Spherically Symmetric Spacetimes

A spherically symmetric ST is a 4D manifold $m = S^2 \times (S^2)^{\perp}$ where $(S^2)^{\perp}$ typically $\mathbb{R}^2$.
$\quad\quad\quad\quad\quad\quad\quad\;\; {}^{\urcorner}$2sphere $\; {}^{\urcorner}$2space

Let $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ be metric on S^2

Then if T, R are coords on $(S^2)^{\perp}$

metric

$$\boxed{ds^2 = -e^{2\Phi(T,R)}dT^2 + e^{2\Lambda(T,R)}dR^2 + r^2(T,R)\,d\Omega^2} \quad (1).$$

$\uparrow$ g_{ij} defined

Also obtained by symmetry requirements on g_{ij} ~~and choice coords.~~

Calculate Einstein tensor $G^a{}_b$.

[(1) Immediately from (1).

(2) Easier: Define $\omega_T = e^{\Phi}dT \quad \omega_R = e^{\Lambda}dR \quad \omega_\theta = r d\theta \quad \omega_\phi = r\sin\theta\, d\phi$,

$e_T = e^{-\Phi}\frac{\partial}{\partial T},\ e_R = e^{-\Lambda}\frac{\partial}{\partial R},\ e_\theta = r^{-1}\frac{\partial}{\partial\theta},\ e_\phi = r^{-1}\operatorname{cosec}\theta\frac{\partial}{\partial\phi}$ $\Big\}$ (2)

$\Rightarrow (\omega_T, \omega_R, \omega_\theta, \omega_\phi), (e_T, e_R, e_\theta, e_\phi)$ form orthonormal basis of T_P^*, T_P]

Results: If $r = \text{constant} \quad G^\theta{}_\theta = G^\phi{}_\phi = r^2$

If r not constant $G^\theta{}_\theta = G^\phi{}_\phi$ linear combination of $G^R{}_R, G^R{}_T, G^T{}_T$ $\Big\}$ (3).

Set $\mathring{f} = e_T(f) = e^{-\Phi}\frac{\partial f}{\partial T}$; $f' = e_R(f) = e^{-\Lambda}\frac{\partial f}{\partial R}$, Then

urvature tensor

$$\tfrac{1}{2}r^2\left(G^T{}_T - G^R{}_R\right) = re^{\Phi}\left(e^{-\Phi}r'\right)' + re^{\Lambda}\left(e^{-\Lambda}\mathring{r}\right)^{\circ} \quad (4)$$

$$\tfrac{1}{2}r^2\left(G^T{}_T + G^R{}_R\right) = e^{-\Phi}\left(e^{\Phi}rr'\right)' - e^{-\Lambda}\left(e^{\Lambda}r\mathring{r}\right)^{\circ} - 1 \quad (5)$$

$$\tfrac{1}{2}re^{\Lambda-\Phi}\,G^R{}_T = \mathring{r}\Phi' - (r')^{\circ} = \mathring{\Lambda}r' - (\mathring{r})' \quad (6)$$

[All expressions for Γ and R for Schwarzschild can be found in Weinberg p178].

[wrt g_{ij} defined by (1) and orthonormal basis;

$R_{ijk\ell} = \frac{1}{2}\left(g_{i\ell,jk} + g_{kj,\ell i} - g_{j\ell,ik} - g_{ik,j\ell}\right) \Rightarrow G^a{}_b$]

Einsteins Vacuum eqns : solns $\quad G_{ij} = 0$

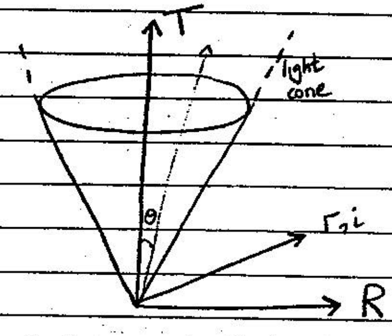
If $r = \text{constant}$, (3) $\Rightarrow G^\theta{}_\theta \neq 0 \Rightarrow$ contradiction

Thus

$r_{,i}$ defines nonzero vector (wrt some defined basis)

which can be $\left\{\begin{array}{l}\text{spacelike}\\ \text{timelike}\\ \text{lightlike.}\end{array}\right.$ ($r_{,i}$ can be represented in ST diag)

Case I $\;r_{,i}$ spacelike

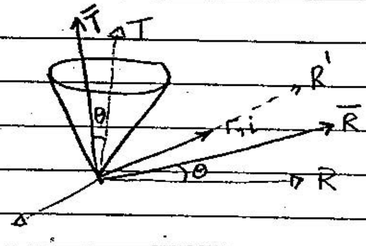


(null cones not affected by Lorentz transfm)

Make a Lorentz rotation to set $r = R$ (can do this since T would still lie in light cones)

Then
$r' = e^{-\Lambda}$
$\mathring{r} = 0.$

under Lorentz rotn



r is scalar field st $r_{,i}$ nonzero spacelike vector

Rotate R st $\bar{R}'$ aligned along $r_{,i}$

$\left[\bar{R}' = e^{-\Lambda}\frac{\partial R}{\partial R}\right]$ ($\bar{R}$ axis along $r_{,i}$)

Then $(6) \Rightarrow \dot{\Lambda} = 0 \Rightarrow \Lambda = \Lambda(R)$ $\boxed{= \Lambda(r)}$ $\left[G^R_T = 0 \text{ and } \dot{r} = 0 \right]$

$$(4) \Rightarrow e^{-\Phi} r' = f(T) = e^{-\Phi - \Lambda} \quad \left[\begin{array}{l} G^{ij} = 0, \dot{r} = 0 \\ \Rightarrow (e^{-\Phi} r')' = 0 \\ \Rightarrow e^{-\Phi} r' = f(T) \end{array} \right]$$

Suppose we define $t = t(T) = \int \frac{dT}{f(T)}$ $\left[\frac{dt}{dT} = \frac{1}{f} \right]$

Making transf^n $T \to t$ $\quad (dT^2 \to (\frac{\partial T}{\partial t})^2 dt^2)$

$$g_{TT} \to g_{tt} = \left(\frac{\partial T}{\partial t}\right)^2 . g_{TT} = f^2 g_{TT}$$

$$= e^{-2\Phi - 2\Lambda} e^{2\Phi} = e^{-2\Lambda}$$

Thus under this transf^n we have set $\Phi \to -\Lambda$ $\boxed{\text{or } \Phi + \Lambda = 0}$ $\left. \begin{array}{l} \text{originally } g_{00} = e^{2\Phi} \\ \text{, now } g_{00} = e^{-2\Lambda} \end{array} \right\}$

[Note that above $\Rightarrow \Phi + \Lambda = f(T) \rightarrow \Lambda = \Lambda(r) \Rightarrow \Phi + \Lambda = \text{const}$]

Finally $(5) \Rightarrow$

$$e^{-\Phi} \left[e^{\Phi} r r' \right]' + 0 - 1 = 0$$

But $f'_{,r} \equiv e^{-\Lambda} \frac{\partial f}{\partial r}$

$$\Rightarrow e^{-\Phi - \Lambda} \frac{\partial}{\partial r} \left(e^{\Phi} r r' \right) = 1$$

But $\Phi + \Lambda = 0$, $r' = e^{-\Lambda} = e^{\Phi}$

$$\Rightarrow \boxed{\frac{d}{dr} \left(e^{2\Phi} r \right) = 1}$$

Solving $r e^{2\Phi} = r - 2m$ [constant integration chosen as $2m$ *]

$$e^{2\Phi} = 1 - \frac{2m}{r}$$

$\Rightarrow$ metric

$$\boxed{ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)} \quad (7)$$

— Schwarzschild metric 1916

* constant of integration $C = -2m$

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \ldots$$

Note that we have solved EFE's in vacuum where $T_{ij} = 0$. But $g_{ij} \neq 0$ —because $T_{ij} \neq 0$ elsewhere (generating grav. field) Effect of grav. field being felt via b.c's (through m)

m arb constant - fixed by b.c's (matter distribution outside vacuum)

(1) No matter $\Rightarrow m = 0 \Rightarrow$ Minkowski SR.

(2) In linearized theory - for pt mass $\bar{m}$ (radial dist r from m)

sol^n $ds^2 = \left(-1 + \frac{2\bar{m}}{r}\right) dt^2 + \left(1 + \frac{2\bar{m}}{r}\right)(dr^2 + r^2 d\Omega^2)$ $\left[\begin{array}{l} h_{0\alpha} = 0 \\ \varepsilon h_{00} = 2\bar{m}/r \\ \varepsilon h_{\alpha\beta} = 2\bar{m}/r \, \delta_{\alpha\beta} \end{array} \right.$

For large r $\left(1 + \frac{2\bar{m}}{r}\right) \simeq \left(1 - \frac{2\bar{m}}{r}\right)^{-1}$ + higher order

Thus to "initial order $\Rightarrow ds^2 \simeq -\left(1 - \frac{2\bar{m}}{r}\right) dt^2 + \left(1 - \frac{2\bar{M}}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$

Where $\bar{m}$ is mass of centrally located pt. source.

