The category of topological spaces and open maps does not have products

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Theorem (Bezhanishvili, K). **TopOpen** does not have the product $\mathbb{S} \times \mathbb{S}$.

Definition. We define the category **TopOpen**.

1. An object is any topological space.
2. A morphism $f : X \to Y$ is a continuous map such that

$$U \subseteq X \text{ is open} \quad \Longrightarrow \quad f[U] \text{ is open.}$$

Definition. $\mathbb{S}$ consists of one closed point and one open point.

Remark. The theorem is also true if the spaces are assumed to be

1. T_0, or
2. sober.

Theorem (Bezhanishvili, K). **PreOpen** does not have the product $\mathbb{S} \times \mathbb{S}$.

Definition. We define the category **PreOpen**.

1. An object of is any preordered set.
2. A morphism $f\colon X \to Y$ is a monotone map such that

$$U \subseteq X \text{ is a downset} \quad \Longrightarrow \quad f[U] \text{ is a downset.}$$

Definition. $\mathbb{S}$ consists of one top element and one bottom element.

Remark. **PreOpen** is the category of Alexandrov spaces and open maps.

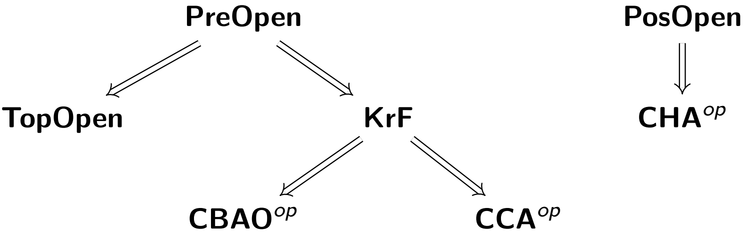
$$\mathbf{PreOpen} \; \underset{\dashleftarrow}{\overset{\hookrightarrow}{\bot}} \; \mathbf{TopOpen}$$

Theorem. Similarly, **PosOpen** does not have the product $\mathbb{S} \times \mathbb{S}$.

The proofs of these two theorems are essentially identical.

Neither theorem appears to be an easy corollary of the other.

Corollary. The following categories do not have binary products.



Definition. We define the category **CHA**.

1. An object is a complete Heyting algebra.
2. A morphism is a complete lattice homomorphism that preserves implication.

Theorem (De Jongh). The coproduct $\hat{H} + \hat{H}$ does not exist in **CHA**, where $\hat{H}$ is the Rieger-Nishimura lattice with a top.

Theorem (Bezhanishvili, K). The coproduct $\mathbf{3} + \mathbf{3}$ does not exist in **CHA**, where $\mathbf{3}$ is the three-element chain.

Theorem (Gaifman; Hales). The category **CBA** of complete Boolean algebras does not have countable coproducts.

Definition. We define the category **KrF**.

1. An object of is a Kripke frame, i.e., a set with a binary relation.
2. A morphism $f\colon (X, R) \to (Y, S)$ is a p-morphism, i.e., a map with

$$f[R[x]] = S[f(x)].$$

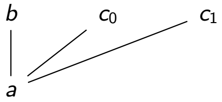
Remark. f is a p-morphism iff it induces a hom. of modal algebras.

Definition. We define the category **CCA**.

1. An object is a complete closure algebra, i.e., a complete Boolean algebra B with a map $\Diamond : B \to B$ that satisfies Kuratowski's axioms.
2. A morphism $\varphi\colon (B, \Diamond_1) \to (B_2, \Diamond_2)$ is a complete homomorphism such that $\varphi(\Diamond_1 a) = \Diamond_2 \varphi(a)$.

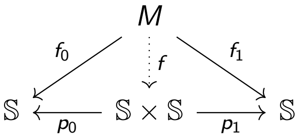
Theorem. **PreOpen** does not have the product $\mathbb{S} \times \mathbb{S}$.

