rho$

density

Special Relativity (Absence gravity).
Einstein (1905)

Principal of Relativity All inertial frames are equivalent for all physical laws

Maxwell's equations (laws electromagnetism) are invariant under Lorentz transformations. (rotations | transl^{n}s (not Galilean)

$$x' = \gamma(x - vt) \quad y' = y \quad z' = z \quad t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-}$$

Two observers do not agree on spatial and temporal displacements, but do agree that interval ds (proper time) between 2 neighbouring events is the same (invariant).

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

ST in SR is $\mathbb{R}^4$, 4 dimensional Euclidean space of signature +2 - pseudo-metric

- Minkowski metric $\eta_{ab} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Newton's first Law becomes: A body under no forces moves along geodesic of ST.

Problems: No gravity included.
"ST continuum not itself influenced by physical cond^{n}s".

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3.
of $\mathbb{R}^3_3$

Problems with Newton's TG

(1) Experimental   Even as early as 19th century discrepances existed w.r.t observation

(i) No perihelion procession of mercury
 - contrary to observation

Note
~~Newtonian TG predicted~~
(a) Neptune - observed / found
(b) planet between Mars & Jupiter (explained by GR)

(ii) Light deflection
"Position" of stars (light rays pass close to sun) altered

(2) Practical   Newton's laws not valid at high velocities
- need to unify with SR - insurmountable difficulties;
(a) esp NTG in $\mathbb{R}^3$, SR in $\mathbb{R}^4$
(b) NTG galilean invariant, SR Lorentz invariant.

(3) Philosophical

(i) See 2(b). Poisson's eq^n not Lorentz invariant → gravitational field (or at least changes in it) propagate at infinite velocities (action at-a-distance) $[\nabla^2 \text{ not } \nabla^2 - \frac{\partial^2}{\partial t^2}]$
(ii) Inertial frames. It is physically impossible to determine such coord frames (mach).
(iii) Absolute time/Absolute space. Contradicts observations (via SR).

Conclusion

Newton's theory Gravity valid in weak grav. field / low velocity ← Correspondance principal(s)

Require   Correspondance Principals

$$\begin{tikzcd} \boxed{\text{Relativistic Theory Gravity}} \arrow[r, "\text{weak grav}"', "\text{low vel}"'] \arrow[d, "\text{no grav.}"'] & \boxed{\text{Newton's TG}} \arrow[d, "\text{no grav.}"] \\ \boxed{\text{SR}} \arrow[r, "\text{low vel}"'] & \boxed{\text{Newton's laws in absence grav.}} \end{tikzcd}$$

<u>Requirements of relativistic theory of gravity</u>

(1) Agrees (locally) with SR. (Thus locally Lorentz invariant – no absolute time – no action-at-a distance)

(2) Singles out class of preferred coord frames. (that, unlike inertial frames, are physical)

[(3) Admits a definition of a tidal field]

(4) Covariant theory (tensor equations).
(5) Agrees with observations [where NTG inadequate]
(6) NTG obtained in weak-field / low velocity limit.

<u>Discussion</u> Physical and Philosophical background.

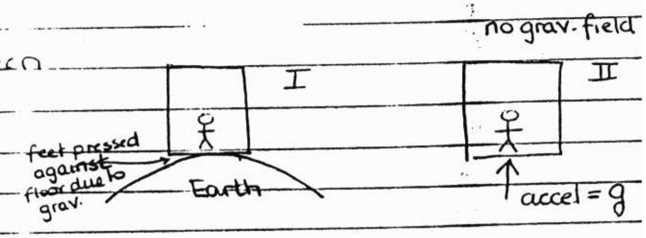
(A) Assume continuum structure – that is, spacetime can be modelled by a connected, 4D Hausdorff C^∞ paracompact differentiable manifold without boundary (whose sig is +2 as is SR (can then satisfy (1)) and (4) satisfied).

(B) <u>Einstein's equivalence principle</u> (1908).

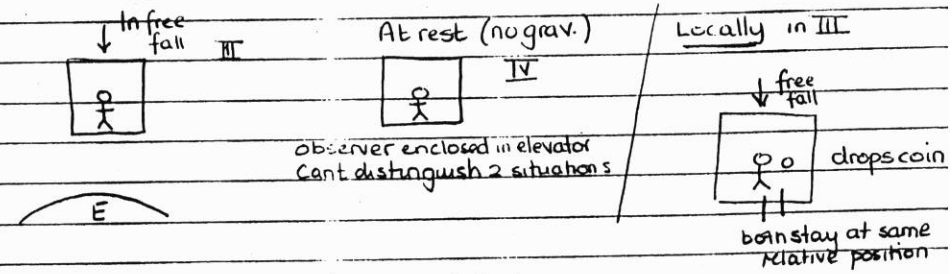
A uniform gravitational field is indistinguishable from a uniformly accelerating reference frame (i.e inertial forces).