We use linearized solⁿ to determine m via b.c's.
By comparison of linearized solⁿ (for large r) with schwarzschild solⁿ (7)
We see that (7) is exact solⁿ of (full, i.e nonlinear) EFE's
for a point mass at origin of SS grav. field.

$$\Rightarrow \text{B.c's} \Rightarrow m \text{ mass of object.}$$

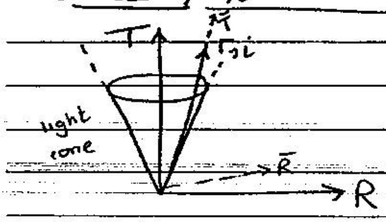
Note solⁿ outside boundary of SS star (vacuum) radius r_0 with total mass energy m^* – solⁿ $r > r_0$ given by (7) – where $m = m^*$ Grav. field outside SS star given by Schwarzschild solⁿ

Recall, this is valid iff $r_{,i}$ spacelike.
i.e.

$$g_{rr} > 0 \quad \Rightarrow \quad 1 - \frac{2m}{r} > 0 \quad = \begin{cases} m < 0 < r \\ \text{or} \\ 0 < 2m < r \end{cases}$$

Note that g_{ij} is t independent, i.e. static, and
That $g_{ij} \to \eta_{ij}$ as $r \to \infty$ [solⁿ due to SS central mass].

Case II ; $r_{,i}$ timelike



1. Rotate to set T axis along $r_{,i}$
2. defⁿ st $r = T$.

Then (assignment ?)

$$ds^2 = \frac{-dT^2}{\frac{2m}{T} - 1} + \left(\frac{2m}{T} - 1\right)dR^2 + T^2 d\Omega^2$$

$r_{,i}$ timelike $\Rightarrow g_{TT} < 0$

$$\Rightarrow \begin{cases} m < 0 < T \\ \text{or} \\ 0 < T < 2m. \end{cases}$$

This space homogeneous (no R dependence). As $R \to \infty$ $g_{ij} \not\to \eta_{ij}$
solⁿ represents anisotropic cosmological model.

Case III, $r_{,i}$ lightlike

$$g^{ij} r_{,i} r_{,j} = 0 \quad \Rightarrow \quad r'^2 = \dot{r}^2 \Rightarrow r' = \pm \dot{r}$$

$$\left.\begin{aligned} (6) &\Rightarrow r'' + r'\Phi' - \ddot{r} - \dot{\Lambda}\dot{r} = 0 \\ (5) &\Rightarrow r'' + r'\Phi' - \ddot{r} - \dot{\Lambda}\dot{r} = 1/r \end{aligned}\right\} \text{no sol}^n.$$

Thus $r_{,i}$ cannot be lightlike.

Birkhoff's Theorem
Spherical symmetry + Vacuum + asymptotic flatness ($g_{ij} \to \eta_{ij}$ as $r \to \infty$) implies Schwarzschild solⁿ.

Note 1. Vacuum + asymptotic flatness. Spherical symmetry $\Rightarrow$ staticity (Schwarzschild static)
Note 2. physical interpretation. ∃ no monopole gravitational radiation : ~~∃~~ monopole grav. rad from bounded source is spherically symmetric, vacuum, asymptotically flat and non static. (this means that if star collapses - no way for gravitational influence of radial collapse to propogate outward

Spherically Symmetric Matter

Consider perfect fluid
[Require $T_{ij}{}^{;j} = 0$]

$$T^i{}_j = \rho u^i u_j + p\,(\underbrace{g^i{}_j + u^i u_j}_{h^i{}_j})$$

demand that $\rho > 0$, $\rho + p > 0$

Back to original eq^{n}s – this time align T axis with U_i

U is timelike and is spherically sym, thus U lies in the RT plane. Make a transformation to set $U \parallel T$ axis i.e

$$U = e_T = e^{-\Phi}\frac{d}{dT}$$

i.e
$$U^i = (e^{-\Phi}, 0, 0, 0)$$
$$U_i = (-e^{+\Phi}, 0,0,0) = -e^{+\Phi} T_{,i} \quad \{T_{,i} = (1,0,0$$

Notice
$$\mathring{f} = e^{-\Phi} f_{,T}$$
$$= f_{,i}U^i \quad = \text{proper time derivative of } f \text{ along fluid streamlines}$$

Define 'radial velocity' $= V = \mathring{r}$

Next compute "acceleration" $\mathring{U}_i$

Defined by: $\mathring{U}_i \equiv U_{i;j}U^j = U_{i,j}U^j$, can show all christofel symbols cancel or vanish.

$$= (-e^{\Phi}T_{,i})_{,j}U^j$$
$$= T_{,i}(-e^{\Phi})_{,j}U^j - e^{\Phi}T_{,ij}U^j$$
$$= -e^{\Phi}T_{,i}\,\Phi_{,j}U^j - e^{\Phi}T_{,ij}U^j$$

But $\mathring{\Phi} = \Phi_{,j}U^j$ by def^n, $U_i = -e^{\Phi}T_{,i}$ $\Big\}$
$$= \mathring{\Phi}U_i - e^{\Phi}T_{,ij}U^j$$
$$= \mathring{\Phi}U_i - e^{\Phi}T_{,ji}U^j$$

can interchange since scalar and all christofels vanish

$$= \mathring{\Phi}U_i - e^{\Phi}\{(T_{,j}U^j)_{,i} - T_{,j}U^j{}_{,i}\}$$
$$= \mathring{\Phi}U_i - e^{\Phi}(\overset{\text{def}^n}{\mathring{T}})_{,i} + e^{\Phi}T_{,j}U^j{}_{,i}$$
$$= \mathring{\Phi}U_i - e^{\Phi}\mathring{T}_{,i} - \underbrace{U_jU^j{}_{,i}}_{\text{zero, } (U_jU^j)_{,i} = 0}$$

Note that $\mathring{T} = e^{-\Phi}\frac{\partial T}{\partial T} = e^{-\Phi}$ and $\mathring{T}_{,i} = (e^{-\Phi})_{,i} = -e^{-\Phi}\Phi_{,i}$

Hence $\mathring{U}_i = \mathring{\Phi}U_i + e^{\Phi}e^{-\Phi}\Phi_{,i}$

$$\underline{\underline{\mathring{U}_i = \mathring{\Phi}U_i + \Phi_{,i}}}$$

$h_i{}^j \equiv g_i{}^j + U_iU^j$ projection tensor.

Also $\mathring{U}_i = \mathring{\Phi}U_i + \Phi_{,i}$

can be written $\mathring{U}_i = h_i{}^j\Phi_{,j}$

Proof;
$$\mathring{U}_i = h_i{}^j\Phi_{,j}$$
$$= g_i{}^j\Phi_{,j} + U_iU^j\Phi_{,j}$$
$$= g_{ik}g^{kj}\Phi_{,j} + U_i\{\Phi_{,j}U^j\}$$
$$= \delta_i{}^j\Phi_{,j} + U_i\{\mathring{\Phi}\}$$
$$= \Phi_{,i} + \mathring{\Phi}U_i$$

Now we consider $U_{i;j} + \mathring{U}_iU_j$

Def^{n}s:

vorticity tensor
$\omega_{ij} = U_{[i;j]} + \mathring{U}_{[i}U_{j]}$

Strain tensor
$\theta_{ij} = U_{(i;j)} + \mathring{U}_{(i}U_{j)}$

expansion
$\theta = \theta_i{}^i$

shear tensor
$\sigma_{ij} = \theta_{ij} - \frac{1}{3}\theta h_{ij}$

$$U_{i;j} + \mathring{U}_iU_j = (-e^{-\Phi}T_{,i})_{;j} + (\mathring{\Phi}U_i + \Phi_{,i})U_j$$
$$= -e^{-\Phi}T_{,ij} + T_{,i}e^{-\Phi}\Phi_{,j} + \Phi_{,i}U_j + \mathring{\Phi}U_iU_j$$
$$= -e^{-\Phi}T_{,ij} + U_i\Phi_{,j} + \Phi_{,i}U_j + \mathring{\Phi}U_iU_j$$
$$\{= -e^{-\Phi}T_{,(ij)} + 2U_{(i}\Phi_{,j)} + \mathring{\Phi}U_{(i}U_{j)}\}$$

The last line is written to indicate that this is symmetric in i,j [i.e $\omega_{ij} = 0$ and hence no vorticity. ∃ shear eq^{n}s but very nasty and no point in bringing them in]

However (contracting on i,j)

expansion: $\theta \equiv U^i{}_{;i}$ $\{= g^{il}U^l{}_{;j} = U_{i;j}$, contract i and j$\}$

$= \mathring{\Lambda} + \frac{2\mathring{r}}{r}$ mathematically ⊛ (Exercise?).

$\Big\{$ Also physically $\theta = \frac{(\Delta V)^\circ}{\Delta V} \Big\}$

⊛ An indication of validity of $\theta = U^i{}_{;i} = \mathring{\Lambda} + \frac{2\mathring{r}}{r}$ will be shown.

