Discrete quantum graphs and discrete quantum groups

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"First quantization is a mystery, but second quantization is a functor."

— Edward Nelson (attributed)

Question. Is the noncommutative metaphor a heuristic or a structure?

Equivalently, is it a funny way of speaking that helps us find interesting theorems, or is it an intuition of a mathematical universe that we do not fully grasp yet?

This question motivates much of my research. My main conclusions so far:

1. For discrete structures, mathematical quantization is a functor.
2. The role of C^*-algebras in the metaphor is surprisingly unclear.

For me, quantum maps and quantum groups were the major turning points.

To quantize sets X, we allow $\ell^\infty(X)$ to be noncommutative.

Theorem (K; Chirvasitu). For a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, t.f.a.e.:

1. every masa is isomorphic to $\ell^\infty(X)$ for some set X,
2. every von Neumann subalgebra $\mathcal{N} \subseteq \mathcal{M}$ is atomic,
3. every self-adjoint $a \in \mathcal{M}$ has an orthonormal basis of eigenvectors,
4. the set of unitary operators in $\mathcal{M}$ is ultraweakly compact,
5. $\mathcal{M} \cong \bigoplus_{i \in I}^{\ell^\infty} M_{n_i}(\mathbb{C})$ for some indexed family $(n_i \in \mathbb{Z}_+ \mid i \in I)$.

Definition (K). In this case, $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is hereditarily atomic (condition 2).

These are essentially discrete quantum spaces (Podleś, Woronowicz).

Proposition (Weaver). There is a one-to-one correspondence between

1. relations R from X to Y,
2. quantum relations $\mathcal{E}$ from $\ell^\infty(X)$ to $\ell^\infty(Y)$.

Definition. A relation from X to Y is a subset $R \subseteq X \times Y$.

Definition (Weaver). A quantum relation from $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ to $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ is an ultraweakly closed subspace $\mathcal{E} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{N}'\mathcal{E}\mathcal{M}' \subseteq \mathcal{E}$.

Definition. We define categories **Rel** and **qRel**:

1. for $X \xrightarrow{R} Y \xrightarrow{S} Z$, define $S \circ R = \{(x, z) \mid \exists y,\ (x, y) \in R,\ (y, z) \in S\}$,
2. for $\mathcal{M} \xrightarrow{\mathcal{E}} \mathcal{N} \xrightarrow{\mathcal{F}} \mathcal{O}$, define $\mathcal{F} \circ \mathcal{E} = \overline{\operatorname{span}}\{fe \mid e \in \mathcal{E}, f \in \mathcal{F}\}$.

The identity morphisms are given by $\mathrm{Id}_X = \{(x, x) \mid x \in X\}$ and $\mathrm{Id}_{\mathcal{M}} = \mathcal{M}'$.

Definition. The "inclusion" functor Inc: **Rel** $\to$ **qRel** is defined by

$$\mathrm{Inc}(X) = \ell^\infty(X) \subseteq \mathcal{B}(\ell^2(X)),$$

$$\mathrm{Inc}(X \xrightarrow{R} Y) = \overline{\mathrm{span}}\{e_{y,x} \mid (x,y) \in R\}.$$

Proposition. This functor is full and faithful and preserves

1. transposes, $\mathrm{Inc}(R^T) = \mathrm{Inc}(R)^*$,
2. complements, $\mathrm{Inc}(R^C) = \mathrm{Inc}(R)^\perp$,
3. Cartesian products, $\mathrm{Inc}(R \times S) = \mathrm{Inc}(R) \mathbin{\overline{\otimes}} \mathrm{Inc}(S)$,
4. order, $R \subseteq S \iff \mathrm{Inc}(R) \subseteq \mathrm{Inc}(S)$, etc.

In short, Inc is a functor of dagger compact quantaloids (Jenča, Lindenhovius).

This structure is an expressive language that is equivalent to first-order logic.

Framework. We quantize a class $\mathfrak{C}$ of discrete structures in the following way:

1. choose a definition of $\mathfrak{C}$ in **Rel**,
2. reinterpret that definition in **qRel** to obtain $\mathfrak{C}'$.

Automatically,

1. $\mathfrak{C}'$ is a class of discrete quantum structures,
2. the commutative members of $\mathfrak{C}'$ are exactly $\mathfrak{C}$.

