A new measure of entropy for quantum states

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This talk will sketch a “conceptual application”

- of noncommutative geometry
- to quantum information theory.

A question in one subject leads to a theorem in the other.

This talk is based on two papers:

[1] A. Kornell, *Characterizations of homomorphisms among unital completely positive maps*, Linear Algebra Appl. **709** (2025).

[2] F. Bayeh and A. Kornell, *Localizing quantum information*, forthcoming preprint.

In the finite setting,

1. noncommutative geometry is the symm. mon. category **qFinSet**,
2. quantum information theory is the symm. mon. category **qFinStoch**.

Definition. **qFinSet** is the opposite of the following category:

1. an object is a finite-dimensional C^*-algebra $\mathcal{A}$,
2. a morphism is a unital $*$-homomorphism.

Definition. **qFinStoch** is the following category:

1. an object is a finite-dimensional C^*-algebra,
2. a morphism is a trace-preserving completely positive map.

In noncommutative geometry,

1. **qFinSet** generalizes the symm. mon. category **FinSet**,
2. **qFinStoch** generalizes the symm. mon. category **FinStoch**,

where

1. **FinSet** consists of finite sets and maps,
2. **FinStoch** consists of finite sets and stochastic maps.

The following diagram summarizes the setting.

$$\begin{tikzcd}
\mathbf{FinSet} \arrow[r, hook] \arrow[d, hook] & \mathbf{FinStoch} \arrow[d, hook] \\
\mathbf{qFinSet} \arrow[r, hook] & \mathbf{qFinStoch}
\end{tikzcd}$$

Definition. A stochastic map $s\colon X \to Y$ is given by probabilities

$$s(y|x) \in [0,1].$$

Definition. The inclusion functor **FinSet** $\hookrightarrow$ **FinStoch** is defined by

$$f\colon X \to Y \qquad \mapsto \qquad w_f\colon X \to Y, \qquad w_f(y|x) = \begin{cases} 1 & y = f(x), \\ 0 & y \neq f(x). \end{cases}$$

Definition. The inclusion functor **qFinSet** $\to$ **qFinStoch** is defined by

$$\pi\colon \mathcal{A} \leftarrow \mathcal{B} \qquad \mapsto \qquad \varphi_\pi\colon \mathcal{A} \to \mathcal{B}, \qquad \varphi_\pi = \pi^*.$$

The adjoint is computed for the standard trace:

$$\mathrm{tr}(p) = 1 \text{ for all minimal projections } p.$$

Proposition. Let $w\colon X \to Y$ be a stochastic map. T.F.A.E.:

1. $w = w_f$ for some map $f\colon X \to Y$,
2. $S(w \circ p) \leq S(p)$ for every stochastic map $p\colon \{*\} \to X$,

where S denotes Shannon entropy.

Definition. The Shannon entropy of p is defined by

$$S(p) = -\sum_{x \in X} p(x) \log p(x),$$

where $p(x) = p(x|*)$.

Question. Does this theorem have a quantum analogue?

Question. Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a t.p.c.p. map. A.T.F.E.:

1. $\varphi = \varphi_\pi$ for some unital $*$-hom. $\pi\colon \mathcal{A} \leftarrow \mathcal{B}$,
2. $S(\varphi \circ \rho) \leq S(\rho)$ for every t.p.c.p. map $\rho\colon \mathbb{C} \to \mathcal{A}$,

where S denotes von Neumann entropy?

Definition. The von Neumann entropy of ρ is defined by

$$S(\rho) = -\mathrm{tr}(\rho \log \rho),$$

where $\rho = \rho(1)$.

Proposition. The answer to this question is no.

Theorem(K). Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a t.p.c.p. map. T.F.A.E.:

1. $\varphi = \varphi_\pi$ for some unital $*$-hom. $\pi\colon \mathcal{A} \leftarrow \mathcal{B}$,
2. $\tilde{S}((\mathrm{id} \otimes \varphi) \circ \rho) \leq \tilde{S}(\rho)$ for every t.p.c.p. map $\rho\colon \mathbb{C} \to M_n(\mathbb{C}) \otimes \mathcal{A}$,

where $\tilde{S}$ denotes adjusted von Neumann entropy.

Definition. The adjusted von Neumann entropy of ρ is defined by

$$\tilde{S}(\rho) = S(\rho) + \mathrm{tr}(\rho \log \zeta_{\mathcal{A}}),$$

where $\zeta_{\mathcal{A}} \in \mathcal{A}$ is the central element such that

$$\zeta_{\mathcal{A}}^2 p = \dim(p\mathcal{A})p$$

for all minimal central projections $p \in \mathcal{A}$.

