Constructing the category of quantum graphs

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Talk outline.

1. Review the matrix construction of **FinHilb**.
2. Use **FinHilb** to define $\mathbf{Q}\mathbb{B}$, a quantum analogue of $\mathbf{B}\mathbb{B}$.
3. Use $\mathbf{Q}\mathbb{B}$ to define **qRel**.
4. Use **qRel** to define **qGph**.
5. Relate **qGph** to the graph homomorphism game.

The categories **FinRel** and **FinHilb** are dagger-compact and dagger-kernel.

Definition (Abramsky, Coecke). A dagger-compact category is

1. a compact closed category,
2. a dagger category,

such that each coherence isomorphism is unitary and $\eta_X^\dagger \circ \sigma_{X,\overline{X}} = \varepsilon_X$.

Definition (Heunen, Jacobs). A dagger-kernel category is

1. a dagger category,
2. with a zero object,

such that every morphism has a kernel k satisfying $k^\dagger \circ k = 1_{\mathrm{dom}(k)}$.

The categories **FinRel** and **FinHilb** come from the matrix construction.

Definition (Mac Lane; Lawvere). Every **CMod**-category **C** has a free finite biproduct completion $\mathrm{Mat}(\mathbf{C})$ whose

1. objects are tuples of objects of **C**,
2. morphisms are matrices of morphisms of **C**.

If **C** is a dagger-**CMod**-category, then so is $\mathrm{Mat}(\mathbf{C})$.

Example. $\mathbf{FinHilb} \simeq \mathrm{Mat}(\mathbf{B}\mathbb{C})$.

Example. $\mathbf{FinRel} = \mathrm{Mat}(\mathbf{B}\mathbb{B})$.

Definition. $\mathbf{qFinRel} = \mathrm{Mat}(\mathbf{Q}\mathbb{B})$.

The dagger-**CMod**-category $\mathbf{Q}\mathbb{B}$ will be defined. It is analogous to $\mathbf{B}\mathbb{B}$.

Let **C** be a dagger-compact category and a dagger-kernel category.

Definition. A dagger kernel in **C** is a morphism $k\colon X \to Y$ such that

1. k is the kernel of some morphism $f\colon Y \to Z$,
2. k satisfies $k^\dagger \circ k = 1_X$.

Dagger kernels are intuitively the embeddings in **C**.

Definition. A bridge in **C** from X to Y is a dagger kernel into $Y \otimes \overline{X}$.

Example. A bridge in **FinRel** from X to Y is a relation from X to Y.

Example. A bridge in **FinHilb** from X to Y is a subspace of $L(X, Y)$.

Definition. We can compose bridges $R \xrightarrow{i} Y \otimes \overline{X}$ and $S \xrightarrow{j} Z \otimes \overline{Y}$.

$$\begin{tikzcd} S \circ R \arrow[r] \arrow[d, "k"'] & S \otimes R \arrow[d, "j \otimes i"] \\ Z \otimes \overline{X} \arrow[r, "1_Z \otimes \eta_Y \otimes 1_{\overline{X}}"'] & Z \otimes \overline{Y} \otimes Y \otimes \overline{X} \end{tikzcd}$$

The composition $S \circ R \xrightarrow{k} Z \otimes \overline{X}$ is obtained as a pullback.

Definition. The dagger of $R \xrightarrow{i} Y \otimes \overline{X}$ is obtained as a composition

$$\overline{R} \xrightarrow{\bar{i}} \overline{Y} \otimes X \xrightarrow{\sigma_{\overline{Y},X}} X \otimes \overline{Y}.$$

Starting with **FinRel**, we get **FinRel**. Starting with **FinHilb**, we get $\mathbf{Q}\mathbb{B}$.

In effect, $\mathbf{Q}\mathbb{B}$ is the dagger sym. monoidal category of matrix subspaces.

I don't know whether we get a dagger-monoidal category in general.

Like $\mathbf{B}\mathbb{B}$, $\mathbf{Q}\mathbb{B}$ is enriched over complete join-semilattices.

(Thus, $\mathbf{B}\mathbb{B}$ and $\mathbf{Q}\mathbb{B}$ are both monoidal 2-categories.)

Because they are enriched over complete join-semilattices, we can define

$$\mathrm{Mat}_\infty(\mathbf{B}\mathbb{B}) \qquad \text{and} \qquad \mathrm{Mat}_\infty(\mathbf{Q}\mathbb{B}).$$

Example. $\mathrm{Mat}_\infty(\mathbf{B}\mathbb{B}) = \mathbf{Rel}$.

Definition (K). $\mathrm{Mat}_\infty(\mathbf{Q}\mathbb{B}) = \mathbf{qRel}$.

Proposition. Since $\mathbf{B}\mathbb{B}$ is a dagger-**CSLatt**-subcategory of $\mathbf{Q}\mathbb{B}$, we find that **Rel** is a dagger-**CSLatt**-subcategory of **qRel**.

We obtain the full and faithful "inclusion" functor $\mathbf{Rel} \to \mathbf{qRel}$.

Proposition. Both **Rel** and **qRel** are dagger-compact categories, and

1. this structure is enriched over complete join-semilattices,
2. this structure is preserved by the "inclusion" functor **Rel** $\to$ **qRel**.