<u>Thought experiment</u>: Einstein's elevator.

Cant distinguish between



Alternatively:



<u>Galileo</u> Weak equivalence principle.

Newtons equations $m_I\vec{a} = m_g\vec{g} + \vec{F}$ — m_I inertial mass, m_g grav mass.

Galileo found that two bodies of different composition and different "mass" falls with same acceleration.

i.e. $m_I = m_g = m$ (Eötvös 1870's Dicke).

Thus $\vec{F} = m(\vec{g} - \vec{a})$ (*)

cannot measure $\vec{g}$ and $\vec{a}$ independently

That is, we do not know what part of acceleration is due to gravitation and what part is due to the choice of coordinate system (i.e. term on right hand would include terms due to the acceleration of coordinate frame).

Must abandon inertial frames

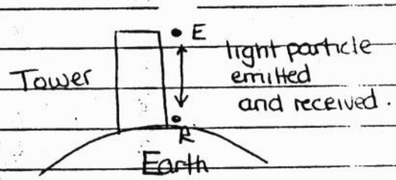
Replace with local inertial frames or Newtonian frames.

Look at coordinate frames, called Newtonian frames, in which
(*)

$$\frac{d^2x^\alpha}{dt^2} + \phi_{,\alpha} = 0$$

These are physically realisable, they are the local freely falling frames of Einstein.

Grav Redshift experiment   Princeton – Pound and Rebka



Exp: Light undergoes frequency shift as it "climbed" out of grav. field (due to energy loss).

a) Coord frame attached to nonmoving tower – SR predicts no redshift which is false
b) Coord frame attached to freely falling particle (no redshift in this frame ~ SR), when transform to the coordinate frame in which R nonmoving ⇒ predicts redshift

$\frac{\Delta\lambda}{\lambda} \simeq \frac{gl}{c^2}$ – correct to 1% that observed.

Thus

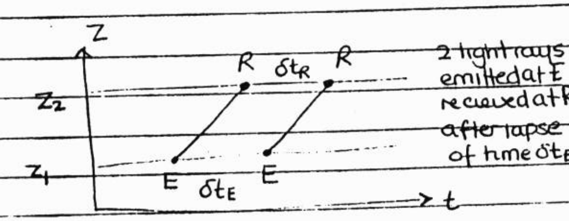
Newtonian / freely falling frames are local inertial frames
⇒ Frames in which SR valid.

Notes Freely falling frames = local inertial frames.
(1) = Newtonian frames – replace inertial frames.
(2) Frames in which gravity ignored and SR valid
– replaces inertial frames of SR

[(3) in GR – local inertial frames ~~and~~ are exponential or normal coords of Riemannian geometry]

B) More physics / motivation

(i) Redshift exp revisited

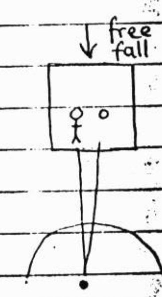


Due to redshift, measured lapse of time by R, δt_R is s.t $\delta t_E \neq \delta t_R$

From diag → suggests Non-Euclidean laws of geometry apply in ST.

(ii) Einstein's elevator revisited:

consider III again



locally:
coin and observer
~~remain same~~
relative position

But

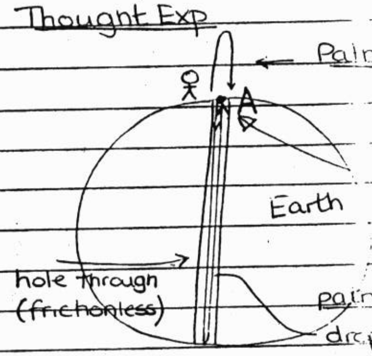
geodesics (free fall) will converge ⟺ tidal gravitational field

Suggests Grav field connected with geodesic deviation
$\Rightarrow R^a{}_{bcd} \neq 0$.