Definition (Freyd, Scedrov). A quantum relation $\mathcal{M} \xrightarrow{\mathcal{E}} \mathcal{N}$ is a quantum map if

$$\mathcal{E}^* \circ \mathcal{E} \supseteq \mathcal{M}', \qquad \mathcal{E} \circ \mathcal{E}^* \subseteq \mathcal{N}'.$$

Theorem (K). There is a canonical one-to-one correspondence between quantum maps $\mathcal{M} \to \mathcal{N}$ and unital normal $*$-homomorphisms $\mathcal{N} \to \mathcal{M}$.

Convention. In a graph, loops are allowed, and multiple edges are forbidden.

Definition (Duan, Severini, Winter; Weaver). A quantum graph consists of

1. a hereditarily atomic von Neumann algebra $\mathcal{M}$,
2. a quantum relation $\mathcal{M} \xrightarrow{\mathcal{E}} \mathcal{M}$ such that $\mathcal{E}^* = \mathcal{E}$.

In other words, a quantum graph is a graph object in **qRel**.

This is the canonical quantum generalization of graphs in NCG.

If we allow arbitrary $\mathcal{M}$, then we have a quantum and measurable generalization.

Definition (de Santiago, McNamara). The box product of quantum graphs is

$$(\mathcal{M}_1, \mathcal{E}_1) \,\square\, (\mathcal{M}_2, \mathcal{E}_2) = (\mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{M}_2, \mathcal{E}_1 \mathbin{\overline{\otimes}} \mathcal{M}_2' \mathbin{\overline{+}} \mathcal{M}_1' \mathbin{\overline{\otimes}} \mathcal{E}_2).$$

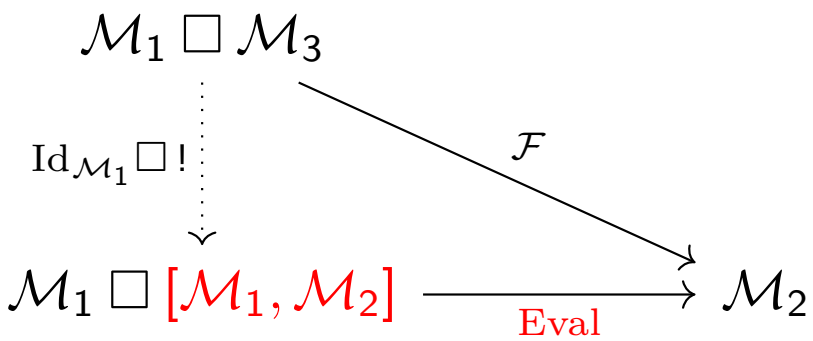
Definition (Musto, Reutter, Verdon; K, Lindenhovius). A quantum homomorphism

$$(\mathcal{M}_1, \mathcal{E}_1) \xrightarrow{\mathcal{F}} (\mathcal{M}_2, \mathcal{E}_2)$$

is a quantum map $\mathcal{M}_1 \xrightarrow{\mathcal{F}} \mathcal{M}_2$ such that $\mathcal{F} \circ \mathcal{E}_1 \subseteq \mathcal{E}_2 \circ \mathcal{F}$.

Equivalently, this is a unital normal $*$-homomorphism $\phi\colon \mathcal{M}_2 \to \mathcal{M}_1$ such that $\phi \mathbin{\overline{\otimes}} \phi\colon \mathcal{M}_2 \mathbin{\overline{\otimes}} \mathcal{M}_2^{\mathrm{op}} \to \mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{M}_1^{\mathrm{op}}$ satisfies $[\mathcal{E}_1] \subseteq (\phi \mathbin{\overline{\otimes}} \phi)([\mathcal{E}_2])$.

Theorem (K, Lindenhovius). The box product makes **qGph**, the category of quantum graphs and quantum homomorphisms, symmetric monoidal closed:



In **Gph**, this universal property defines the graph of homomorphisms $G_1 \to G_2$.

Proposition. In **qGph**, the commutative summand of $[\mathcal{M}_1, \mathcal{M}_2]$ is isomorphic to

$$\ell^\infty(\{\mathcal{F} \mid \mathcal{F} \text{ is a quantum homomorphism } \mathcal{M}_1 \to \mathcal{M}_2\}).$$

Theorem (K, Lindenhovius). Let G_1 and G_2 be simple finite graphs. Then, t.f.a.e.:

1. $[\mathrm{Inc}(G_1), \mathrm{Inc}(G_2)]$ is not isomorphic to $\mathrm{Inc}(\varnothing)$,
2. the (G_1, G_2)-homomorphism game has a quantum winning strategy.

