

Quantizing discrete structures

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This talk describes a quantization scheme for classes of discrete structures.

For each class of discrete structures, quantization is an injective function

$$\begin{array}{c} \text{(class of discrete structures)}/\text{isomorphism} \\ \downarrow \\ \text{(class of equipped operator algebras)}/\text{isomorphism} \end{array}$$

whose range consists of exactly the commutative algebras.

This scheme is prescriptive when classes are understood intentionally, i.e., as definitions. Equivalent definitions may yield distinct quantizations.

In practice, the choice of a suitable definition is straightforward.

Talk outline.

1. basic principles underlying the quantization scheme,
2. the quantization of binary relations,
3. the quantization scheme in terms of the graphical language,
4. example: quantum graphs,
5. example: quantum maps,
6. example: quantum groups,
7. example: (quantum graph) homomorphisms,
8. example: quantum (graph isomorphism),
9. example: quantum metrics.

Mathematicians can differ in

1. the structures that they study,
2. the questions that they ask.

The primary object of study is the source of motivating questions.

Three approaches to a mathematical subject:

1. **Pure.** Structures are the primary objects of study.
2. **Applied.** Another subject is the primary object of study.
3. **Foundational.** The subject itself is the primary object of study.

In this sense, “foundational” does not mean “pioneering” or “influential.”

Examples of foundational work in noncommutative geometry:

1. Pedersen's Borel algebras,
2. Woronowicz's pseudospaces,
3. Weaver's quantum relations.

This work shed more light on the nature of the noncommutative metaphor than it did on the structure of C^* -algebras, von Neumann algebras, etc.

Views on the noncommutative metaphor range between

1. a useful but flawed heuristic for investigating operator algebras,
2. a robust mathematical structure that is yet to be fully understood.

Does the noncommutative metaphor have mathematical reality?

Previously, the evidence was mixed. Now, the evidence points to “yes.”

In the case of discrete structures, the noncommutative metaphor can already be formalized as a single mathematical structure.

Almost all known examples fit into this formalized framework.

There are a couple of loose ends:

1. different notions of quantum metric structure,
2. different notions of counting measure, e.g., δ -forms.

This talk covers this formalized framework, albeit informally.

Discrete structures are also known as first-order structures.

The basic ingredients are sets and relations. We quantize these ingredients.

Intuition. A quantum set is a discrete quantum space.

Podleś and Woronowicz defined discrete quantum spaces in 1990.

Let us approach the quantization of sets *a priori*.

Proposition. Each set X is determined by the C^* -algebra $\ell^\infty(X)$.

Intuition. A quantum set \mathbb{X} should be determined by a C^* -algebra $\ell^\infty(\mathbb{X})$.

Proposition. For a C^* -algebra \mathcal{M} , the following are equivalent:

1. every masa of \mathcal{M} is isomorphic to $\ell^\infty(X)$ for some set X ,
2. $\mathcal{M} \cong \bigoplus_{i \in I}^{\ell^\infty} M_{n_i}(\mathbb{C})$ for some indexed family $(n_i \in \mathbb{Z}_+ \mid i \in I)$.

Proof. For the forward direction, we apply the theory of AW^* -algebras.

1. \mathcal{M} is an AW^* -algebra (Saitô, Wright).
2. \mathcal{M} is type I.
3. \mathcal{M} is a von Neumann algebra (Kaplansky).
4. \mathcal{M} is an ℓ^∞ -direct sum of type I factors.
5. The factors of \mathcal{M} are finite.

Theorem (K; Chirvasitu). For a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, t.f.a.e.:

1. $\mathcal{M} \cong \bigoplus_{i \in I}^{\ell^\infty} M_{n_i}(\mathbb{C})$ for some indexed family $(n_i \in \mathbb{Z}_+ \mid i \in I)$,
2. every masa is isomorphic to $\ell^\infty(X)$ for some set X ,
3. every von Neumann subalgebra $\mathcal{N} \subseteq \mathcal{M}$ is atomic,
4. every self-adjoint $a \in \mathcal{M}$ has an orthonormal basis of eigenvectors,
5. the set of unitary operators in \mathcal{M} is ultraweakly compact.

Definition (K). In this case, $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is **hereditarily atomic**.

Proposition. The commutative hereditarily atomic von Neumann algebras are exactly those isomorphic to $\ell^\infty(X)$ for some set X .

Basic principles.

1. Quantum sets \mathbb{X} correspond to her. atom. von Neumann algebras \mathcal{M} .

$$\mathcal{M} \cong \ell^\infty(\mathbb{X})$$

2. Their Cartesian product corresponds to the spatial tensor product.

$$\ell^\infty(\mathbb{X} \times \mathbb{Y}) = \ell^\infty(\mathbb{X}) \otimes \ell^\infty(\mathbb{Y})$$

3. A set X is quantized to both a quantum set \mathbb{X} and its opposite \mathbb{X}° .

$$\ell^\infty(\mathbb{X}^\circ) = \ell^\infty(\mathbb{X})^\circ$$

These are basic principles and not axioms in the modern sense.

