# Modal and tense operators on a complete Boolean algebra

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Definition. A **Stone space** $X$ is a compact Hausdorff space such that every connected component is a point.

Definition. A **continuous relation** on $X$ is a closed subset $R \subseteq X \times X$ such that

1. $R[x]$ is closed for all points $x \in X$,
2. $R^{-1}[U]$ is clopen for all clopens $U \subseteq X$.

(Continuous relations on $X$ generalize continuous functions on $X$.)

Definition.

$$\mathcal{CR}(X) = \{R \mid R \text{ is a continuous relation on } X\}$$

$$R \leq S \qquad \Longleftrightarrow \qquad R \subseteq S$$

Proposition. $\mathcal{CR}(X)$ has joins.

$$R \vee S = R \cup S$$

Let $\alpha\mathbb{N}$ be the one-point compactification of $\mathbb{N}$.

Proposition. $\mathcal{CR}(\alpha\mathbb{N})$ does not have meets.

Proposition. $\mathcal{CR}(\alpha\mathbb{N})$ is not distributive as a join-semilattice.

Definition. A join-semilattice $L$ is **distributive** if

$$a \leq b \vee c \qquad \Longrightarrow \qquad (\exists b' \leq b)\,(\exists c' \leq c)\; a = b' \vee c'.$$

Definition. A Stone space $X$ is **extremally disconnected** if the closure of every open set is open.

Proposition. In this case, $\mathcal{CR}(X)$ is a complete lattice.

$$\bigvee_{i \in I} R_i = \mathrm{cl}\left(\bigcup_{i \in I} R_i\right)$$

Theorem(Bezhanishvili, K). In this case, $\mathcal{CR}(X)$ is a coframe.

Definition. A complete lattice $L$ is a **coframe** if

$$a \vee \left(\bigwedge_{i \in I} b_\alpha\right) = \bigwedge_{i \in I} a \vee b_\alpha.$$

Proposition. For general extremally disconnected $X$, the frame $\mathcal{CR}(X)^{op}$

1. is zero-dimensional,
2. need not be algebraic,
3. need not be spatial.

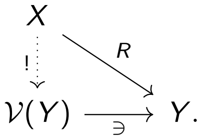
Let $\beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$.

Theorem(Bezhanishvili, K). $\mathcal{CR}(\beta\mathbb{N}) \cong \Gamma(\mathbb{N} \times \beta\mathbb{N})$.

Corollary. The frame $\mathcal{CR}(\beta\mathbb{N})^{op}$ is

1. algebraic,
2. spatial,
3. not a coframe,
4. not a Boolean algebra.

Definition. The **Vietoris space** $\mathcal{V}(Y)$ is $\Gamma(Y)$ with the topology such that




Theorem(Bezhanishvili, K). $\mathcal{CR}(\beta\mathbb{N}) \cong \Gamma(\mathbb{N} \times \beta\mathbb{N})$.

Proof.

$$\mathcal{CR}(\beta\mathbb{N}) \cong \mathcal{C}(\beta\mathbb{N}, \mathcal{V}(\beta\mathbb{N})) \cong \mathcal{C}(\mathbb{N}, \mathcal{V}(\beta\mathbb{N})) \cong \Gamma(\mathbb{N} \times \beta\mathbb{N})$$

□

Proposition. The function $\cap\colon \mathcal{V}(\beta\mathbb{N}) \times \mathcal{V}(\beta\mathbb{N}) \to \mathcal{V}(\beta\mathbb{N})$ is not continuous!

Definition. A **necessity operator** $\Box \in \mathcal{NO}(B)$ on a Boolean algebra $B$ is

$$\Box\colon B \to B, \qquad \Box 0 = 0, \qquad \Box(a_1 \wedge a_2) = \Box a_1 \wedge \Box a_2.$$

Stone Duality. $B \cong \Omega(\mathrm{Spec}(B)) \cap \Gamma(\mathrm{Spec}(B))$.

Jónsson-Tarski Duality. $\mathcal{NO}(B) \cong \mathcal{CR}(\mathrm{Spec}(B))^{op}$.

$$\Box_1 \leq \Box_2 \qquad \Longleftrightarrow \qquad (\forall a \in B)\, \Box_1 a \leq \Box_2 a$$

Corollary. In general, $\mathcal{NO}(B)$ is a meet-semilattice that

1. need not have joins,
2. need not be distributive.

Corollary. When $B$ is complete, $\mathcal{NO}(B)$ is a complete Heyting algebra.

