Quantum entropy and Bell states

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This talk is about analogies between classical and quantum information theories.

These analogies are motivated by noncommutative geometry.

Main point. We can guess new theorems about quantum information by putting old theorems about classical information through the "quantum metaphor."

Finite-dimensional C^*-algebras are the quantum analogue of finite sets.

My research is centered on the foundations of the quantum metaphor.

My motivation is the opinion that the striking cohesion of the quantum metaphor and its many connections to quantum theory cannot be a coincidence.

The quantum metaphor captures something essential about quantum theory.

Let X, Y, and Z be finite sets.

Definition. A stochastic map $X \xrightarrow{f} Y$ is given by mapping probabilities

$$\mathbb{P}_f(y|x) \in [0,1], \qquad \sum_{y \in Y} \mathbb{P}_f(y|x) = 1.$$

Definition. The composition of stochastic maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ is defined by

$$\mathbb{P}_{g \circ f}(z|x) = \sum_{y \in Y} \mathbb{P}_g(z|y) \mathbb{P}_f(y|x).$$

Definition. It is deterministic if $\mathbb{P}_f(y|x) \in \{0,1\}$ for all $x \in X$ and $y \in Y$.

Deterministic stochastic maps $X \to Y$ are essentially just maps $X \to Y$.

"A stochastic map is deterministic iff it imports no information."

Definition. A probability distribution on X is just a stochastic map $\{*\} \xrightarrow{m} X$.

Definition. The Shannon entropy of a probability distribution m is given by

$$S(m) = -\sum_{x \in X} \mathbb{P}_m(x) \log \mathbb{P}_m(x) \qquad \mathbb{P}_m(x) = \mathbb{P}_m(x|*).$$

Theorem. For a stochastic map $X \xrightarrow{f} Y$, t.f.a.e.:

1. f is deterministic,
2. $S(f \circ m) \leq S(m)$ for all probability distributions m on X.

Question. What is the analogue of this theorem in quantum information theory?

Definition. A bipartite probability distribution is one of the form $\{*\} \xrightarrow{m} X \times Y$.

Definition. It is perfectly correlated if for every $A \subseteq X$ there exists $B \subseteq Y$ with

$$\mathbb{P}_m(A \times B) + \mathbb{P}_m(A^c \times B^c) = 1$$

and vice versa.

Theorem. For a bipartite probability distribution $\{*\} \xrightarrow{m} X \times Y$, t.f.a.e.:

1. m is perfectly correlated,
2. $S(m_X) = S(m) = S(m_Y)$.

Here, $\{*\} \xrightarrow{m_X} X$ and $\{*\} \xrightarrow{m_Y} Y$ are the marginal probability distributions.

Question. What is the analogue of this theorem in quantum information theory?

We now describe a joint setting for classical and quantum information theory.

Definition. A $*$-algebra is an algebra over $\mathbb{C}$ with an operation $A \mapsto A^*$ such that

$$(zA)^* = \overline{z}A^*, \qquad (A+B)^* = A^* + B^*, \qquad (AB)^* = B^*A^*, \qquad (A^*)^* = A.$$

Proposition. For a finite-dimensional $*$-algebra $\mathcal{A}$, t.f.a.e.:

1. $A = 0$ whenever $A^*A = 0$,
2. $\mathcal{A} \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C})$ for some $n_1, \ldots, n_s \in \mathbb{Z}_+$.

Definition. In this case, we say that $\mathcal{A}$ is a finite-dimensional C^*-algebra.

Definition. The naive trace $\mathcal{A} \xrightarrow{\text{tr}} \mathbb{C}$ is then defined by

$$\text{tr}(A_1 \oplus \cdots \oplus A_s) = \text{tr}(A_1) + \cdots + \text{tr}(A_s).$$

Definition. An element of a f.d. C^*-algebra is positive if it is of the form A^*A.

Definition. A linear map $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ between f.d. C^*-algebras is

1. positive if $\varphi(A)$ is positive for all positive $A \in \mathcal{A}$,
2. completely positive if $M_n(\mathcal{A}) \xrightarrow{M_n(\varphi)} M_n(\mathcal{B})$ is positive for all $n \geq 1$,
3. a channel if it is completely positive and trace-preserving.

Proposition. There is a canonical one-to-one correspondence between

1. channels $\mathbb{C}^m \xrightarrow{\varphi} \mathbb{C}^n$,
2. stochastic maps $\{1, \ldots, m\} \xrightarrow{f} \{1, \ldots, n\}$.

In this sense, channels generalize stochastic maps.

Definition. A state on $\mathcal{A}$ is just a channel $\mathbb{C} \xrightarrow{\mu} \mathcal{A}$.

States $\mathbb{C} \xrightarrow{\mu} \mathcal{A}$ are in bijection with density operators $\rho \in \mathcal{A}$ via $\rho = \mu(1)$.