These principles explicitly quantize sets. They suffice to quantize relations.

Definition (Weaver). A **quantum relation** is a projection

$$r \in \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n.$$

Proposition. $\text{Pr}(\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n)$ is a complete modular ortholattice.

$$r \perp s \iff rs = 0$$

Definition. For $p \in \text{Pr}(\mathcal{M}_k)$, we write

$$p_k = \underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes p \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-k}.$$

Similarly, we use $r_{kl} \in \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ for $r \in \mathcal{M}_k \otimes \mathcal{M}_l$.

Every feature of first-order logic can now be quantized:

1. connectives (Birkhoff, von Neumann),
2. quantifiers (Weaver),
3. equality (K).

Hence, we can quantize any class of first-order structures.

First-order structures may also be called **discrete structures**.

To keep the language more familiar, we quantize **Rel** instead.

The dagger compact structure of **Rel** is equivalent to first-order logic.

We now define the dagger compact category **Rel** and then quantize it.

Definition. The symmetric monoidal category **Rel**:

1. an object is a set,
2. a morphism from X to Y is a subset of $X \times Y$,
3. the composition of morphisms $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is

$$S \circ R = \{(x, z) \mid \exists y, (x, y) \in R, (y, z) \in S\} \subseteq X \times Z,$$

4. the identity morphism on X is $\Delta_X = \{(x, x) \mid x \in X\}$,
5. the monoidal product of X and Y is $X \boxtimes Y = X \times Y$,
6. the monoidal unit is $\{*\}$,
7. the monoidal product of $R \subseteq X_1 \times X_3$ and $S \subseteq X_2 \times X_4$ is

$$R \boxtimes S = R_{13} \cap S_{24} \subseteq (X_1 \times X_2) \times (X_3 \times X_4).$$

Definition. The dagger structure on **Rel**: for a morphism $R \subseteq X \times Y$,

$$R^\dagger = \{(y, x) \mid (x, y) \in R\} \subseteq Y \times X.$$

Definition. Dual objects in **Rel**:

1. the dual of an object X is X ,
2. the unit of the duality is the morphism $\{*\} \times \Delta_X \subseteq \{*\} \times (X \times X)$,
3. the counit of the duality is the morphism $\Delta_X \times \{*\} \subseteq (X \times X) \times \{*\}$.

Proposition. **Rel** is enriched over ortholattices with

$$R \perp S \iff R \cap S = \emptyset,$$

and this structure can be defined in terms of the dagger compact structure.

Which quantum relation is the diagonal?

Proposition. 0 is the only projection $r \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ such that

$$r \perp p \otimes q \quad \text{whenever} \quad p \perp q \in \text{Pr}(M_2(\mathbb{C})).$$

Definition (K). $\delta_{\mathcal{M}}$ is the largest projection in $\mathcal{M}^{\circ} \otimes \mathcal{M}$ such that

$$\delta_{\mathcal{M}} \perp p^{\circ} \otimes q \quad \text{whenever} \quad p \perp q \in \text{Pr}(\mathcal{M}).$$

Proposition. $0_{\mathcal{M}}$ is the only projection $p \in \mathcal{M}$ such that $p^{\circ} \otimes 1_{\mathcal{M}} \perp \delta_{\mathcal{M}}$.

Intuition. $\delta_{\mathcal{M}}$ is the diagonal quantum relation in $\mathcal{M}^{\circ} \otimes \mathcal{M}$.

Definition. A quantum relation **from** \mathcal{M} **to** \mathcal{N} is a projection in $\mathcal{M}^{\circ} \otimes \mathcal{N}$.

Let $r \in \mathcal{M}^\circ \otimes \mathcal{N}$ and $s \in \mathcal{N}^\circ \otimes \mathcal{O}$ be quantum relations.

Definition. $s \circ r$ is the smallest projection in $\mathcal{M}^\circ \otimes \mathcal{O}$ such that

$$(1 - s \circ r)_{14} \wedge (\delta_{\mathcal{N}})_{32} \perp r \otimes s.$$

Proposition. When $\mathcal{M} = \ell^\infty(X)$, $\mathcal{N} = \ell^\infty(Y)$, and $\mathcal{O} = \ell^\infty(Z)$, then

$$\text{supp}(s \circ r) = \text{supp}(s) \circ \text{supp}(r).$$

Theorem (Weaver). For $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$, there is a o.t.o.c.::

1. projections $r \in \mathcal{M}^\circ \otimes \mathcal{N}$,
2. ultraweakly closed subspaces $\mathcal{V} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{N}'\mathcal{V}\mathcal{M}' = \mathcal{V}$.