(iii) Lemma: λ curve with tangent vector X. Suppose Y and Z parallely transported along X. Suppose we are in Riemannian manifold with metric g. Then the angle between Y and Z remains constant along λ.

$$\cos(a(Y,Z)) = \frac{g(Y,Z)}{[g(Y,Y)g(Z,Z)]^{1/2}} \qquad \text{But } \nabla_X g = 0,\ \nabla_X Y = 0,\ \nabla_X Z = 0$$

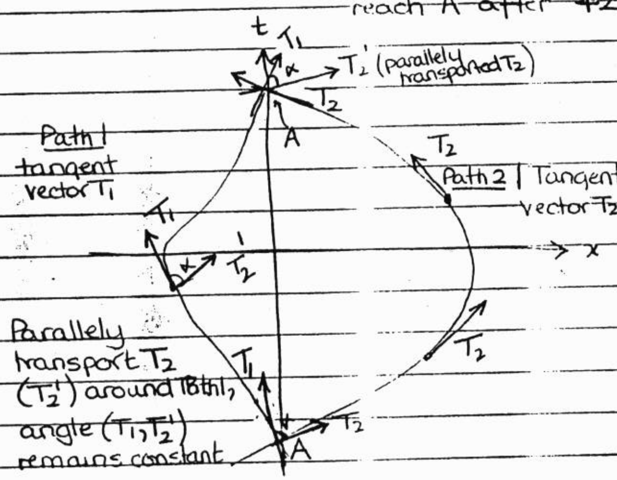
Thought Exp



Path 2 throws ball 2 with such velocity that arrives back at A at same time as ball 1

2 balls have different vel at $t=0$ ⇒ 2 paths have different tangent vectors at A.

drop ball 1 (frictionless) - falls due to grav - accelerates through centre, decelerates - reaches opposite pole - returns just to reach A after 42 minutes.



Clearly parallel transport of T_2 around T_1 and T_2 not equivalent

Suggests (parallel transport around closed path st $X_P^* - X_P \neq 0$)
$\Rightarrow R^a{}_{bcd} \neq 0$

(D) Equivalence principles

1) Einstein's equivalence principle (B). (EEP)
2) Weak equivalence principle (UFF) (WEP)
} ⇒ local inertial frames.

3) Strong Equivalence Principle (SEP)

In the neighbourhood of each point in spacetime (anywhere and anytime) ∃ preferred coordinate systems - called local inertial systems - s.t relative to these systems locally the laws of physics take on their standard SR form.

— Tells us how to derive laws of physics in GR.

Experimental evidence:
For (1) and (2) extremely strong
For (3), some partial support.

Note/speculation: SEP - at any pt in ST we may erect LIF st laws of physics obey SR: Analogy - in non Euclidean manifold M, at each pt. in M ∃ nhbd locally homeomorphic to $\mathbb{R}^n$ ⇒ suggests laws of Grav. resemble formulae in Riemannian geometry.

(E) Philosophical

Suggests (with hindsight) Riemannian manifold.
Due to: presence of grav field in "NTG" ∃ family of freely falling curves in $\mathbb{R}^4$, the curves are "curved" due to grav. field.
Reinterpret: the entirety of curves form a 4D surface. Regard gravitation as intrinsic property of ST - preferred curves are now "straight" lines in non Euclidean manif

[Thus: In grav. field: $R^a{}_{bcd} \neq 0 \rightarrow g^{ab}, \Gamma^a{}_{bc} \neq 0$]

Derivation 1 (standard-type)

Require: (1) Agrees (locally) with SR
(2) Singles out class of preferred world lines representing freely falling particles
(3) Admits tidal gravitational field
(4) Covariant
[(5) observation see later].

(A) assume ST 4D Differentiable manifold → (4) satisfied.

(B) Local inertial frames (LIF) = freely falling frames → (2) satisfied.

Postulate: SR valid in local inertial frames → (1) satisfied.

In LIF $ds^2 = \eta_{ij}dx^i dx^j$
motion "freely falling" particles $\frac{d^2x^i}{d\tau^2} = 0$ } (SEP)

$\Rightarrow$ In LIF, $\eta_{ij,k}$ and $\mathring{\Gamma}^i_{jk} = 0$ [And from above these occur in same coord syst.].

Now change to arbitrary coord system (make coord transf^n) (accelerating rel. to LIF in general)

$\eta_{ij} \to g_{ij}$
$\mathring{\Gamma}^i_{jk} \to \Gamma^i_{jk}$ (not zero in new coords)
$(, \Rightarrow ;)$

By EEP, inertial forces (due to change in coords) indistinguishable locally from grav. field → grav. field enters in exactly the same way.

