Quantum graphs and quantum nonlocality

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Many mathematical structures have quantum analogues, e.g.,

1. groups $\rightsquigarrow$ quantum groups,
2. spaces $\rightsquigarrow$ quantum spaces,
3. graphs $\rightsquigarrow$ quantum graphs.

This "noncommutative metaphor" is gradually developed heuristically.

There are two mysteries here:

1. Why are these quantum analogues closely related to each other?
2. Why are these quantum analogues closely related to physics?

I will hint at an answer to the first question and illustrate the second.

Two scientists with expensive hardware and a plan are spatially separated.

Each encounters an element of X. Each responds with an element of Y.

Definition. A **correlation** is a resulting probabilistic map $X \times X \to Y \times Y$.

Theorem (Bell). There exists a quantum correlation $\mathcal{B}$ that is not classical.

Theorem (Clauser, Horne, Shimony, Holt). A skeptic can distinguish $\mathcal{B}$ from the set of all classical correlations using data points (x_1, x_2, y_1, y_2).

Theorem (Cleve, Høyer, Toner, Watrous). There is a game that the scientists can win more often using $\mathcal{B}$ than using any classical correlation.

Definition. In this context, a correlation is called a **strategy**.

(We are implicitly identifying plans that produce the same correlation.)

Definition (Mančinska, Roberson). Let G and H be finite simple graphs. The (G, H)-**homomorphism game** is played by two players who are cooperating against an adverse environment. The play proceeds as follows:

1. the players agree on a strategy,
2. the players are separated,
3. each player $i \in \{1, 2\}$ encounters a random vertex x_i of G,
4. each player $i \in \{1, 2\}$ responds with a vertex y_i of H,
5. the players win unless one of the following happens:
   (a) $x_1 \sim x_2$ but $y_1 \not\sim y_2$,
   (b) $x_1 = x_2$ but $y_1 \neq y_2$.

Definition. A graph is **simple** if no vertex is adjacent to itself.

Proposition. The following are equivalent:

1. there exists a winning strategy for the (G, H)-homomorphism game,
2. there is a homomorphism $G \to H$.

Theorem (Mančinska, Roberson). There exist finite simple graphs

1. G on 14 vertices,
2. H on 4 vertices

such that the (G, H)-homomorphism game

1. has no winning strategy,
2. has a winning quantum strategy.

We express these results in terms of the categories **Gph** and **qGph**.

We will shortly define

1. **Gph**, the category of graphs and homomorphisms,
2. **qGph**, the category of quantum graphs and homomorphisms.

These are both **symmetric monoidal categories**: each is equipped with

1. a binary product functor $- \square -$,
2. a unit object K_1,
3. natural transformations witnessing various identities.

Definition. We define the symmetric monoidal category **Gph**:

1. an object is a graph, i.e., a set with a symmetric binary relation,

$$G = (V_G, E_G), \qquad\qquad x_1 \sim x_2 \iff (x_1, x_2) \in E_G$$

2. a morphism $G \to H$ is a graph homomorphism from G to H, i.e.,

$$\phi\colon V_G \to V_H \qquad\qquad x_1 \sim x_2 \implies \phi(x_1) \sim \phi(x_2),$$

3. the monoidal product of G and H is their box product $G \mathbin{\square} H$, i.e.,

$$V_{G \square H} = V_G \times V_H \qquad (x_1, y_1) \sim (x_2, y_2) \iff \begin{cases} x_1 \sim x_2 \text{ and } y_1 = y_2, \\ x_1 = x_2 \text{ and } y_1 \sim y_2, \end{cases}$$

4. the monoidal unit is the complete simple graph on one vertex, i.e.,

$$V_{K_1} = \{*\} \qquad\qquad * \not\sim *.$$

Definition. The homomorphism graph $[G, H]$ is defined by

$$V_{[G,H]} = \{\phi\colon G \to H\}, \qquad \phi_1 \sim \phi_2 \iff \phi_1(x) \sim \phi_2(x) \text{ for all } x \in V_G.$$

Proposition. The symmetric monoidal category **Gph** is closed.

$$\begin{tikzcd}
K \,\square\, G \arrow[d, dotted, "! \times \mathrm{id}_G"'] \arrow[dr, "\phi"] & \\
{[G, H] \,\square\, G} \arrow[r, "\mathrm{eval}"'] & H
\end{tikzcd}$$

Proposition. For finite simple graphs G and H, t.f.a.e.:

1. the (G, H)-homomorphism game has a winning strategy,
2. $[G, H]$ is not the empty graph, i.e., $[G, H] \not\cong K_0$.

To obtain quantum analogues of these propositions, we

1. reformulate the definition of **Gph** in **Rel**,
2. reinterpret that definition in **qRel**.

We obtain the category **qGph** of quantum graphs and homomorphisms.

In this sketch,

1. **Rel** is the category of sets and relations,
2. **qRel** is the category of quantum sets and relations,
3. “quantum” refers to the noncommutative metaphor,
4. “reinterpret” refers to internalization in category theory.