Definition (Jenča, Lindenhovius). In other words, they are dagger-compact quantaloids, and the "inclusion" is a homomorphism of these.

This structure can be viewed as formalizing first-order logic.

We can generalize definitions from **Rel** to **qRel**, "quantizing" them.

This is "allegorical" rather than "categorical" internalization.

This quantization is in the sense of noncommutative geometry.

For example, we can obtain discrete quantum groups in this way!

Definition. We define the symmetric monoidal category **qGph** in **qRel**:

1. an object is a pair $G = (V_G, E_G\colon V_G \to V_G)$ such that $E_G^\dagger = E_G$,
2. a morphism $G \to H$ is a morphism $\phi\colon V_G \to V_H$ such that
$$\phi^\dagger \circ \phi \geq 1_{V_G}, \qquad \phi \circ \phi^\dagger \leq 1_{V_H}, \qquad \phi \circ E_G \leq E_H \circ \phi,$$
3. the monoidal product $G \,\Box\, H$ is defined by
$$V_{G \Box H} = V_G \otimes V_H, \qquad E_{G \Box H} = (E_G \otimes 1_{V_H}) \vee (1_{V_G} \otimes E_H),$$
4. the monoidal unit is defined by $K_1 = (1, 0_1)$.

There is also an initial object, which is defined by $K_0 = (0, 0_0)$.

The same definition defines the category **Gph** in **Rel**.

(In this setting, loops are allowed, and multiple edges are forbidden.)

We have a full and faithful monoidal "inclusion" functor **Gph** $\to$ **qGph**.

Proposition. The monoidal category **Gph** is closed.

Theorem (K, Lindenhovius). The monoidal category **qGph** is closed.

Proposition. For finite simple graphs G and H, t.f.a.e.:

1. the (G, H)-homomorphism game has a winning strategy,
2. $[G, H] \ncong K_0$ in **Gph**.

Theorem (K, Lindenhovius). For finite simple graphs G and H, t.f.a.e.:

1. the (G, H)-homomorphism game has a winning quantum strategy,
2. $[G, H] \ncong K_0$ in **qGph**.

Graph homomorphism games demonstrate quantum nonlocality.

Definition (Mančinska, Roberson). Let G and H be finite simple graphs. The (G, H)-homomorphism game is played by two players who are cooperating against an adverse environment. The play proceeds as follows:

1. the players agree on a strategy,
2. the players are separated,
3. each player $i \in \{1, 2\}$ encounters a random vertex x_i of G,
4. each player $i \in \{1, 2\}$ responds with a vertex y_i of H,
5. the players win unless one of the following happens:
   (a) $x_1 \sim x_2$ but $y_1 \not\sim y_2$,
   (b) $x_1 = x_2$ but $y_1 \neq y_2$.

Proposition. The following are equivalent:

1. there exists a winning strategy for the (G, H)-homomorphism game,
2. there is a homomorphism $G \to H$.

Theorem (Mančinska, Roberson). There exist finite simple graphs

1. G on 14 vertices,
2. H on 4 vertices

such that the (G, H)-homomorphism game

1. has no winning strategy,
2. has a winning quantum strategy.

Definition. A quantum strategy may use quantum entanglement.

Theorem (K, Lindenhovius). For finite simple graphs G and H, t.f.a.e.:

1. the (G, H)-homomorphism game has a winning quantum strategy,
2. $[G, H] \not\cong K_0$ in **qGph**.

To sketch a proof of this theorem, we appeal to the following two facts.

Proposition. The category **qGph** is canonically enriched over **Gph**, and the "inclusion" functor **Gph** $\to$ **qGph** has right adjoint $G \mapsto \mathbf{qGph}(K_1, G)$.

Theorem (Mančinska, Roberson). For simple graphs G and H, t.f.a.e.:

1. the (G, H)-homomorphism game has a winning quantum strategy,
2. there is a homomorphism $G \to \mathbf{qGph}(Q_d, H)$ for some $d > 0$.

Here, $Q_d = (\mathbb{C}^d, 0_{\mathbb{C}^d})$. This is a "quantum discrete graph."

Proof. We have the following natural bijections:

$$\begin{aligned}
\mathbf{qGph}(Q_d, [G, H]) &\cong \mathbf{qGph}(Q_d \,\square\, G, H) \\
&\cong \mathbf{qGph}(G \,\square\, Q_d, H) \\
&\cong \mathbf{qGph}(G, [Q_d, H]) \\
&\cong \mathbf{Gph}(G, \mathbf{qGph}(K_1, [Q_d, H])) \\
&\cong \mathbf{Gph}(G, \mathbf{qGph}(K_1 \,\square\, Q_d, H)) \\
&\cong \mathbf{Gph}(G, \mathbf{qGph}(Q_d, H)).
\end{aligned}$$

We now reason that

1. $\mathbf{qGph}(Q_d, [G, H]) = \varnothing$ for all $d > 0$ iff $[G, H] \cong K_0$ in $\mathbf{qGph}$,
2. $\mathbf{Gph}(G, \mathbf{qGph}(Q_d, H)) = \varnothing$ for all $d > 0$ iff the (G, H)-hom. game has no winning quantum strategy. $\square$

Theorem (Vaes). Discrete quantum groups are in one-to-one correspondence with allegorical group objects in **qRel**.