Proposition. When $\mathcal{A}$ is commutative, $\tilde{S}(\rho) = S(\rho)$ for all $\rho \in \mathcal{A}$.

Adjusted von Neumann entropy generalizes Shannon entropy

1. in noncommutative geometry.
2. in quantum information theory?

Question. What characterizes the t.p.c.p. maps $\rho\colon \mathbb{C} \to \mathcal{A} \otimes \mathcal{B}$ such that

$$\tilde{S}((\mathrm{tr} \otimes \mathrm{id}) \circ \rho) = \tilde{S}(\rho)?$$

Proposition. Let $p\colon \{*\} \to X \times Y$ be a stochastic map. T.F.A.E.:

1. $S((! \times \mathrm{id}) \circ p) = S(p)$,
2. for each map $f\colon X \to \{0,1\}$, there is a map $g\colon Y \to \{0,1\}$ such that

$$\mathrm{Ran}\,(w_f \times w_g) \circ p \subseteq \{(0,0),(1,1)\}.$$

Theorem(Bayeh, K). Let $\rho\colon \mathbb{C} \to \mathcal{A} \otimes \mathcal{B}$ be a t.p.c.p. map. T.F.A.E.:

1. $\tilde{S}((\mathrm{tr} \otimes \mathrm{id}) \circ \rho) = \tilde{S}(\rho)$,
2. for each unital $*$-homomorphism $\pi\colon \mathcal{A} \leftarrow \mathbb{C}^2$, there is a unital $*$-homomorphism $\chi\colon \mathcal{B} \leftarrow \mathbb{C}^2$ such that

$$\mathrm{Ran}\,(\varphi_\pi \otimes \varphi_\chi) \circ \rho \subseteq \mathrm{span}\{e_0 \otimes e_0, e_1 \otimes e_1\}.$$

Definition. A right-Bell state is a t.p.c.p. map ρ that satisfies condition 2.

A right-Bell state is an EPR state relative to $(\mathcal{A}, \mathrm{Proj}\,\mathcal{A})$ in the sense of

[3] Arens and Varadarajan, *On the concept of Einstein-Podolsky-Rosen states and their structure*, J. Math. Phys. **41** (2000).

Theorem(Bayeh, K). Adjusted von Neumann entropy $H = \tilde{S}$ is the unique quantum entropy H such that the previous theorem holds, i.e., such that for all t.p.c.p. maps $\rho\colon \mathbb{C} \to \mathcal{A} \otimes \mathcal{B}$, T.F.A.E.:

1. $H(\mathcal{B}, (\mathrm{tr} \otimes \mathrm{id}) \circ \rho) = H(\mathcal{A} \otimes \mathcal{B}, \rho)$,
2. ρ is a right-Bell state.

(We have made the ambient C^*-algebras explicit.)

Definition. A quantum entropy is a function on pairs $(\mathcal{A}, \rho)$ such that

1. $H(\mathcal{A}, \rho) = H(\mathcal{B}, \sigma)$ whenever $\sigma = \pi \circ \rho$ for some $*$-iso. $\pi\colon \mathcal{A} \to \mathcal{B}$,
2. $H(\mathcal{A} \oplus \mathcal{B}, \rho \oplus 0) = H(\mathcal{A}, \rho)$,
3. $H(\mathcal{A} \otimes \mathcal{B}, \rho \otimes \sigma) = H(\mathcal{A}, \rho) + H(\mathcal{B}, \sigma)$,
4. $H(\mathcal{A}, p_1\rho_1 + p_2\rho_2) = p_1 H(\mathcal{A}, \rho_1) + p_2 H(\mathcal{A}, \rho_2) - p_1 \log p_1 - p_2 \log p_2$.

Proposition. Von Neumann entropy is a quantum entropy.

Proposition. Von Neumann entropy is not monotonic.

Definition. A quantum entropy is monotonic if

$$H(\mathcal{B}, (\mathrm{tr} \otimes \mathrm{id}) \circ \rho) \leq H(\mathcal{A} \otimes \mathcal{B}, \rho).$$

(Shannon entropy is not a quantum entropy, but it is monotonic.)

Theorem(Bayeh, K). Adjusted von Neumann entropy $\tilde{S}$ is monotonic, and for every monotonic quantum entropy H, we have $H(\mathcal{A}, \rho) \geq \tilde{S}(\mathcal{A}, \rho)$.

Proposition. A quantum entropy is monotonic iff, for any multipartite state, it yields an outer measure via marginalization. (This is localization.)

We cannot ask for a measure due to information redundancy.