Definition. We define the dagger symmetric monoidal category **Rel**:

1. an object is a set,
2. a morphism $X \to Y$ is a relation from X to Y, i.e., $R \subseteq X \times Y$,
3. relations $R\colon X \to Y$ and $S\colon Y \to Z$ are composed like functions, i.e.,
$$S \circ R = \{(x, z) \mid (x, y) \in R \text{ and } (y, z) \in S \text{ for some } y\}$$
4. the monoidal product of X and Y is their Cartesian product $X \times Y$,
5. the monoidal unit is a singleton $\{*\}$,
6. the dagger of a relation $R\colon X \to Y$ is its converse $R^\dagger\colon Y \to X$, i.e.,
$$R^\dagger = \{(y, x) \mid (x, y) \in R\},$$
7. the set of relations $X \to Y$ is a complete ortholattice with
$$R \leq S \iff R \subseteq S, \qquad \neg R = (X \times Y) \setminus R.$$

Definition. We reformulate the definition of **Gph** in **Rel**:

1. a graph G is a pair (V_G, E_G), where
   - V_G is an object in **Rel**,
   - E_G is a morphism $V_G \to V_G$ in **Rel** such that $E_G^\dagger = E_G$,
2. a homomorphism from G to H is a morph. $\phi\colon V_G \to V_H$ in **Rel** with

$$\phi^\dagger \circ \phi \geq \mathrm{id}_{V_G}, \qquad \phi \circ \phi^\dagger \leq \mathrm{id}_{V_H}, \qquad \phi \circ E_G \leq E_H \circ \phi,$$

3. the box product of G and H is the graph that is defined by

$$V_{G \square H} = V_G \times V_H, \qquad E_{G \square H} = (E_G \times \mathrm{id}_{V_H}) \vee (\mathrm{id}_{V_G} \times E_H).$$

4. the complete simple graph on one vertex is defined by
   - V_{K_1} is the monoidal unit of **Rel**,
   - E_{K_1} is the minimum morphism $V_{K_1} \to V_{K_1}$ in **Rel**.

Definition. We define **qGph** in **qRel**:

1. a quantum graph G is a pair (V_G, E_G), where
   - V_G is an object in **qRel**,
   - E_G is a morphism $V_G \to V_G$ in **qRel** such that $E_G^\dagger = E_G$,
2. a homomorphism from G to H is a morph. $\phi\colon V_G \to V_H$ in **qRel** with
$$\phi^\dagger \circ \phi \geq \mathrm{id}_{V_G}, \qquad \phi \circ \phi^\dagger \leq \mathrm{id}_{V_H}, \qquad \phi \circ E_G \leq E_H \circ \phi,$$
3. the box product of G and H is the quantum graph that is defined by
$$V_{G \square H} = V_G \times V_H, \qquad E_{G \square H} = (E_G \times \mathrm{id}_{V_H}) \vee (\mathrm{id}_{V_G} \times E_H).$$
4. the complete simple quantum graph on one vertex is defined by
   - V_{K_1} is the monoidal unit of **qRel**,
   - E_{K_1} is the minimum morphism $V_{K_1} \to V_{K_1}$ in **qRel**.

Definition. We define the dagger symmetric monoidal category **qRel**:

1. an object is a quantum set X, i.e., a pair $(\mathrm{At}(X), \mathrm{dim})$, where
   - $\mathrm{At}(X)$ is a set, which is "the set of atoms of X,"
   - $\mathrm{dim}\colon \mathrm{At}(X) \to \mathbb{Z}_+$ is a map,
2. a morphism $R\colon X \to Y$ is a relation, i.e., a choice of subspaces
$$R_{xy} \subseteq M_{m\times n}(\mathbb{C})$$
for $x \in \mathrm{At}(X)$ and $y \in \mathrm{At}(Y)$, where $n = \mathrm{dim}(x)$ and $m = \mathrm{dim}(y)$,
3. the composition of relations $R\colon X \to Y$ and $S\colon Y \to Z$ is defined by
$$(S \circ R)_{xz} = \mathrm{span}\{sr \mid r \in R_{xy} \text{ and } s \in S_{yz} \text{ for some } y \in \mathrm{At}(Y)\}.$$

We continue to define the dagger symmetric monoidal category **qRel**:

4. the monoidal product of X and Y is their Cartesian product $X \times Y$,

$$\mathrm{At}(X \times Y) = \mathrm{At}(X) \times \mathrm{At}(Y), \qquad \dim(x, y) = \dim(x) \dim(y),$$

5. the monoidal unit I is defined by $\mathrm{At}(I) = \{*\}$ with $\dim(*) = 1$,
6. the dagger of a relation $R\colon X \to Y$ is its converse $R^\dagger\colon Y \to X$, i.e.,

$$(R^\dagger)_{yx} = (R_{xy})^\dagger$$

7. the set of relations $X \to Y$ is a complete ortholattice with

$$R \leq S \iff R_{xy} \subseteq S_{xy}, \qquad (\neg R)_{xy} = (R_{xy})^\perp.$$

Proposition. There is a full and faithful structure-preserving functor

$$\mathbf{Rel} \to \mathbf{qRel}$$

that maps each set X to the quantum set X' such that

$$\mathrm{At}(X') = X, \qquad \mathrm{dim}(x) = 1 \text{ for all } x.$$

We identify **Rel** with its image in **qRel**.

Consequently, we identify **Gph** with its image in **qGph**.

“Every set is a quantum set, and every graph is a quantum graph.”

This is a standard feature of the noncommutative metaphor.

Theorem (K, Lindenhovius). The symmetric monoidal category **qGph** is closed: for all quantum graphs G and H, there exists a "quantum homomorphism graph" $[G, H]$ that has the relevant universal property.

$$
\begin{tikzcd}
K \mathbin{\Box} G \arrow[d, dotted, "! \times \mathrm{id}_G"'] \arrow[dr, "\phi"] & \\
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Theorem (K, Lindenhovius). For finite simple graphs G and H, t.f.a.e.:

1. the (G, H)-homomorphism game has a winning quantum strategy,
2. $[G, H]$ is not the empty graph, i.e., $[G, H] \ncong K_0$.