Definition. The von Neumann entropy of a state $\mathbb{C} \xrightarrow{\mu} \mathcal{A}$ is given by

$$S(\mu) = -\operatorname{tr}(\rho \log \rho).$$

Definition (K). The adjusted von Neumann entropy of a state $\mathbb{C} \xrightarrow{\mu} \mathcal{A}$ is given by

$$\tilde{S}(\mu) = S(\mu) + \operatorname{tr}(\rho \log \zeta_{\mathcal{A}}).$$

Definition. For $\mathcal{A} = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C})$, we define $\zeta_{\mathcal{A}} = n_1 I_{n_1} \oplus \cdots \oplus n_s I_{n_s}$.

For a pure state $\mathbb{C} \xrightarrow{\mu} M_n(\mathbb{C})$, the adjusted von Neumann entropy is $\tilde{S}(\mu) = \log n$.

Intuitively, the term $\operatorname{tr}(\rho \log \zeta_{\mathcal{A}})$ is the "entropy of quantum uncertainty."

Proposition. For a channel $\mathbb{C}^m \xrightarrow{\varphi} \mathbb{C}^n$, t.f.a.e.:

1. the corresponding stochastic map $\{1, \ldots, m\} \xrightarrow{f} \{1, \ldots, n\}$ is deterministic,
2. the adjoint $\mathbb{C}^m \xleftarrow{\varphi^*} \mathbb{C}^n$ is a homomorphism. ($\mathrm{tr}(\varphi(a)b) = \mathrm{tr}(a\varphi^*(b))$)

Definition. A channel $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ is deterministic if $\mathcal{A} \xleftarrow{\varphi^*} \mathcal{B}$ is a homomorphism.

Theorem (K). Let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ be a channel. Then, t.f.a.e.:

1. φ is deterministic,
2. $\tilde{S}(M_n(\varphi) \circ \mu) \leq \tilde{S}(\mu)$ for all $n \geq 1$ and all states $\mathbb{C} \xrightarrow{\mu} M_n(\mathcal{A})$.

In what sense is a deterministic channel deterministic?

To test for literal determinism, we need duplicated inputs.

Quantum information cannot be duplicated (no-broadcasting theorem).

However, duplicated quantum information can be prepared.

Definition. A bipartite state is one of the form $\mathbb{C} \xrightarrow{\mu} \mathcal{A} \otimes \mathcal{B}$.

Measurements of $P \in \mathrm{Proj}(\mathcal{A})$ and $Q \in \mathrm{Proj}(\mathcal{B})$ always agree when

$$(*) \qquad \mathbb{P}_\mu(P \otimes Q) + \mathbb{P}_\mu(P^\perp \otimes Q^\perp) = 1, \qquad \mathbb{P}_\mu(R) = \mathrm{tr}(\mu(1)R).$$

Definition. A bipartite state $\mathbb{C} \xrightarrow{\mu} \mathcal{A} \otimes \mathcal{B}$ is

1. right-Bell if for each $P \in \mathrm{Proj}(\mathcal{A})$ there exists a $Q \in \mathrm{Proj}(\mathcal{B})$ such that $(*)$,
2. left-Bell if for each $Q \in \mathrm{Proj}(\mathcal{B})$ there exists a $P \in \mathrm{Proj}(\mathcal{A})$ such that $(*)$,
3. Bell if it is both left-Bell and right-Bell.

Proposition. For a channel $\mathcal{A}_1 \xrightarrow{\varphi} \mathcal{A}_2$, t.f.a.e.:

1. φ is deterministic,
2. for every Bell state $\mathbb{C} \xrightarrow{\mu} \mathcal{A}_1 \otimes \mathcal{B}$, we obtain a right-Bell state by composing

$$\mathbb{C} \xrightarrow{\mu} \mathcal{A}_1 \otimes \mathcal{B} \xrightarrow{\varphi \otimes \mathrm{id}} \mathcal{A}_2 \otimes \mathcal{B} \ .$$

"After Alice applies φ and performs a measurement on her part of the system, Bob can still learn the outcome by performing a measurement on his part."

Intuition. A deterministic φ introduces no randomness.

Example. We prepare a qubit in the $+$-state and measure it in the standard basis.

$$\mathbb{C} \xrightarrow{\mu} M_2(\mathbb{C}) \xrightarrow{\varphi} \mathbb{C}^2$$

Definition. For a bipartite state $\mathbb{C} \xrightarrow{\mu} \mathcal{A} \otimes \mathcal{B}$, we define



Theorem (Bayeh, K). For a bipartite state $\mathbb{C} \xrightarrow{\mu} \mathcal{A} \otimes \mathcal{B}$, t.f.a.e.:

1. μ is right-Bell,
2. $\tilde{S}(\mu) = \tilde{S}(\mu_{\mathcal{B}})$.

Intuition. All of the information exists in the right part.

x Corollary. A bipartite state $\mathbb{C} \xrightarrow{\mu} \mathcal{A} \otimes \mathcal{B}$ is Bell iff $\tilde{S}(\mu_{\mathcal{A}}) = \tilde{S}(\mu) = \tilde{S}(\mu_{\mathcal{B}})$.

Theorem (Bayeh, K). $\tilde{S}$ is the unique quantum entropy for which this is true.