Proof sketch. Suppose otherwise. Let $f_0, f_1 \colon M \to \mathbb{S}$ be given by

$M =$ 

$$f_i(x) = \begin{cases} 0 & \text{if } x = a, \\ 1 & \text{if } x = b, \\ i \oplus j & \text{if } x = c_j. \end{cases}$$

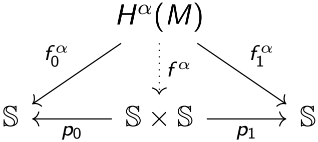
Both f_0 and f_1 are open maps.



The map $f = (f_0, f_1)$ is injective, so $\#(\mathbb{S} \times \mathbb{S}) \geq 4$.

Proof sketch continues. For each ordinal α, we construct

1. a poset $H^\alpha(M)$,
2. open maps $f_0^\alpha, f_1^\alpha \colon H^\alpha(M) \to \mathbb{S}$.



We then show that

1. $\# H^\alpha(M)$ grows without bound as $\alpha \to \infty$,
2. the maps $f^\alpha \colon H^\alpha(M) \to \mathbb{S} \times \mathbb{S}$ are injective.

We obtain an immediate contradiction. □

We construct each $H^\alpha(M)$ to be a subset of V, the class of hereditary sets.

The class V is partially ordered by

$$x < y \quad \Longleftrightarrow \quad x \in z_1 \in \cdots \in z_n \in y.$$

Definition. An antichain is a subset $A \subset V$ such that

$$x \not< y \text{ and } x \not> y$$

for distinct $x, y \in A$.

Definition. An antichain A is nontrivial if $\#A \geq 2$.

Definition. For each subset $N \subseteq V$, we define

$$H(N) = N \cup \{A \subseteq N \mid A \text{ is a nontrivial antichain}\}.$$

Definition. We define $M = \{a, b, c_0, c_1\}$, where

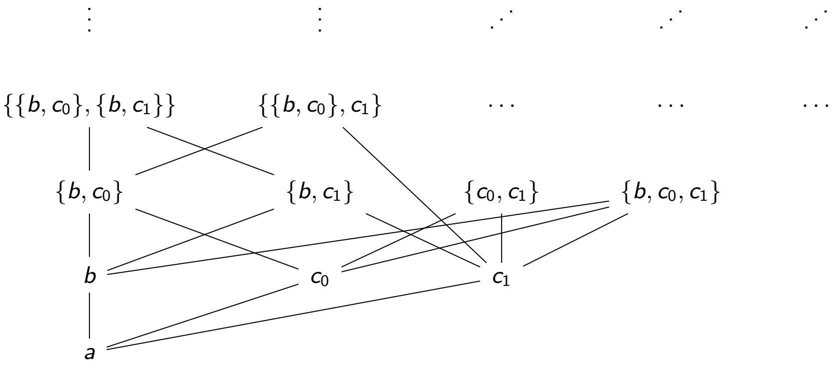
$$a = \{\{\varnothing\}\}, \qquad b = \{a, 1\}, \qquad c_0 = \{a, 2\}, \qquad c_1 = \{a, 3\}.$$

Definition. We define the cumulative hierarchy $H^\alpha(M)$.

1. $H^0(M) = M$,
2. $H^{\alpha+1}(M) = H(H^\alpha(M))$,
3. $H^\lambda(M) = \bigcup_{\alpha<\lambda} H^\alpha(M)$ for limit ordinals λ,
4. $H(M) = \bigcup_{\alpha} H^\alpha(M)$.

Theorem. $H(M)$ is a proper class.

We visualize the construction of $H(M)$:



The technical core of the proof is the following lemma.

Lemma. Let $N \subseteq V$ be such that

1. $y \in N$ whenever $x < y < z$ with $x, z \in N$,
2. $x \leq y$ or $y \geq x$ whenever $x, y < z$ with $x, y, z \in N$.

Let α be an ordinal, and let P be a preordered set. For all open maps

$$f \colon H^{\alpha}(N) \to P$$

if f is injective on N, then f is injective.

Main idea. For $x, y \in H^{\alpha+1}(M) \setminus H^{\alpha}(M)$,

$$f(x) = f(y) \;\Rightarrow\; f[\downarrow x] = f[\downarrow y] \;\Rightarrow\; \downarrow x \setminus \{x\} = \downarrow y \setminus \{y\} \;\Rightarrow\; x = y.$$