$\Rightarrow$ In general coord system we have metric g_{ij} (Riemannian manifold)
$ds^2 = g_{ab}\,dx^a dx^b$
Equations of motion of particles in grav. field are geodesic eq^{n}s

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0.$$

RATHER: than having particles following "curved" lines in $\mathbb{R}^4$ as in Newton's theory — we have particles follow "straight lines" in Riemannian manifold.

(*) Note: in LIF

$$\mathring{g}_{ij;k} = \mathring{g}_{ij,k} - \mathring{\Gamma}^m_{ki}\,\mathring{g}_{mj} - \mathring{\Gamma}^m_{kj}\,\mathring{g}_{im} = 0$$

$\Rightarrow \mathring{g}_{ij;k} = 0$ in LIF (Each term RHS zero in LIF). $\mathring{g}_{ij,k} = 0$ $\mathring{\Gamma}^i_{jk} = 0$

(but tensor) $\Rightarrow g_{ij;k} = 0$ in any coord system
$\Rightarrow \nabla g = 0$
$\Rightarrow \Gamma$ is metric-induced / Levi-Civita connection.

(*)

SO: We are in Riemannian manifold
(C) suggests Riemann tensor plays a part.

Also (i) We need field equations to generate grav. field — in NTG Poisson's eq^n — 2nd derivs of ϕ — require equations in 2nd derivatives of g_{ab}.

(ii) From above particles in grav field obey geodesic eq^n. From (C) we find that geodesic deviation governed by $R^a{}_{bcd}$. We have argued that tidal grav. field enters via geodesic deviation (Gravit^n manifests itself in relative acceleration of nhbring particles)

Gravitational field $\Leftrightarrow$ $R^a{}_{bcd}$ – satisfies (3)

Field equations for g_{ab} ?

LHS	=	RHS
$\updownarrow$		$\updownarrow$
grav. field	–	sources for grav
$\updownarrow$		$\updownarrow$
terms involving $R^a{}_{bcd}$		T^{ij} "mass-energy" tensor.
		$\hookrightarrow$ (see derivation 2).

(1) Suppose $G^{ij} = \kappa T^{ij}$

(i) G^{ij} depends on grav. field, i.e g_{ab} [$G^{ij} = 0$ when $g^{ij} = \eta^{ij}$ when $T^{ij} = 0$]

(ii) $\kappa = 8\pi G$ – demand we get NTG (Poisson's eq^n) in appropriate limit.

(iii) Conservation of energy $T^{ij}{}_{;j} = 0 \Rightarrow G^{ij}{}_{;j} = 0$.

$\boxed{G^{ij}?}$

(iv) Assume constructed from $R^a{}_{bcd}$

(v) Assume G^{ij} contains (no higher than) 2nd derivatives of g_{ab} and is linear in 2nd derivatives of g_{ab} (analogy with Poisson's eq^n)

(vi) G^{ij} symmetric and is tensor (like T^{ij})

(viii) Riemann tensor only tensor satisfying (v) – G^{ij} "linear" in $R^a{}_{bcd}$ (not $R^a{}_{bcd} R^b{}_{aef}$ etc.) [see Weinberg p 133/134]

$$\Rightarrow \quad G^{ij} = R^{ij} + aRg^{ij} + bg^{ij}$$

a, b. constants.

(ix) Find a, b – When no matter present we get $G^{ij} = 0$ and we would want ST to be flat ($R^a{}_{bcd} = 0$). Note that if $b \neq 0$ ($g^{ij} = \eta^{ij}$)

$\Rightarrow$ $R^{ij} + aRg^{ij} \neq 0$ (contracting) $\Rightarrow (1+4a)R \neq 0$. Thus $b = 0$.

[Note: $1+4a \neq 0$ otherwise LHS of EFE's trace-free]

[Note: Einstein later added in just such a term λg^{ab} to EFE's to {avoid empty space sol^{n}s – mach's principle / produce steady state cosmologies} – "biggest blunder of my life"
– λ called cosmological constant & must be extremely small s.t Newton's eq^{n}s valid in appropriate limit; usually λ taken to be zero]

(x) $G^{ij}{}_{;j} = 0 \Rightarrow a = -\frac{1}{2}$ [Recall $(R^{ij} - \frac{1}{2}Rg^{ij})_{;j} = 0$ – Bianchi Identity]

EFE's $\boxed{G^{ij} \equiv R^{ij} - \frac{1}{2}Rg^{ij} = 8\pi G T^{ij}}$

(2) Hilbert (1915)