These games are a key example of quantum nonlocality (Mančinska, Roberson).

Intuition. $[\mathcal{M}_1, \mathcal{M}_2]$ is the quantum graph of homomorphisms $\mathcal{M}_1 \to \mathcal{M}_2$.

The factors of $[\mathrm{Inc}(G_1), \mathrm{Inc}(G_2)]$ are likely in bijection with the finite-dimensional irreducible representations of the game C^*-algebra $\mathcal{A}(G_1, G_2)$ (Ortiz, Paulsen).

Convention. A semigroup has an identity element.

Definition. A quantum semigroup consists of

1. a hereditarily atomic von Neumann algebra $\mathcal{M}$,
2. quantum maps $\mathbb{C} \xrightarrow{\mathcal{E}} \mathcal{M}$ and $\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{M} \xrightarrow{\mathcal{F}} \mathcal{M}$ such that

$$\mathcal{F} \circ (\mathcal{F} \mathbin{\overline{\otimes}} \mathcal{M}') = \mathcal{F} \circ (\mathcal{M}' \mathbin{\overline{\otimes}} \mathcal{F}), \quad \mathcal{F} \circ (\mathcal{E} \mathbin{\overline{\otimes}} \mathcal{M}') = \mathcal{M}' = \mathcal{F} \circ (\mathcal{M}' \mathbin{\overline{\otimes}} \mathcal{E}).$$

Classically, groups have two equivalent definitions:

1. a group is a semigroup in which every element has an inverse,
2. a group is a semigroup with an inversion map (a unary operation).

Classically, the distinction between these two definitions is relevant only in logic.

In the quantum setting, these definitions are not equivalent.

Definition. The top predicate on $\mathcal{M}$ is the maximum quantum relation $\mathcal{M} \xrightarrow{\mathcal{T}} \mathbb{C}$.

Concretely, $\mathcal{T} = \mathcal{B}(\mathcal{H}, \mathbb{C})$, where $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$.

Theorem (Vaes). Up to isomorphism, there is a canonical one-to-one correspondence between

1. quantum groups, i.e., discrete quantum groups,
2. quantum semigroups $(\mathcal{M}, \mathcal{E}, \mathcal{F})$ such that

$$\mathcal{E}^* \circ \mathcal{F} \circ (\mathcal{T}^* \mathbin{\overline{\otimes}} \mathcal{M}') = \mathcal{T} = \mathcal{E}^* \circ \mathcal{F} \circ (\mathcal{M}' \mathbin{\overline{\otimes}} \mathcal{T}^*),$$

where $\mathcal{M} \xrightarrow{\mathcal{T}} \mathbb{C}$ is the top predicate on $\mathcal{M}$.

In **Rel**, these conditions express directly that each element has an inverse.

Quantum logic says that if inverses are unique, then there is an inversion map...

Definition (K). The diagonal predicate on $\mathcal{M} \overline{\otimes} \mathcal{M}^{op}$ is the maximum q. relation

$$\mathcal{M} \overline{\otimes} \mathcal{M}^{op} \xrightarrow{\mathcal{D}} \mathbb{C}$$

such that $\mathcal{D}(p \otimes (1-p)) = \{0\}$ for all projections $p \in \mathcal{M}$.

Theorem (Chirvasitu, K). Up to isomorphism, there is a canonical one-to-one correspondence between

1. quantum groups of Kac type, i.e., discrete quantum groups of Kac type,
2. quantum semigroups $(\mathcal{M}, \mathcal{E}, \mathcal{F})$ with a quantum map $\mathcal{M}^{\mathrm{op}} \xrightarrow{\mathcal{S}} \mathcal{M}$ such that

$$\mathcal{E}^* \circ \mathcal{F} \circ (\mathcal{S} \overline{\otimes} \mathcal{M}') \supseteq \mathcal{D}_1, \qquad \mathcal{E}^* \circ \mathcal{F} \circ (\mathcal{M}' \overline{\otimes} \mathcal{S}) \supseteq \mathcal{D}_2,$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the diagonal predicates on $\mathcal{M}^{\mathrm{op}} \overline{\otimes} \mathcal{M}$ and $\mathcal{M} \overline{\otimes} \mathcal{M}^{\mathrm{op}}$.

In **Rel**, these conditions express directly that $\mathcal{S}$ is the inversion map.