Proposition. Let $B$ be a complete Boolean algebra, and let $\Box \in \mathcal{NO}(B)$. The following are equivalent:

1. $\Box \left( \bigwedge_{i \in I} a_i \right) = \bigwedge_{i \in I} \Box a_i$,
2. $\Box a_1 \vee a_2 = 1$ iff $a_1 \vee \Box^\dagger a_2 = 1$ for some $\Box^\dagger \in \mathcal{NO}(B)$.

Definition. In this case, $\Box$ is a **tense necessity operator**, $\Box \in \mathcal{TNO}(B)$.

Proposition. $\mathcal{TNO}(B)$ is a frame.

Theorem (Bezhanishvili, K). Let $B$ be a complete Boolean algebra. The following are equivalent:

1. $\mathcal{TNO}(B)$ is a spatial frame,
2. $\mathcal{TNO}(B)$ is a complete atomic Boolean algebra,
3. $B$ is atomic.

Definition. A complete Boolean algebra $B$ is **measurable** if there exists

$$\mu\colon B \to [0, \infty]$$

1. $\mu(a) = 0$ iff $a = 0$,
2. $\mu(a_1 \vee a_2) = \mu(a_1) + \mu(a_2)$ whenever $a_1 \wedge a_2 = 0$,
3. $\mu\left(\bigvee_{n\in\omega} a_n\right) = \bigvee_{n\in\omega} \mu(a_n)$ whenever $a_0 \leq a_1 \leq \cdots$,
4. $0 < \mu(a_1)$ implies that $0 < \mu(a_0) < \infty$ for some $a_0 \leq a_1$.

Example. The complete Boolean algebra $\mathcal{R}\Omega(\mathbb{R})$ is not measurable.

Theorem (folk?). The following are equivalent:

1. $B$ is measurable,
2. $B \cong M(X)/N(X)$ for some decomposable measure space $X$.

Let $B$ be a measurable complete Boolean algebra. We paraphrase.

Definition (Weaver). A **measurable relation** on $B$ is a relation $R \subseteq B \times B$ such that

$$\left( \bigvee_{i \in I} a_i, \bigvee_{j \in J} b_j \right) \in R \quad \Longleftrightarrow \quad (\exists i \in I)\,(\exists j \in J)\,(a_i, b_j) \in R.$$

Proposition. There is a one-to-one correspondence between

1. measurable relations on $B$,
2. tense necessity operators on $B$.

Theorem (Weaver). Let $\Box \in \mathcal{TNO}(B)$. Let $c_1, c_2 \in B$ with $\Box c_1 \vee c_2 \neq 1$. There exist complete Boolean algebra homomorphisms $\pi_i \colon B \to B'$ with

$$\pi_i(c_i) = 0, \qquad \Box a_1 \vee a_2 = 1 \quad \Longrightarrow \quad \pi_1(a_1) \vee \pi_2(a_2) = 1.$$

Let $B$ be a measurable complete Boolean algebra. We paraphrase.

Definitions (Weaver). A tense necessity operator $\Box \in \mathcal{TNO}(B)$ is

1. a **measurable graph structure** if $\Box \leq \mathrm{id}$ and $\Box^\dagger = \Box$,
2. a **measurable poset structure** if $\Box \leq \Box \circ \Box$ and $\Box \vee \Box^\dagger = \mathrm{id}$,
3. a **measurable equivalence relation** if $\Box \leq \mathrm{id}$, $\Box^\dagger = \Box$, and $\Box^\dagger = \Box$.
4. a **measurable function** if $\mathrm{id} \leq \Box \circ \Box^\dagger$ and $\Box^\dagger \circ \Box \leq \mathrm{id}$.

Proposition. There is a one-to-one correspondence between

1. measurable equivalence relations on $B$,
2. complete Boolean subalgebras of $B$.

Proposition. There is a one-to-one correspondence between

1. measurable functions on $B$,
2. complete Boolean algebra homomorphisms on $B$.

We compare two perspectives on complete Boolean algebras.

Spectral perspective.

1. Complete Boolean algebras are dual to extr. disc. Stone spaces.
2. Tense necessity operators are dual to interior relations.

Localic perspective.

1. Complete Boolean algebras are dual to discrete locales.
2. Tense necessity operators are dual to relations on those locales.

Complete Boolean algebras generalize sets in point-free topology.

Similarly, quantum OMLs generalize sets in noncommutative geometry.