~~$U^i{}_{;i} = g^{il}U_{l;i} = g^{il}\{U_{l,i} - \Gamma^k_{li}U_k\} = g^{il}\{U_{l,i} + \frac{1}{2}g^{li}(g_{...} + g_{ki,l} - g_{li,k})U_k\}$~~

~~Now: $ds^2 = -e^{2\Phi}dT^2 + e^{2\Lambda}dR^2 + r^2d\Omega^2$ $\quad g_{ij} = \begin{pmatrix} -e^{2\Phi} & & \\ & e^{2\Lambda} & \\ & & r^2 \end{pmatrix}$ $\quad g^{ij} = \begin{pmatrix} -e^{-2\Phi} & & \\ & e^{-2\Lambda} & \\ & & 1/r^2 \end{pmatrix}$~~

~~Nonvanishing $g_{ij,k}$ are~~

~~(*) $g_{TT,T} = -2e^{2\Phi}\frac{\partial\Phi}{\partial T}$ $\quad g_{RR,T} = 2e^{2\Lambda}\frac{\partial\Lambda}{\partial T}$ $\quad g_{\theta\theta,T} = 2r\frac{\partial r}{\partial T}$~~

~~$g_{TT,R} = -2e^{2\Phi}\frac{\partial\Phi}{\partial R}$ $\quad g_{RR,R} = 2e^{2\Lambda}\partial\Lambda/\partial R$ $\quad g_{\theta\theta,R} = 2r\frac{\partial r}{\partial R}$~~

(*) $\Theta = \dot{\Lambda} + \frac{2\dot{r}}{r}$

$$u^i{}_{;i} = g^{il} u_{l;i} = g^{il}\{u_{l,i} - \Gamma^k{}_{li} u_k\}$$

$$= g^{il}(u_{l,i}) - \tfrac{1}{2} g^{il} g^{km} u_k (g_{lm,i} + g_{im,l} - g_{il,m})$$

$$ds^2 = -e^{2\Phi} dT^2 + e^{2\Lambda} dR^2 + r^2 d\Omega^2 \qquad g_{ij} = \begin{pmatrix} -e^{2\Phi} & & \\ & e^{2\Lambda} & \\ & & r^2 \end{pmatrix} \quad g^{ij} = \begin{pmatrix} -e^{-2\Phi} & & \\ & e^{-2\Lambda} & \\ & & \frac{1}{r^2} \end{pmatrix}$$

nonvanishing $g_{ij,k}$ are
$$\begin{cases} g_{TT,T} = -2e^{+2\Phi}\frac{\partial\Phi}{\partial T} & g_{RR,T} = 2e^{2\Lambda}\frac{\partial\Lambda}{\partial t} \\ g_{TT,R} = -2e^{+2\Phi}\frac{\partial\Phi}{\partial R} & g_{RR,R} = 2e^{2\Lambda}\frac{\partial T}{\partial R} \\ g_{rr,T} = 2r\frac{\partial r}{\partial T} & g_{rr,R} = 2r\frac{\partial r}{\partial T} \end{cases}$$

$$g^{il} u_{l,i} = g^{iT}(-e^{\Phi})_{,i}$$

$$= g^{TT}(-e^{\Phi})_{,T} = e^{-2\Phi}.e^{\Phi}\,\Phi_{,T} = e^{-\Phi}\frac{\partial\Phi}{\partial t}$$

$$= \dot{\Phi}$$

$u_i = (-e^{\Phi}, 0, 0, 0)$

$g^{il} = 0 \quad \text{if } i \neq l$

$$-\tfrac{1}{2} g^{il} g^{Tm} (g_{lm,i} + g_{im,l} - g_{il,m}) u_T$$

$$= -\tfrac{1}{2} g^{il} g^{TT} (g_{lT,i} + g_{iT,l} - g_{il,T}) u_T$$

$$= -\tfrac{1}{2} g^{TT} u_T \{ g^{TT} g_{TT,T} + g^{RR}(-g_{RR,T}) + 2g^{\theta\theta}(-g_{\theta\theta,T}) \}$$

$$= +\tfrac{1}{2} e^{-2\Phi} \cdot -e^{\Phi} \{ +2e^{-2\Phi} e^{2\Phi} \frac{\partial\phi}{\partial T} - 2e^{-2\Lambda} e^{2\Lambda}\frac{\partial\Lambda}{\partial T} - 2\frac{1}{r^2} 2r\frac{\partial R}{\partial T} \}$$

$$= -e^{-\Phi}\{ \frac{\partial\Phi}{\partial t} - \frac{\partial\Lambda}{\partial T} = \frac{2}{r}\frac{\partial r}{\partial T} \}$$

$$= -\dot{\Phi} + \dot{\Lambda} + \frac{2\dot{r}}{r}$$

Thus $\Theta = \dot{\Phi} + \{ -\dot{\Phi} + \dot{\Lambda} + \frac{2\dot{r}}{r} \} = \dot{\Lambda} + \frac{2\dot{r}}{r}$

First examine $T_{ij}{}^{;ij} = 0$

From earlier, for perfect fluid
$$\boxed{\begin{array}{l} \overset{\circ}{\rho} + (\rho+p)\Theta = 0 \quad (\\ (\rho+p)\overset{\circ}{u}{}^i + h^{ik} p_{,k} = 0 \quad (\end{array}}$$

Proofs $\quad T^{ij}{}_{;j} = [(\rho+p)u^i u^j + p g^{ij}]_{;j} = (\rho+p)_{,j} u^i u^j + (\rho+p)\{u^i{}_{;j}u^j +$

$+ p_{,j} g^{ij} = 0$

(1) Multiply by u_i (use $u^i{}_{;k} u_i = 0$ from $(u^i u_i)_{;k} = 0$ where $u_i u^i = -1$)

$$-(\rho+p)_{,j} u^j + (\rho+p)(-u^j{}_{;j}) + p_{,j} u^j = 0$$

$$\Rightarrow \rho_{,j} u^j + (\rho+p) u^j{}_{;j} = 0$$

$$\Rightarrow \dot{\rho} + (\rho+p)\Theta = 0.$$

(2) Regrouping $\quad \rho_{,j} u^i u^j + (\rho+p)\{u^i{}_{;j}u^j + u^j{}_{;j}u^i\} + (p)_{,j} h^{ij} = 0$

$$\Rightarrow (\rho_{,j} u^j + \overset{\nearrow}{(\rho+p)} u^j{}_{;j}) u^i + (\rho+p) u^i{}_{;j} u^j + p_{,j} h^{ij}$$

$$\equiv (\rho+p)\overset{\circ}{u}{}^i + p_{,j} h^{ij} = 0.$$

For $i = T \qquad \Rightarrow \qquad \dot{\rho} + (\rho+p)\Theta = 0$

$$\Rightarrow \boxed{\overset{\circ}{\rho} + (\rho+p)\left(\dot{\Lambda} + \frac{2\dot{r}}{r}\right) = 0 \quad (}$$

$i = 2,3,4 \qquad (\rho+p)\dot{u}_i + h_{ij} p^{,j} = 0$

$$h_i{}^j \Phi_{,j}$$

$$(\rho+p) h_i{}^j \Phi_{,j} + h_i{}^m p_{,m} = 0$$

$$h_i{}^j \left[\Phi_{,j} + \frac{p_{,j}}{(\rho+p)} \right] = 0$$

$(\rho+p) > 0$

Define Enthalpy of a fluid to be $W = \int \frac{dp}{\rho + p}$
(assuming an eq^n of state $\rho = \rho(p)$
or $p = p(e)$)

Note that $W_{;j} = \left(\int \frac{dp}{(e+p)}\right)_{;j} = \int \left(\frac{1}{e+p}\right)_{;j} dp$

$$= \int \frac{\partial}{\partial p}\left(\frac{1}{e+p}\right) p_{;j}\, dp = \left[\frac{p_{;j}}{e+p}\right] - \int \frac{1}{(e+p)} \frac{\partial}{\partial p}(p_{;j}) dp$$

But $\frac{\partial}{\partial p}(p_{;j}) = \left(\frac{\partial p}{\partial p}\right)_{;j} = (1)_{;j} = 0$. Hence $W_{;j} = \frac{p_{;j}}{e+p}$.

Hence $h_i^{\ j}(\phi_{;j} + W_{;j}) = 0$

Thus on any surface T = constant

$$d\phi + dW = 0.$$

$$\therefore \quad \phi(R,T) = \underset{\substack{\uparrow \\ \text{arbitrary "constant"}}}{\phi(0,T)} - \int_0^R \left(\frac{\partial W}{\partial R}\right) dR$$

Centre of fluid is where 2 sphere S^2 has zero radius i.e $r = 0$ and we fix origin of R coord by $R = 0$ there. Choose T to be proper time at centre.

$$e^{\phi(0,T)} = 1 \quad \text{hence } \phi(0,T) = 0.$$

$$\therefore \quad \phi(R,T) = -\int_0^R \left(\frac{\partial W}{\partial R}\right)_T dR \quad — \quad (9)$$

For perfect fluid we have solved $T^{ij}{}_{;j} = 0$. Still have to solve einstein's eq^{n}s $G_{ij} = 8\pi G T_{ij}$.