EFE's Derived from invariance principle $\int (\mathcal{L}_g + \mathcal{L}_m) \underbrace{\sqrt{-g}\, d^4x}_{\text{invariant volume element.}}$

$\mathcal{L}_g$ – scalar – depending on 2nd derivs of g (linear) $= R$

Euler-Lagrange Eq^{n}s $\Rightarrow$ $G^{ij} = 8\pi G T^{ij}$.

$$\uparrow$$

$$= \frac{\partial \mathcal{L}_g \sqrt{g}}{\partial g_{ab}}$$

Derivation 2 (physics based)

Require ① Agrees (locally) with SR
② Singles out class of preferred world lines representing free fall.
③ Admits tidal gravitational field
④ Covariant
[⑤ observation – see later].

Ⓐ Assume ST 4D Diff' manifold – ④ satisfied.

Try to derive GR by analogy with NTG and SR

Newton's eq^n's choose coord systems that are called Newtonian frames designed by the fact that they are freely falling frames (thus we have preferred frames with physical meaning)

In NTG take $x^i = (t, x^\alpha)$ (Abs. time / Abs. position)

$ST \equiv \mathbb{R} \otimes \mathbb{R}^3$

Thus $\dot{x}^i = (1, v^\alpha)$ $v^\alpha = \frac{dx^\alpha}{dt}$

In Newtonian Frames $\ddot{x}^\alpha + \phi_{,\alpha} = 0$

or $\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$

Not^n: $\alpha, \beta = 1,2,3$; $i, j = 1,2,3,4$

Where Γ^i_{jk} is symmetric connection st $\left.\begin{array}{l}\Gamma^\alpha_{00} = \phi_{,\alpha} \\ \text{else } \Gamma^i_{jk} = 0.\end{array}\right\}$ ⊛

With this connection ⊛, we find that in a Newtonian frame (assignment).

$R^\alpha{}_{0\beta 0} = \phi_{,\alpha\beta}$ else $R^i{}_{jkl} = 0$

Ricci tensor

And thus: $R_{00} = \delta^{\alpha\beta}\phi_{,\alpha\beta}$ else $R_{ij} = 0$ $[R_{ij} = R^k{}_{ikj}]$

Since $t_{,i} = \delta_i^0$ → write this as $R_{ij} = \underbrace{(\delta^{\alpha\beta}\phi_{,\alpha\beta})}_{\nabla^2\phi} t_{,i}t_{,j}$

and Poisson's eq^n becomes $R_{ij} = (4\pi G\rho)\, t_{,i}t_{,j}$

Suppose we have that the matter is homogeneous, with density ρ and pressure p (perfect fluid), then in a Newtonian frame we can write down Newton's laws of physics in the following form:

$u^i = (1, v^\alpha)$

$\Gamma^\alpha_{00} = \phi_{,\alpha}$, else $\Gamma^i_{jk} = 0$

$h^{ab} \stackrel{def}{=} \begin{pmatrix} 0 & 0\;0\;0 \\ \begin{smallmatrix}0\\0\\0\end{smallmatrix} & \delta^{\alpha\beta} \end{pmatrix}$ "flat metric" $\mathbb{R}^3$

$T^{ab} = (\rho + p)u^a u^b + p h^{ab}$ (Energy-momentum tensor)

Laws: $T^{ab}{}_{;b} = 0$

covariant deriv wrt Γ – laws expressed in grav. field.

(i) $a = 0 \Rightarrow$ continuity eq^n

(ii) $a = \alpha \Rightarrow$ Navier stokes eq^n

Special Relativity

In Newtonian frame (or in absence of gravity) – call them local inertial frames (LIF)

light moves in straight line $\ddot{x}^i = 0$

– $0 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{ij}dx^i dx^j$

Lorentzian flat metric (in LIF $\eta_{ij} = \mathrm{diag}(-1,1,1,1)$)

(gives conformal structure of ST).

Absolute time thrown away, coord time replaces it (proper time $d\tau^2 = \eta_{ij}dx^i dx^j$ invariant). In general $t_{,i} \neq \delta_i^0$

In SR, 4velocity vr for u^i – world line of particles

$x^i = x^i(\tau)$,

$$u^i = \frac{dx^i}{d\tau} \quad \text{with} \quad \eta_{ij}u^i u^j = -1 \text{ (normalized timelike vector).}$$

Define $t = x^0$, $v^i = \frac{dx^\alpha}{dt}$, $\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

so that $u^0 = \gamma$, $u^\alpha = \frac{dx^\alpha}{dt}\frac{dt}{d\tau} = \gamma v^\alpha \Rightarrow u^i = \gamma(1, v^\alpha)$.