Summary so far:

$$ds^2 = -e^{2\phi} dT^2 + e^{2\Lambda} dR^2 + r^2(R,T)\, d\Omega^2$$

For perfect fluid;

$$\left.\begin{array}{l} T_{ij} = e u_i u_j + p h_{ij} \\ u^i = (1,0,0,0) \end{array}\right\} \quad T_{ij}{}^{;j} = 0 \Rightarrow \begin{array}{ll} i=0 & \overset{\circ}{e} + (p+e)\left(\overset{\circ}{\Lambda} + \frac{2\overset{\circ}{r}}{r}\right) = 0 \\ i=1,2,3 & \phi(R,T) = -\int_0^R \left(\frac{\partial W}{\partial R}\right)_T dR \end{array}$$

Field eq^{n}s in Matter.

We have deduced all possible, we must now turn to Einstein's eq^{n}s.

Using $T_{ij} = e u_i u_j + p h_{ij}$ we find

$$\left.\begin{array}{l} T^R{}_T = 0 \\ T^T{}_T = -e \\ T^R{}_R = p \\ \text{And so from } G_{ij} = 8\pi G T_{ij} \text{ we deduce (setting } G = 1) \end{array}\right\} \text{Se N}$$

$$\boxed{\begin{array}{llll} G^R{}_T & = 0 & — & (10) \\ G^T{}_T - G^R{}_R & = -8\pi(e+p) & — & (11) \\ G^T{}_T + G^R{}_R & = -8\pi(e-p) & — & (12) \end{array}}$$

in appropriate form to compare with eq^{n}s (4), (5) & (

Now we add (11) and (12) and using (4) & (5) with $\overset{\circ}{r} = V$ we obtain, see Note (ii) for proof.

$$\boxed{8\pi e r^2 = 1 + V^2 - r'^2 - 2rr'' + \underbrace{2rV\overset{\circ}{\Lambda}}_{}}$$

chosen "combinatio of EFE's th does not contai

And using (10) and (6)

$$\boxed{V' = \overset{\circ}{\Lambda} r'}$$

Substitute $\overset{\circ}{\Lambda} = \frac{V'}{r'}$ to eliminate $\overset{\circ}{\Lambda}$ from above

eq^n ⇒ $8\pi\rho r^2 r' = r' + r'V^2 - r'^3 - 2rr''r' + 2rVV'$

Note that $(r[1+V^2-r'^2])' = r' + r'V^2 + 2rV'V - r'^3 - 2rr'r''$

Hence $8\pi\rho r^2 r' = (r[1+V^2-r'^2])'$

to obtain $8\pi e r^2 r' = (r[1+V^2-r'^2])'$ — (

Field eq^ns with matter. Note (i).

$T^i{}_j = eu^iu_j + ph^i{}_j$

$h_{ij} = g_{ij} + u_iu_j \qquad h^i{}_j = g^{il}h_{lj} = g^{il}g_{lj} + u^iu_j$
$\qquad\qquad = \delta^i{}_j + u^iu_j$

$\therefore\ T^i{}_j = (e+p)u^iu_j + p\delta^i{}_j.$

Now $u_i = (-e^{\phi},0,0,0)$
$\quad u^i = (e^{-\phi},0,0,0).$

$\therefore\ T^R{}_T = 0$

$T^T{}_T = (e+p)\,u_Tu^T + p$
$\quad = (e+p)-e^{-\Phi}e^{\Phi}+p$
$\quad = -(e+p)+p = -e$

$T^R{}_R = p$

$\therefore$ Taking $G=1$ $\quad G^i{}_j = 8\pi G T^i{}_j$ yields

$G^R{}_T = 0$
$G^T{}_T = -8\pi e$
$G^R{}_R = 8\pi p.$

Hence
$G^R{}_T = 0$
$G^T{}_T - G^R{}_R = -8\pi(p+e)$
$G^T{}_T + G^R{}_R = -8\pi(e-p).$

Note (ii) Note that $f' = e^{-\Lambda}\frac{\partial f}{\partial R}$ $\quad \mathring{f} = e^{-\phi}\frac{\partial f}{\partial T}$ $\quad \mathring{r} = V$

Eq^n (4) $\Rightarrow \frac{1}{2}r^2(G^T{}_T - G^R{}_R) = re^{\phi}(e^{-\phi}r')' + re^{\Lambda}(e^{-\Lambda}\mathring{r}$

Eq (5) $\Rightarrow \frac{1}{2}r^2(G^T{}_T + G^R{}_R) = e^{-\phi}(e^{\phi}rr')' - e^{-\Lambda}(e^{\Lambda}r\mathring{r})^{\circ} -$

Adding we get $(G^T{}_T = -8\pi e)$

$$8\pi e r^2 = -re^{\phi}(e^{-\phi}r')' - re^{\Lambda}(e^{-\Lambda}\mathring{r})^{\circ}$$
$$- e^{-\phi}(e^{\phi}rr')' + e^{-\Lambda}(e^{\Lambda}r\mathring{r})^{\circ} +$$

$$= 1 + \{-r\cancel{e^{\phi}e^{-\phi}}^{=1}r'' - rr'e^{\phi}(e^{-\phi})'\} - \{e^{\Lambda}e^{-\Lambda}\overset{\circ\circ}{r} + re^{\Lambda}\mathring{r}(e$$
$$- \{e^{-\phi}e^{\phi}(rr')' - rr'e^{-\phi}(e^{\phi})'\} + \{e^{-\Lambda}e^{\Lambda}(r\mathring{r})^{\circ} + e^{-\Lambda}r\mathring{r}($$

$$= 1 + \{-rr'' + rr'e^{\phi}\cancel{e^{-\phi}}e^{-\Lambda}\frac{\partial\phi}{\partial R}\} - \{r\overset{\circ\circ}{r} + re^{\Lambda}\mathring{r}(e^{-\Lambda}$$
$$- \{\overset{-rr''+r'^2}{(rr')'} - rr'e^{-\phi}\cancel{e^{-\Lambda}}e^{\phi}\frac{\partial\phi}{\partial R}\} + \{(r\mathring{r})^{\circ} + e^{-\Lambda}r\mathring{r}($$

$$= 1 - 2rr'' - r'^2 + \{-r\overset{\circ\circ}{r} - re^{\Lambda}\mathring{r}(e^{-\Lambda})^{\circ} + (r\mathring{r})^{\circ} + e^{-\Lambda}r\mathring{r}(e$$

$$= \text{"} \quad + \{-r\overset{\circ\circ}{r} - re^{\Lambda}\mathring{r}e^{-\phi}\frac{\partial}{\partial T}(e^{-\Lambda}) + (r\mathring{r})^{\circ} + e^{-\Lambda}r\mathring{r}e$$

$$= \text{"} \quad + \{-r\overset{\circ\circ}{r} - re^{\Lambda}\mathring{r}e^{-\Lambda}\cdot -1\ e^{-\phi}\frac{\partial\Lambda}{\partial T} + (r\mathring{r})^{\circ} + e^{-\Lambda}r\mathring{r}e^{-\phi}\frac{\partial}{\partial}$$

$$= \text{"} \quad + \{-r\overset{\circ\circ}{r} + r\mathring{r}\mathring{\Lambda} + (r\mathring{r})^{\circ} + r\mathring{r}\mathring{\Lambda}\}$$

$$= \text{"} \quad + \{-\cancel{r\overset{\circ\circ}{r}} + r\mathring{r}\mathring{\Lambda} + \cancel{r\overset{\circ\circ}{r}} + \mathring{r}^2 + r\mathring{r}\mathring{\Lambda}\}$$

$$= 1 - 2rr'' - r'^2 + 2r\mathring{r}\mathring{\Lambda} + \mathring{r}^2.$$

$$= 1 - 2rr' - r'^2 + 2rV\mathring{\Lambda} + V^2 \quad \text{with } \mathring{r} = V.$$

Define $m(R,T) = \int_{0\ T\text{const}}^{R} 4\pi r^2 e\, dr$ (14)

which is essentially the 'mass' inside sphere of radius R.

Now (13) can be integrated

$$8\pi r^2 e r' = (r[1+V^2-r'^2])'$$

$' \equiv e^{-\Lambda}\frac{\partial}{\partial R}$ $\Rightarrow e^{-\Lambda}\left\{2 \quad 4\pi r^2 e \frac{dr}{dR}\right\} = \frac{d}{dR}\left[r[1+V^2-r'^2]\right]\Big\} e^{-\Lambda}$

cancel $e^{-\Lambda}$ $\int$ate $\Rightarrow$ $2\int_0^R 4\pi r^2 e \frac{dr}{dR} dR = r[1+V^2-r'^2] + f(T)$ arbitary fn of T from integration

$$2\int_0^R 4\pi r^2 e\, dr = r[1+V^2-r'^2] + f(T)$$

Hence $2m = r[1+V^2-r'^2] + f(T)$ — (15)

The <u>physical Interpretation</u> of metric (1) is that of sphere centre r=0 and radius given by

$$\text{radius} = \int_0^R e^{\Lambda} dR.$$

and area $4\pi r^2(R,T)$.

In the limit $R \to 0$ we expect to recover Euclidean geometry, i.e $r \to$ radius

As $R \to 0$ radius $= \int_0^R e^{\Lambda} dR \to e^{\Lambda(0,T)} R + O(R^2)$ (*).

$e^{\Lambda} = e^{\Lambda(0,T)} + O(R)$

And, by def^n. $r' = e^{-\Lambda}\frac{\partial r}{\partial R}$

For small R we can integrate to obtain $Rr' \sim e^{-\Lambda} r \to r \sim e^{\Lambda} R r'$

Thus, for small R, using $r \sim e^{\Lambda} R r'$ and series expansion

$$r(R,T) = e^{\Lambda(0,T)} r'(0,T) R + O(R^2) \quad (**)$$

Thus, for consistency, from (*) and (**) $r'(0,T) = 1$ at centre.
[This is cond^n that local geometry at centre is Euclidean].

Thus. B.c's at centre $R = r = 0$

$\Phi = 0$

$r' = 1$

$m = 0$ (want mass inside sphere of zero radius to be zero).

$\frac{2m}{r} \to 0$ (assuming e continuous

Using B.c's in eq^n (15) yields $f(T) = 0$.

Hence $\boxed{r'^2 = 1 + V^2 - \frac{2m}{r}}$ (16)

Now $e^{\Lambda} dR = \frac{dr}{r'}$, and so on any T = constant surface the metric is

$$(ds^2)_{T\text{const}} = \frac{dr^2}{\left(1+V^2-\frac{2m}{r}\right)} + r^2 d\Omega^2 \quad (17)$$

[Full metric includes $-\underbrace{e^{-2\Phi}}_{\text{Known}} dT^2$

Thus looks like schwarzschild, so consider interpretation of m

$$dV = 4\pi r^2 d(\text{radius})$$ {Note r ≠ radius – but $4\pi r^2$ is area}

$$= 4\pi r^2 e^{\Lambda} dR$$ $\left(\text{radius} = \int_0^R e^{\Lambda} dR\right)$

$$= \frac{4\pi r^2 dr}{r'}$$ $\left(r' = e^{-\Lambda}\frac{dr}{dR}\right)$

Thus $m = \int 4\pi r^2 \rho \, dr = \int \rho r' dV$

$$= \int \rho \left(1 + v^2 - \frac{2m}{r}\right)^{1/2} dV$$

$$\simeq \int \underbrace{\rho}_{\text{rest mass + thermal energy}} \Big(1 + \underbrace{\tfrac{1}{2}v^2}_{\text{kinetic energy}} - \underbrace{\tfrac{m}{r}}_{\text{Newtonian potential energy}}\Big) dV$$

Assuming $\frac{v^2}{c^2} \ll 1$, $\frac{m}{r} \ll 1$ using binomial expansion

⇒ mass m = total energy

Time evolution of star

Need equations for $\dot{m}, \dot{v}, \dot{\Phi}$

For $\dot{v}$ look at $(T_T ; R_R)$ eqⁿs (11) and (4) ⇒

$$4\pi r^2(\rho + p) = r\dot{r}\dot{\Lambda} + rr'\Phi' - r\ddot{r} - rr''$$

assuming $r' \neq 0$, $v' = \dot{\Lambda} r'$, (10) and (6) we obtain

$$-4\pi r r'(\rho + p) = \tfrac{1}{2}(r'^2 - v^2)' - r'^2\Phi' + r'\dot{v}$$

$v = \dot{r}$, ∴ $\ddot{r}$ replaced by $\dot{v}$, $(r'^2)' = 2r'r''$ etc

---

Using (16) $\tfrac{1}{2}(r'^2 - v^2)' = -\frac{m'}{r} + \frac{m}{r^2}r' = -4\pi r r'\rho + \left(\frac{m}{r^2}\right)r$

$= \frac{1}{2}\left(1 - \frac{2m}{r}\right)'$

Thus

$$-4\pi r r'(\cancel{\rho} + p) = -4\pi \cancel{r r'\rho} + \left(\frac{m}{r^2}\right)r' - r'^2\Phi' + r$$

or $$\boxed{\underbrace{\dot{v}}_{\text{radial acceleration ~~outwards~~}} = \underbrace{r'\Phi'}_{\text{geom. term}} - \underbrace{\frac{m}{r^2}}_{\text{looks like Newtonian attraction (G=1)}} - \underbrace{4\pi r p}_{\text{pressure term}}} \quad (18)$$

In GR pressure is attractive. Press cannot halt coll of star in GR

For $\dot{m}$ multiply (18) by rv

$$-4\pi r^2 p v = \frac{mv}{r} + rv\dot{v} - rr'\Phi' v.$$

$$= \frac{mv}{r} + rv\dot{v} - rr'(r')^{\circ}$$ $\left\{ (r')^{\cdot} = \Phi' v \text{ from (10) and (}\right.$

$$= \frac{mv}{r} + rv\dot{v} - \tfrac{1}{2}r(r'^2)^{\cdot}$$ using (16) fo

$$= \frac{mv}{r} + rv\dot{v} - \tfrac{1}{2}r\left\{2v\dot{v} - \frac{2\dot{m}}{r} + \frac{2m\dot{r}}{r^2}\right\}$$ $\dot{r} = v$

$$= \dot{m}$$

Hence $$\boxed{\underbrace{\dot{m}}_{\text{rate of change of total energy}} = \underbrace{-4\pi r^2 p v}_{\text{Rate of working of pressure}}} \quad (19)$$

{ $dU = -pdV$ 1st law of thermod (consequence of EFE's

For $\dot{\Phi}$ to proceed further need to specify $\dot{\Phi}$ (no way of finding it – must be given).

↳ specifies type of matter...

We now examine (non physical example) of a pressure free star. We note that since in (18) we have $-p$ term, the consequence of including pressure in model would be thus ; without pressure the solution will predict (stellar) collapse, inclusion of pressure will tend to drive (help) collapse. (due to $-p$ term). Thus investigating pressure free matter we obtain essential theory.

Pressure Free Collapse.

$$p = 0 \qquad e = e(T) \text{ (indep of R).} \begin{cases} \text{i.e for a dust cloud} \\ T^{ab} = e u^a u^b \end{cases}$$

Tolman (1934), oppenheimer & Snyder (1939)

$$p = 0 \Rightarrow W = 0 = \int \frac{dp}{p+e} \Rightarrow \Phi = \Phi(T) \text{ alone}$$

But from eqⁿ (9) $\Phi(0,T) = 0 \Rightarrow \boxed{\Phi = 0}$

Now the $T^1{}_0 = 0$ eqⁿ gives : Note that $\begin{cases} {}^{\circ} = e^{-\phi}\frac{\partial}{\partial T} \overset{\text{since } \phi=0}{=} \frac{\partial}{\partial T} \\ {}' = e^{-\Lambda}\frac{\partial}{\partial R} \end{cases}$

$$\mathring{\Lambda} = \frac{V'}{r'} = \frac{e^{-\Lambda}\frac{\partial}{\partial R}(V)}{e^{-\Lambda}\frac{\partial}{\partial R}(r)}$$

But $\mathring{\Lambda} = \frac{\partial \Lambda}{\partial T}$ and $V = \mathring{r} = \frac{\partial r}{\partial T}$

Hence $$\frac{\partial \Lambda}{\partial T} = \frac{\frac{\partial}{\partial R}\left(\frac{\partial r}{\partial T}\right)}{\frac{\partial r}{\partial R}}$$

Integrating we get

$$\Lambda = \log\left(\frac{\partial r}{\partial R}\right) + \underset{\nwarrow \text{ due to } \int \text{tion}}{\text{function}(R)}$$

Hence $$e^{2\Lambda} = e^{\{2\log(\frac{\partial r}{\partial R}) + 2\text{function}(R)\}}$$

$$= \left(\frac{\partial r}{\partial R}\right)^2 \cdot \text{function}(R)$$

Write $$e^{2\Lambda} = \frac{\left(\frac{\partial r}{\partial R}\right)^2}{1 + f(R)}$$ where $(1+f) > 0$ $\left(\frac{1}{1+f(R)} = \text{function } R\right)$

$$\therefore \quad (1 + f(R)) = \left(e^{-\Lambda}\frac{\partial r}{\partial R}\right)^2$$

But by defⁿ $r' = e^{-\Lambda}\frac{\partial r}{\partial R}$ Hence $\boxed{(r')^2 = 1 + f(R) \quad — (20)}$

Now with $\Phi = 0, p = 0$

eq(18) $\Rightarrow \mathring{V} = -\frac{m}{r^2}$ (*)

eqⁿ(19) $\Rightarrow \mathring{m} = 0. \Rightarrow m = m(R).$

Integrate (*) to get $V^2 = \frac{2m(R)}{r} + g(R)$ — (21)

Proof; $\mathring{V} = \frac{dV}{\partial T}$ since $\Phi = 0$

$$\frac{\partial V}{\partial T} = -\frac{m(R)}{r^2}$$

Now $V = \mathring{r} = \frac{\partial r}{\partial T}$ since $\Phi = 0$; multiply both sides by V

$$\therefore \quad V\frac{\partial V}{\partial T} = -\frac{m(R)}{r^2}\frac{\partial r}{\partial T}$$

$$\frac{\partial}{\partial T}\left(\tfrac{1}{2}V^2\right) = +m(R)\frac{\partial}{\partial T}\left(\frac{1}{r}\right) = \frac{\partial}{\partial T}\left(\frac{m(R)}{r}\right)$$

Hence $V^2 = \frac{2m(R)}{r} + g(R)$

But eqⁿ(16) $r'^2 = 1 + V^2 - \frac{2m}{r}$

Using eqⁿ(21) $r'^2 = 1 + \left(\frac{2m}{r} + g(R)\right) - \frac{2m}{r}$

$$= 1 + g(R).$$

And eqⁿ(20) $r'^2 = 1 + f(R).$

Hence $f(R) = g(R)$.