In SR, perfect fluid μ is energy density, p pressure, u^i 4-velocity

$$T^{ij} = (\mu + p)u^i u^j + p\eta^{ij}$$

Laws of physics. $T^{ij}{}_{;j} = 0$ (absence gravity)

(i) $i = 0$ – first law of thermodynamics

(ii) ~~$i = \alpha$~~ ~~Relativistic~~ Navier-Stokes eq^n (with internal energy $\mu + p$)

[Note: Fluid description of matter. Energy momentum tensor, describes local "matter" content which will act as source for grav. field. From SR energy is source of grav field ($E = mc^2$) – T^{ij} contains energy, pressure, fluxes of matter – for more information see Weinberg. Note $T^{ij}{}_{;j} = 0$ is conservation of energy momentum – relativistic analogue of conservation of mass].

In GR   In Newtonian frame (LIF) combine above

$$\boxed{T^{ab} = (\mu + p)u^a u^b + p\eta^{ab}}, \quad \boxed{T^{ab}{}_{;b} = 0.}$$

(B) GR (Solution)

i) SEP: a) allows identification of L.I.F's in SR with Newtonian frames. SR valid in L.I.F's

b) (Similar to (A)) – Assume ST manifold endowed with metric (i.e Riemannian manifold) of signature (-1,1,1,1) – allows (1) and (2) to be satisfie

and

$$T^{ab} = (\mu + p)u^a u^b + p\eta^{ab}$$

$$T^{ab}{}_{;b} = 0$$

Note: Again we have metric and affine structure endowed upon M. In a L.I.F $g_{ij} = \eta_{ij}$, $g_{ij,k} = 0$, $\Gamma^i{}_{jk} = 0$ (Again we note that SEP says L.I.F same for NTG and SR – above cond^n's hold in same coord system) $\Rightarrow g_{ij;k} = g_{ij,k} - \Gamma^m{}_{ki}g_{mj} - \Gamma^m{}_{kj}g_{im} = 0 \Rightarrow \nabla g = 0 \Rightarrow \Gamma$ metric induced connecte

Field Eq^n's   Look to NTG and SR. Do for perfect fluid. From NTG and SR, in general frame, $T^{ab} = (\mu + p)u^a u^b + pg^{ab}$. Generalize $T^{ab}{}_{,b} = 0$ in SR, in presence of grav. field (NTG) to $T^{ab}{}_{;b} = 0$

Generalize Poisson's eq^n $R_{00} = 4\pi G\rho$ (*)

(a) LHS (*) $\longrightarrow R_{ij}u^i u^j$ u^i 4-velocity of fluid.

Rationale: from above we wrote Poisson's eq^n in L.I.F as $R_{ij} = (4\pi G\rho)t_{,i}$ (absolute t; $t_{,\alpha} = 0$). Only valid for low velocities. Replace $t_{,i}$ with u_i – "SR" 4-velocity, since $u^i = \gamma(1, v^\alpha)$ and if $v^\alpha \ll 1$, $\gamma \sim 1$ and $u^i \cong (1,0,0,0)$ (note that in SR $t_{,i} \neq 0$). Thus $R_{ij} = R_{00}u_i u_j \Rightarrow R_{ij}u^i u^j = R_{00}(u_i u^i)(u_j u^j) =$

– since $u_i u^i$. This result is true in L.I.F., but is covariant – true in all coord systems.

(b) RHS (✱) ρ is matter density in NTG. In SR (and GR) all energies will generate grav. field ⇒ must replace ρ with "Relativistic" energy-density type term.

In SR $T_{ij}u^i u^j = \{(\mu+p)u_i u_j + p\eta_{ij}\}u^i u^j = (\mu + p - p) = \mu$ = energy density

And $\mu = \rho(1+e)$, where e is internal energy, so that

$$\frac{\mu - \rho}{\rho} = \frac{\rho(1+e) - \rho}{\rho} = \frac{e}{\rho} \sim \frac{v^2}{c^2} \quad \text{(dimensions)}$$

and pressure p,

and $\frac{p}{\rho} \sim \frac{v^2}{c^2}$ (dimensions)

In NTG $\frac{v^2}{c^2} \ll 1 \Rightarrow \mu \sim \rho$, $p \sim 0$ | Thus replace ρ in (✱) with $\mu + \alpha p$ (lin comb. of μ & p)