Now, by defⁿ(14)

$$m(R) = \tfrac{4}{3}\pi r^3 e(T) \qquad (\dagger)$$

i.e. $$r = r(R,T) = a(T)\,b(R)$$

with $\left.\begin{aligned} e(T) &= e_0/a^3(T) \\ m(R) &= \tfrac{4}{3}\pi e_0 b^3(R) \end{aligned}\right\}$ to satisfy $(\dagger)$

$$\overset{\circ}{\Lambda} \left(= \frac{v'}{r'}\right) = \frac{\frac{\partial}{\partial T}\left(\frac{\partial r}{\partial R}\right)}{\frac{\cdot \partial r}{\partial R}} = \frac{\frac{\partial}{\partial T}\left(a(T)\, b'(R)\right)}{a(T)\, b'(R)}$$ where ' denote $\frac{\partial}{\partial R}$

$$= \frac{\overset{\circ}{a}\, b'(R)}{a\, b'(R)} = \frac{\overset{\circ}{a}}{a}$$

So $\overset{\circ}{\Lambda} = \frac{\overset{\circ}{a}}{a}$

Integrating

$$\Lambda = \log a + k(R)$$

$$\Rightarrow e^{\Lambda} = a(T)\, e^{k(R)} = c(R)\, a(T).$$

make transformation $R \rightarrow \bar{R}$ to set $c(R) = 1$.

Hence $\underline{\underline{e^{\Lambda} = a(T)}}$.

Now eq^n (21) $\quad V^2 = f(R) + \frac{2m(R)}{r}$

making substitutions

$$\left(\frac{\partial}{\partial T}(r)\right)^2 = \left(\frac{\partial}{\partial T}(a(T)b(R))\right)^2 = \overset{\circ}{a}^2(T)\, b^2(R)$$

$$= f(R) + 2, \frac{\frac{4}{3}\pi\rho_0\, b^3(R)}{a(T)\, b(R)}.$$

i.e $$\overset{\circ}{a}^2 b^2 = f + \frac{\frac{8}{3}\pi\rho_0\, b^3}{a\, b}$$

$$\underset{\uparrow \atop f_n T}{\left(\frac{\overset{\circ}{a}}{a}\right)^2} = \underset{\uparrow \atop f_n R}{\left(\frac{f}{b^2}\right)} \underset{\uparrow \atop f_n T}{\frac{1}{a^2}} + \underset{\uparrow \atop f_n T}{\frac{\frac{8}{3}\pi\rho_0}{a^3}}$$

Only way this can hold is for $\frac{f}{b^2}$ to be a constant.

Hence $\underline{f = \lambda b^2}$ $\quad\lambda$ constant.

Hence $$\underline{\underline{\left(\frac{\overset{\circ}{a}}{a}\right)^2 = \frac{\lambda}{a^2} + \frac{(8\pi\rho_0/3}{a^3}}} \quad - \quad (22)$$

$\Big\{$ Robertson-Walk
Friedmann

Note that $ds^2 = -e^{-2\Phi}dT^2 + e^{2\Lambda}dR^2 + r^2 d\Omega^2$
where, in case considered, $T^{ij} = \rho(T) u^i u^j$

(i) $\Phi = 0$ (ii) $e\Lambda = a(T)$ (iii) $r(R,T) = a(T)\, b(R)$

From eq^n (20) $\quad (r')^2 = 1 + f(R)$

$$\left\{ r' = e^{-\Lambda}\frac{\partial r}{\partial R} = \frac{1}{a(T)}\frac{\partial}{\partial R}(a(T)b(R)) = \frac{\partial b(R)}{\partial R} \right\}$$

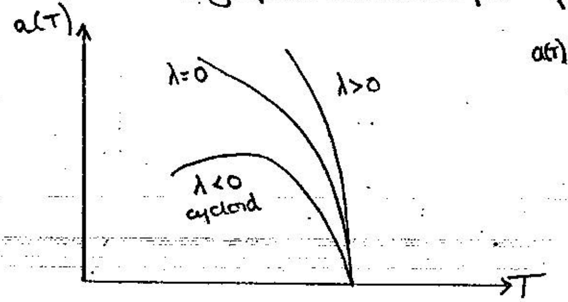
$$\underline{\underline{\left(\frac{db}{dR}\right)^2 = 1 + \lambda b^2}} \quad - \quad (23)$$

By scaling R in (23), i.e $R \rightarrow \alpha R$, we can set $\lambda = -1, 0, \text{or } 1$ depending on its sign. Then solutions are

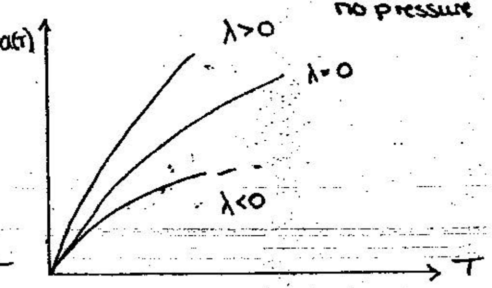
$\lambda = -1 \qquad b = \sin R$
$\lambda = 0 \qquad b = R$
$\lambda = +1 \qquad b = \sinh R$.

[Cosmology comments]

And graphic solutions for (22); homogeneous cosmology, no pressure



If λ<0, star starts from finite radius with zero velocity
λ=0 star collapses from rest with ∞ radius
λ>0 starts from ∞ radius with non zero infall velocity



Interpreting initial conditio differently - essentially mirror of first diagram.

tsd2

(1.1) Possible endpoints of stellar evolution

We are looking for an intuitive result and so the analysis will be non rigorous.

Let us consider a cold non-rotating star (in "ground state").

At low pressure → pure Fe (since Fe is lowest energy state, and most of mass will want to combine to give Fe).

At higher pressures ($\rho \gtrsim 10^5$ gm/cc) — fermion degeneracy (since fermions obey Pauli exclusion principle, at higher densities all lowest electron states are full and so there will be higher resulting pressure).

We shall derive some 'rough' relations.

Let n = no of fermions /cc ; m = mass

Then momentum $\sim \hbar n^{+1/3}$ ~ uncertainty principle $\Delta x \Delta p \sim \hbar$, Δx - uncertainty in $x \sim n^{-1/3}$ $\rightarrow \Delta p \sim \hbar n^{+1/3}$

and

$$\text{velocity} = \begin{cases} \sim \hbar n^{1/3}/m & (\text{non rel}) \; (\text{non rel}, \underline{p} = m\underline{v}) \\ \sim 1 & (\text{rel}) \end{cases}$$

Pressure = flux . momentum
= (no.density × velocity) × Momentum ; certainty has units of pressure $\sim \frac{m}{L^3} \cdot L/s \cdot mL/s \sim \frac{mLs^{-2}}{L^2} = \frac{\text{Force}}{\text{Area}}$
=

$$\therefore \boxed{\text{Pressure} = \begin{cases} \sim \hbar^2 n^{5/3} m^{-1} & (\text{non rel}) \\ \sim \hbar n^{4/3} & (\text{rel}). \end{cases}} \quad (*)$$

If we consider a rough division (not exact but just guidelines) of regions of density against the main sources of pressure we find, we can later discuss collapse of different types of matter depending on density.

density	Main Source of pressure
$10^5 < \rho < 10^7$	non relativistic electrons e^-
$10^7 < \rho < 4\times10^{11}$	relativistic electrons
$4\times10^{11} < \rho < 10^{13}$	non rel. neutrons n
$\rho > 10^{13}$	neutron degeneracy (+ strong interactions)

Now, total mass ≃ mass of neutron × no in sphere of radius R

$$m \simeq m_n \cdot n \cdot R^3 \quad \sim \tfrac{4}{3}\pi \sim 1 .$$

$$\therefore \; R = m^{1/3} m_n^{-1/3} n^{-1/3}$$

And Pressure balance (Newtonian ~ P × Area ~ Force due to grav)

$$P . R^2 \sim \overset{\text{mass} \times \text{acc due to grav}}{M} . M/R^2$$

$$\therefore \boxed{P = m^2/R^4 = m^{2/3} n^{4/3} m_n^{4/3}} \quad (**)$$

Note again that

$$\text{velocity} \sim \begin{cases} \hbar n^{1/3}/m & \text{non rel} \\ 1 & \text{rel} \end{cases}$$

So take as condition of

$$\begin{cases} \text{non rel as } \hbar n^{1/3}/m << 1 & \sim \hbar n^{1/3} < m \\ \text{rel as } \hbar n^{1/3}/m \gtrsim 1 & \sim \hbar n^{1/3} \gtrsim m \end{cases}$$