Note that $T = T_{ij}g^{ij} = \{(\mu+p)u_i u_j g^{ij} + p g_{ij}g^{ij}\} = -\mu + 3p$

so, rather than taking RHS of (✱) to be linear combination of μ and p, we shall take (✱) as linear combination of μ and $(-\mu+3p)$ so that (writing this conveniently)

$$\text{RHS (✱)} \Rightarrow 4\pi G\{\lambda \underbrace{T_{ij}u^i u^j}_{\mu} + (1-\lambda)(\underbrace{-T}_{\mu - 3p})\}$$

$$= 4\pi G\left[\lambda T_{ij} + (1-\lambda)Tg_{ij}\right]u^i u^j$$

$[g_{ij}u^i u^j = -1]$

where λ is an arbitrary, as yet undetermined constant.

Thus Poisson's eq^n becomes

$$R_{ij}u^i u^j = 4\pi G\left[\lambda T_{ij} + (1-\lambda)g_{ij}T\right]u^i u^j$$

the simplest solution of which is

$$R_{ij} = 4\pi G\left[\lambda T_{ij} + (1-\lambda)g_{ij}T\right] \qquad (\dagger)$$

Determine λ.

(i) Take trace of $(\dagger)$ ⇒ $R = 4\pi G(4-3\lambda)T$ $[R = R_{ij}g^{ij}$, $g_{ij}g^{ij} = 4]$.

⇒ $R_{;i} = 4\pi G(4-3\lambda)T_{;i}$

(ii) Take divergence of $(\dagger)$

$$R_{ij}{}^{;j} = 4\pi G\{\lambda(\underbrace{T_{ij}{}^{;j}}_{0\ \text{(conserv}^n\text{ law)}}) + (1-\lambda)g_{ij}(T)^{;j}\}$$

$$= 4\pi G(1-\lambda)T_{;i}$$

But a) T scalar ⇒ $T_{;i} = T_{,i}$ b) Contracted Bianchi identities

⇒ $R_{ij}{}^{;j} = \frac{1}{2}R_{;i}$, thus

$$R_{;i} = 4\pi G(2-2\lambda)T_{;i}$$

Since, in general $T_{;i} \neq 0$, equating two expressions for $R_{;i}$

yields $2 - 2\lambda = 4 - 3\lambda$ ⇒ $\boxed{\lambda = 2}$.

Effectively obtained from conservation laws in GR.

Thus $\boxed{\text{Einsteins eq}^n\text{s are}}$ $\boxed{R_{ij} = 4\pi G(2T_{ij} - Tg_{ij})}$

or using $R = 4\pi G(4-3\lambda)T = -8\pi G T$, equivalently

$$\boxed{\left(R_{ij} - \tfrac{1}{2}Rg_{ij}\right) = 8\pi G T_{ij}}$$

Einstein Field Eq^{n}s of 1915

Problems with approach (only suggestive).

1. We have only "proven" EFE's for perfect fluid – assume true for all sources.
2. We need 10 eq^{n}s for 10 potentials g_{ab}. We have obtain tensor eq^n (10 eq^{n}s) from one scalar eq^n! Justification – seems to agree with experiments. More gen soln's than above can, and in fact have, been taken.

Derivation 3: Quasi-h torical. (Seminar)

The General Theory c Relativity
A Relativi ic Theory of Gravity.

Mathematics

Spacetime is repr ented by a 4-Dimensional, Hausdorff, differentiable manif l M endowed with a Lorentzian metric g_{ab}. The metric c nes measures of length.

An affine struct e is induced on M by g_{ab}. The metric connection is given by $\Gamma^a_{bc} = \frac{1}{2} g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d})$. The straight lines in the T manifold are the geodesics $\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0$ (the curves s.t.: tangent vector everywhere parallel / minimizes distance between 2 ST point).

The metric is n nifestation of gravitational field. Gravity is treated as ntrinsic property of ST by treating ST as Riemannian m anifold.

Einstein: "we therefore arr e at the r sult that the grav. field influences and even determin s the metrical laws of the ST continuum. If the laws of configu thon of ideal rigid bodies are to be expressed geometrically, then in the presence of a grav. field the geometry is not Euclidean".

$g_{\mu\nu}$ is 4x4 symmetric tensor ↔ 10 indep comps. We need 10 field eq^ns to determine the metric — given by Einstein's equations, which in vacuo are $R_{ab} = 0$. Since expressed in terms of Ricci tensor - field eq^ns have geometric quality.