Relativistic, non-relativistic } condition is $\begin{cases} n \gtrsim m^3 \hbar^{-3} & n \sim \\ n < m^3 \hbar^{-3} & n < \end{cases}$

(a) Pressure due to non rel e^- $\qquad n < m^3 \hbar^{-3}$

(*) $P = \hbar^2 n^{5/3} m_e^{-1}$ ~ non rel, due to e^-

(**) $P = m^{2/3} n^{4/3} m_n^{4/3}$

$$\therefore \; \hbar^2 n^{5/3} m_e^{-1} = m^{2/3} n^{4/3} m_n^{4/3}$$

$$n = m^2 m_n^4 m_e^3 \hbar^{-6}$$

Hence $n < m_e^3 \hbar^{-3} \Rightarrow m^2 m_n^4 m_e^3 \hbar^{-6} < m_e^3 \hbar^{-3}$

$$m < \hbar^{3/2} m_n^{-2} = \underline{m_L \sim 1.5 m_\odot}$$

$$\underline{m < 1.5 m_\odot}$$

i.e for there to be endpoint of evolution with pressure-gravity equilibrium

(b) rel e^- $(n \tilde{\lessgtr} m_e^3 \hbar^{-3})$

(*) $P = \hbar n^{4/3}$

(**) $P - m^{2/3} n^{4/3} m_n^{4/3}$

$$\left.\begin{matrix}\end{matrix}\right\} \quad \hbar \sim m^{2/3} m_n^{4/3}$$

$$\underline{\underline{m \sim \hbar^{3/2} m_n^{-2} \sim m_L}}$$

equilibrium endpt if $m \sim m_L$

(c) non rel neutrons n $(n \lessgtr m_n^3 \hbar^{-3})$

(*) $P = \hbar^2 n^{5/3} m_n^{-1}$

(**) $P - m^{2/3} m_n^{4/3} n^{4/3}$

$$\left.\begin{matrix}\end{matrix}\right\} \quad n = m^2 m_n^7 \hbar^{-6}$$

$$\therefore \quad \underline{\underline{m < m_L}}$$

equilibrium endpt if $m \sim m_L$

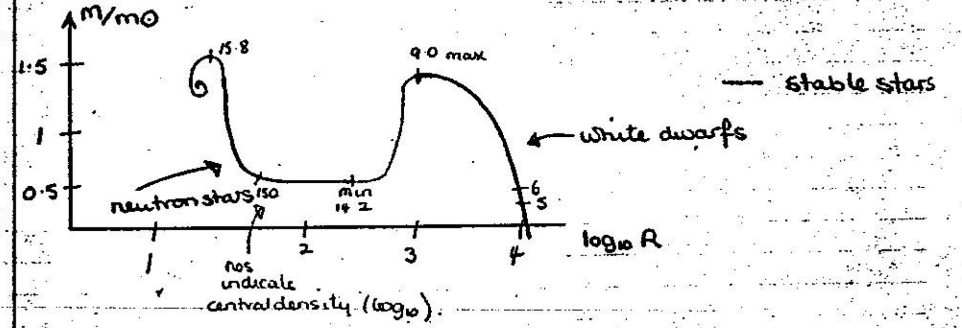
(d) Relativistic neutrons $(n \sim m_e^3 \hbar^{-3})$

(*) $P = \hbar n^{4/3}$

(**) $P = m^{2/3} m_n^{4/3} n^{4/3}$

$$\left.\begin{matrix}\end{matrix}\right\} \quad \underline{\underline{m \sim m_L}}$$

If we were to consider GR instead of Newtonian physics and include strong interactions, would make little qualitative difference, i.e density region would not predict star of mass much greater than $\sim 2m_\odot$.



Hence there are strong indications that there is a maximum mass m_{max} $(\lesssim 5m_\odot)$ for which stars have an equilibrium stellar end point.

Heavier stars must either (when nuclear fuel exhausted & Pressure can no longer hold grav.)

(1) collapse catastrophically.

(2) lose mass in some way during contraction (or before) in order to get below m_{max} & have equilibrium endpt.

[Would be quite coincidence if large masses lost just enough mass to avoid catastrophic collapse.]

Note

Star - spherical → schwarzschild geometry outside star $(r > r_0)$

Let schwarzschild radius $r = 2m$

If star static then $r_0 > 2m$ (since surface of static star must correspond to orbit of a timelike Killing vector & in schwarzschild geometry there only exists timelike Killing vector field when $r_0 > 2m$ - if $r_0 < 2m$ surface would be expanding or contracting).

Examples: Sun - schwarzschild radius = 3.0 km

Earth - " = 1 cm

Spherical Collapse. Summary and discussion

we shall consider the spherical collapse of a star (consisting of matter approximated to be a dust cloud) surrounded by a vacuum. - Oppenheimer and Snyder collapse. (simple idealisation introduces qualitative aspects of subject).

(i) The matter. (Friedmann interior).

The energy-momentum tensor for a dust cloud is by def^n

$$T_{ab} \equiv \rho u_a u_b$$ (Note ~ pressure free perfect fluid).

we have

(i) Conservation of stress energy $T^{ab}{}_{;b} = 0$ i.e $(\rho u^a u^b)_{;b} = 0$

(ii) conservation of baryons $(\rho u^a)_{;a} = 0$

(iii) From (i) and (ii) we can infer that

$$0 = (\rho u^a u^b)_{;b} = \cancel{(\rho u^b)_{;b}}^{0} + \rho u^b u^a{}_{;b}$$

i.e $u^a{}_{;b} u^b = 0$ — eq^n of geodesics, hence dust particles move on geodesics.

We obtain solution inside matter from earlier

we shall give brief outline of proof, in detail done earlier

For spherically symmetric spacetime

$$ds^2 = -e^{2\Phi} dT^2 + e^{2\Lambda} dR^2 + r^2(T,R)(d\theta^2 + \sin^2\theta d\phi^2)$$

Einstein eq^{n}s are solved for a perfect fluid viz

$$T_{ab} = \rho u_a u_b + p h_{ab}$$

(conserv^n laws) and on page 9,10* values for Φ, Λ are obtained.

For the particular case of pressure free collapse $p = 0$ (dust cloud).

$\Phi = 0$ and from earlier (pg 16-19)

Thus $ds^2 = -dT^2 + a^2(T) dR^2 + b^2a^2(d\theta^2 + \sin^2\theta d\phi^2)$ — use a, b to find result

$$\left.\begin{aligned} e^{\Lambda} &= a(T) \\ r &= a(T) b(R) \\ \rho(T) &= \rho_0 / a^3(T) \\ m(R) &= \tfrac{4}{3}\pi \rho_0 b^3(R) \end{aligned}\right\} \text{where} \quad \text{eq}^n(22) \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{\lambda}{a^2} + \frac{8\pi\rho_0/3}{a^3} \qquad \text{eq}^n(23) \quad \left(\frac{db}{dR}\right)^2 = 1 + \lambda b^2$$ (*)

λ can be scaled to -1, 0 and 1 and for $\lambda = -1$ the physica analogue is collapse of star so take $\lambda = -1$. So that

$$\dot{a}^2 = -1 + \frac{8\pi\rho_0/3}{a}$$

$$\left(\frac{db}{dR}\right)^2 = 1 - b^2$$ hence $b = \sin R$ (sol^n)

change to (more standard) notation

$R \to \chi$ $\quad R_0 = 8\pi\rho_0/3$ $\quad a(\tau) = R(\tau)$ $\quad (T \to \tau)$

Then $\left(\frac{dR}{d\tau}\right)^2 = -1 + R_0/R$ $\qquad b = \sin\chi$

The parametric eq^n of this sol^n is $R = \frac{R_0}{2}(1 + \cos\eta)$ $\tau = \frac{R_0}{2}(\eta$

$\left.\begin{aligned} \frac{dR}{d\eta} &= -\frac{R_0}{2}\sin\eta \\ \frac{d\tau}{d\eta} &= \frac{R_0}{2}(1+\cos\eta) \end{aligned}\right\}$

Hence solution $ds^2 = -d\tau^2 + R^2(\tau)\left[d\chi^2 + \underset{\downarrow b^2}{\sin^2\chi}\, d\Omega^2\right]$

Hence solution inside matter i.e $0 \le \chi \le \chi_s < \pi/2$ is given by

$a = R$, $b = \sin R$, $R \to \chi$ $T \to \tau$ in (*)

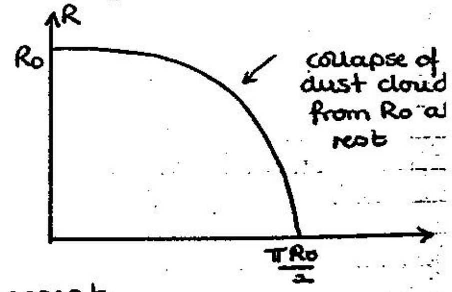
$$ds^2 = -d\tau^2 + R^2(\tau)\left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin\right.$$

where

$R(\eta) = \frac{R_0}{2}(1 + \cos\eta)$

$\tau(\eta) = \frac{R_0}{2}(\eta + \sin\eta)$

(cycloid)



Where R identified with Schwarzschild radial component (i.e surface of star = $4\pi R^2$).

The density and scalar curvature reach infinite val after a finite proper time $\pi R_0/2$ measured along geodesic particle world lines. The geodesic incomplet in future imply space time singularity.

(ii) Want the consider geometry outside matter. (i.e $\chi \ge \chi_s$)

Since spherically symmetric, static and vacuum, b Birkhoffs theorem sol^n must be Schwartschild viz

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2(d\theta^2 + \sin$$

where m is a constant

<u>Inside matter</u>

ASSIGNMENT? ⊛

$$\boxed{\begin{aligned} r &= \frac{R_0}{2}(1+\cos\eta)\sin\chi = R\sin\chi \\ \tau &= \frac{R_0}{2}(\eta+\sin\eta) \end{aligned}}$$

$r = a(\tau)b(R) = R\sin\chi$

Since particles travel on geodesics, thus is eq^n for radially infalling geodesic inside matter.

<u>Outside</u>

Eq^n for radially infalling timelike geodesic in schwarzschild region is given by

$$\dot{r}^2 = E^2 - 1 + \frac{2m}{r}$$

If $\frac{dr}{d\tau} = 0$ when $\tau = 0$ at $r = R$, then $E^2 - 1 = -\frac{2m}{R}$

$$\therefore \frac{dr}{d\tau} = \left(\frac{2m}{r} - \frac{2m}{R}\right)^{1/2} \Rightarrow \int d\tau = \int \left(\frac{2m}{r} - \frac{2m}{R}\right)^{-1/2} dr$$

Put $r = R\cos^2\omega$
$dr = -2R\cos\omega\sin\omega\, d\omega$

$$\Rightarrow \int d\tau = 2R\sqrt{R/2m}\int \frac{1+\cos 2\omega}{2}\, d\omega$$

$$\tau = \frac{R}{2}\sqrt{R/2m}\,(2\omega + \sin 2\omega)$$

and $r = R\cos^2\omega = \frac{R}{2}(1+\cos 2\omega)$

Put $2\omega = \eta$

$$(**) \quad \boxed{\begin{aligned} \tau &= \frac{R}{2}\sqrt{R/2m}\,(\eta + \sin\eta) \\ r &= \frac{R}{2}(1+\cos\eta) \end{aligned}}$$

To <u>match up</u> radially infalling timelike geodesics, compare (*) and (**) at $\chi = \chi_s$

$$\frac{R_0}{2}(1+\cos\eta)\sin\chi_s = \frac{R}{2}(1+\cos\eta) \Rightarrow \frac{R_0\sin\chi_s}{2} = \frac{R}{2}$$

and $\frac{R_0}{2} = \frac{R}{2}\left(\frac{R}{2m}\right)^{1/2}$

$$\Rightarrow \frac{R_0}{2} = \frac{R_0}{2}\sin\chi_s\left(\frac{R_0\sin\chi_s}{2m}\right)^{1/2}$$

$$\Rightarrow \quad m = \frac{R_0\sin^3\chi_s}{2}$$

(iii) We wish to 'match up' geometry outside and inside matter to obtain complete solution for collapse. Hen the 2 regions must be continuous in the sense that a particle at χ_s must have geodesic motion with respect to both regions. Can be shown that if

$$m = \tfrac{1}{2}R_0\sin^3\chi_s \quad \text{then 2 regions matched}$$

(i) Matching done by writing schwarzschild sol^n in form similar to metric ($\lambda = -1$) for collapsing matter and demanding grav. field continuous at surface of star $r = r_s$
(ii) Done properly on pg 25(a) [assignment ?]
(iii) To see consistency: m is constant. If $m = \frac{1}{2}R_0\sin^3\chi_s$ then sol^n for vacuum outside collapsing star. If m = mass, then sol^n outside non collapsing star. However, $m = \frac{1}{2}R_0\sin^3\chi_s = \frac{4}{3}\pi S_0 b^3(\chi_s) =$ <u>mass of st</u>
so that <u>above "matching cond^n" consistent</u> (by analogy)

The internal geometry is well behaved at <u>$r = 2m$</u> ($r = R(\tau)\sin\chi_s$) (only a coordinate singularity) and we must extend schwartschild solution through $r = 2m$

Define the Eddington-Finkelstein ingoing coord V by

$$V = t + r + 2m\ln(r - 2m)$$

So that $\quad ds^2 = -\left(1 - \frac{2m}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin$

Note that V is a null coord (i.e surfaces <u>V = const are null surfaces</u> – metric instrinsic <u>to surface V = const has canonical form (0++)</u>) so that <u>V = const is geodesic of photon</u>.

With this extension the schwarzschild geometry is certainly regular at $r = 2m$.

However, at $r = 0$ something goes wrong. If we consider invariants

$$R_{abcd}R^{abcd} = \frac{48m^2}{r^6}$$

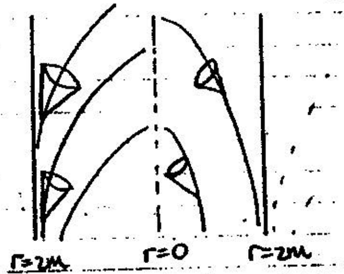
So that there is a singularity at $r = 0$ (this has catastrophic results on Schwartschild geometry outside matter too).

The singularity is not visable from infinity
Since r = 2m is a null surface

$\left\{ ds^2 = -(1-\frac{2m}{r})dv^2 + 2dv\,dr + r^2 d\Omega^2 \right.$
$r = 2m, dr = 0 \quad ds^2 = r^2 d\Omega^2$ has intrinsic geometry $(^0{+}_{+})$ $\}$

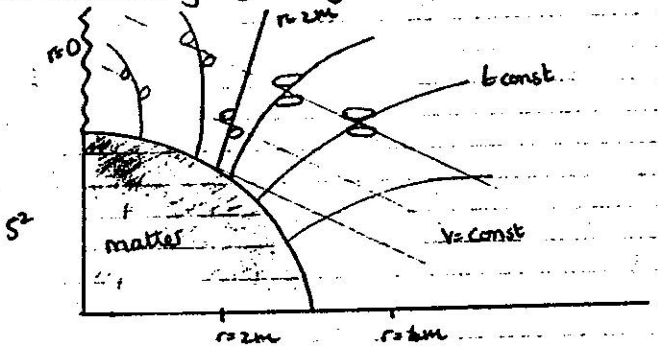
future directed null or timelike curves can only cross r=2m inwards.

In 2D ($\overset{\theta}{\phi}$ suppressed) the light cones with will, since r = 2m is null surface.



we say that $\{r \leq 2m\}$ is the region inside a black hole.

The Boundary $\{r = 2m\}$ is the event horizon.



Singularity inside matter is also invisible from infinity.

Notes

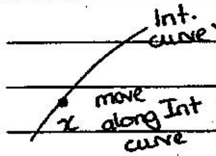
(1) Shown existence black hole in special case (spherical symmetry) - general case?
Q: Generic collapse possess a) ST singularity b) event horizon surrounding singularity
A: a) Hawking penrose singularity Theorems ⇒ a). (with causal condⁿs).
b) circumstantial evidence - cosmic censorship hypothesis.
[Failure (b) ⇒ physics unknowable inside black holes since do not affect outer regions - physics in "disarray".]
(2) Conservative viewpt. Generic collapse settles down to stationary black holes - characterized by mass, charge and angular momentum (Schwarzschild Riesner Nordstrom, Kerr) (No hair theorems)
(3) Bekenstein - entropy of black hole (Hawking - area $\dot{A} \geq 0$)
(4) Hawking radiation.

Schwarzschild solution / spacetime

Sphericalsymmetry + vacuum ST + asympt flat $\xrightarrow{\text{Birkhoff}}$ Schwarzschild and static

To analyse what happens outside r_0 (radius of star) need to investigate geodesics

A ST has a symmetry transfⁿ generated by X^i iff



$$\left(\mathcal{L}_X g\right)_{ij} = 0 = 2X_{(i;j)}$$

Then X^i is called Killing vector

$(\mathcal{L}_X g)_{ij} \equiv g_{ij;k}$
$+ g_{kj} X^k_{;i}$
$+ g_{ik} X^k_{;j}$
$= X_{j;i} + X_{i;}$

Symmetries generate conservation laws.

Lemma 1 Suppose T geodesic vector and K Killing vector fiel (KVF). Then K∘T is conserved along geodesic ⇒

$$\nabla_T (K \circ T) = 0$$

$K_{i;j} + K_{j;i} = 0$
$T^i{}_{;j} T^j = 0$

Proof $\nabla_T(K \circ T) = (K_i T^i)_{;j} T^j = K_{i;j} T^i T^j + T^i{}_{;j} K_i$
$= K_{(i;j)} T^i T^j = 0$

Now consider a radially infalling (Ω invariant) geodesic γ with unit tangent vector T (T∘T = -1 - timelike geodesic) let τ denote propertime along γ and $\dot{}$ denote $\frac{d}{d\tau}$ $\left(\dot{f} = \frac{df}{d\tau}\right)$.

$$\left.\begin{aligned} ds^2 &= -F dt^2 + \frac{1}{F} dr^2 + r^2 d\Omega^2 \\ F &= 1 - \frac{2m}{r} \end{aligned}\right\} \quad (1).$$