Riemann tensor $R_{abcd} = C_{abcd} + \frac{1}{2} g_{ac}R_{bd} - \frac{1}{2} g_{ad}R_{bc} - \frac{1}{2} g_{bc}R_{ad} + \frac{1}{2} g_{bd}R_{ac} + \frac{1}{6} g_{ad}g_{cb}R - \frac{1}{6} g_{ac}g_{bd}R$.

C_{abcd} Weyl tensor. Note $C_{abcd}\,g^{ac} = 0$. Weyl tensor that part of Riemann tensor that doesn't depend on R_{ab} - so that in absence of gravity - when $R_{ab} = 0$ — Riemann tensor non zero which implies that a grav. field can exist in empty space [in 2,3 dimensions $R_{ab} = 0 \Rightarrow R^a{}_{bcd} = 0$ (since Weyl tensor tensor identically zero)].

Physics Gravitational field described by 10 gravitational potentials of the metric tensor, g_{ab}.

(1) Measurements described by g_{ab} via line element

$$\boxed{d\tau^2 = g_{ab}\,dx^a dx^b}$$

Thus gravitational field affects rods and clocks (i.e measuring process) through g. Ex: "ticking" of clocks in grav. field governed by above equation. Thus SR → clocks depend on velocity, GR → clocks depend on position (relation to grav. field).

(2) If particle is acted upon by no external forces it follows geodesic in ST. Thus, the equations of motion of test particles in a gravitational field are

$$\boxed{\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0}$$

(3) Need 10 differential equations for the 10 gravitational potentials, relating the potentials to the "source" of the grav. field - represented by the Energy - momentum tensor T [illegible] by Einstein's field equations.

EFE's $\boxed{R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G\, T_{ab}}$ G – Newtonian Grav. const.

– equations to determine g's from source.

These are 10 coupled, simultaneous, nonlinear second order PDE's for g_{ab}'s (linear in second derivs.). Eq^{n}s must be nonlinear since nonlinearity represents the fact that the gravitational field acts as a source for itself (∃ energy associated with grav. field).

Problem

First sight: 10 eq^{n}s (G_{ab} – EFE's)
– for 10 variables (g_{ab}).

But: 4 Bianchi identities (conservation laws) ⇒ only 6 indep EFE's!

Thus: 4 degree's of freedom available. – this corresponds to the fact that if g_{ab} is sol^n of EFE's, then so is g'_{cd} obtained by general coord transformation.

In GR: 6 EFE's + 4 coord cond^{n}s
↳ must choose coord system.

Analogy Maxwell's eq^{n}s 4 eq^{n}s – 4 EM potentials.
But Maxwell's eq^{n}s gauge invariant – 3 indep eq^{n}s.
– must choose gauge to solve eq^{n}s.

Cauchy Problem

Initial value problem: Given g_{ab} and $g_{ab,0}$ on spacelike hypersurface (everywhere for t = const), do EFE's give formulae for second time derivatives of g_{ab} so that we can establish evolution of g_{ab} (away from spatial hypersurface) for all t?

Need 10 eq^{n}s in 2nd time derivs of g_{ab}'s.

But 4 EFE's do not contain 2nd time derivs of g
– the eq^{n}s are therefore constraints on initial data

To see this: Bianchi identities can be written
$$\frac{\partial}{\partial x^0}(G^{a0}) = -\frac{\partial}{\partial x^\alpha}(G^{a\alpha}) - \Gamma^a_{bc}G^{bc} - \Gamma^b_{bc}G^{ac}$$

– RHS contains no time derivatives higher than $\partial^2/\partial t^2$ → so neither does LHS → G^{a0} contains no time derivatives higher than $\partial/\partial t$ → no information regarding the time evolution of the g_{ab}s from EFE's $G^{a0} = 8\pi G\, T^{a0}$.

Thus: Only six "dynamical" EFE's. $\left[\text{for } \frac{\partial^2 g_{\alpha\beta}}{\partial t^2}\right]$

4 more needed $\left[\text{for } \frac{\partial^2 g_{a0}}{\partial t^2}\right]$

But knew this already since we can always make coord transformation leaving g_{ab}, $g_{ab,0}$ unchanged on t = const but altering g_{ab} everywhere else.

Thus 4 cond^{n}s on $\frac{\partial^2 g^{a0}}{\partial t^2}$ obtained from 4 coord cond^{n}s.