Furthermore, $s \circ r$ corresponds to $\mathcal{W}\mathcal{V} = \overline{\text{span}\{wv \mid v \in \mathcal{V}, w \in \mathcal{W}\}}$.

Definition. The symmetric monoidal category **qRel**:

1. an object is a hereditarily atomic von Neumann algebra \mathcal{M} ,
2. a morphism from \mathcal{M} to \mathcal{N} is a projection in $\mathcal{M}^\circ \otimes \mathcal{N}$,
3. the composition of morphisms $r \in \mathcal{M}^\circ \otimes \mathcal{N}$ and $s \in \mathcal{N}^\circ \otimes \mathcal{O}$ is

$$s \circ r = 1 - \sup\{p \in \text{Pr}(\mathcal{M}^\circ \otimes \mathcal{O}) \mid p_{14}\delta_{32} \perp r_{12}s_{34}\} \in \mathcal{M}^\circ \otimes \mathcal{O},$$

4. the identity morphism on \mathcal{M} is $\delta_{\mathcal{M}} \in \mathcal{M}^\circ \otimes \mathcal{M}$,
5. the monoidal product of \mathcal{M} and \mathcal{N} is $\mathcal{M} \boxtimes \mathcal{N} = \mathcal{M} \otimes \mathcal{N}$,
6. the monoidal unit is \mathbb{C} ,
7. the monoidal product of morphs. $r \in \mathcal{M}_1^\circ \otimes \mathcal{M}_3$ and $s \in \mathcal{M}_2^\circ \otimes \mathcal{M}_4$ is

$$r \boxtimes s = r_{13} \wedge s_{24} \in (\mathcal{M}_1 \otimes \mathcal{M}_2)^\circ \otimes (\mathcal{M}_3 \otimes \mathcal{M}_4).$$

Definition. The dagger structure on **qRel**: for a morphism $r \in \mathcal{M}^\circ \otimes \mathcal{N}$,

$$r^\dagger = \sigma(r)^\circ \in \mathcal{N}^\circ \otimes \mathcal{M},$$

where $\sigma: \mathcal{M}^\circ \otimes \mathcal{N} \rightarrow \mathcal{N} \otimes \mathcal{M}^\circ$ is the flip map.

Definition. Dual objects in **qRel**:

1. the dual of an object \mathcal{M} is \mathcal{M}° ,
2. the unit of the duality is the morphism $1 \otimes \delta_{\mathcal{M}} \in \mathbb{C}^\circ \otimes (\mathcal{M}^\circ \otimes \mathcal{M})$,
3. the counit of the duality is the morphism $\delta_{\mathcal{M}} \otimes 1 \in (\mathcal{M} \otimes \mathcal{M}^\circ)^\circ \otimes \mathbb{C}$.

Theorem (K). **qRel** is a dagger compact category.

Proposition. **qRel** is enriched over complete ortholattices.

Proposition. There is a full and faithful dagger compact functor

$$\mathbf{Rel} \longrightarrow \mathbf{qRel}$$

that maps

1. each set X to $\ell^\infty(X)$,
2. each relation $R \subseteq X \times Y$ to $[R] \in \ell^\infty(X \times Y) \cong \ell^\infty(X) \otimes \ell^\infty(Y)$.

Proposition. It preserves all ortholattice structure too.

Proposition. Its essential range consists of the commutative algebras.

Intuition. This is the “inclusion” functor of **Rel** into **qRel**.

Quantization scheme.

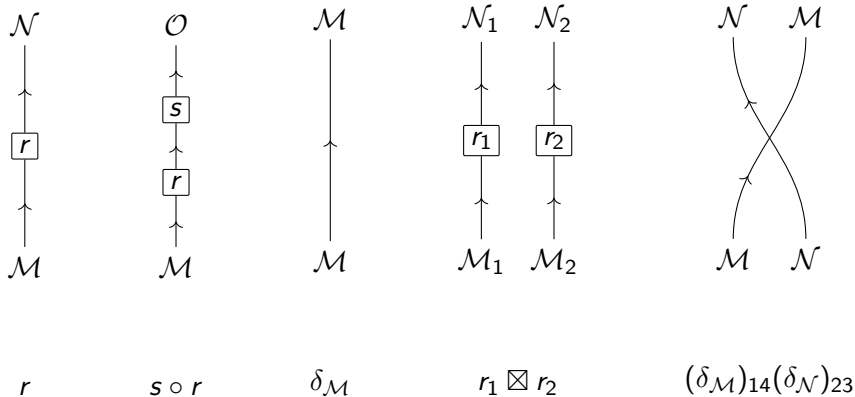
1. Start with a class of discrete structures.
2. Choose a definition of that class in **Rel**.
3. Reinterpret that definition in **qRel**.
4. Obtain a class of equipped operator algebras.

The definition is formulated in terms of the mentioned structure of **Rel**:

1. it is a dagger compact category,
2. it is enriched over complete ortholattices.

Proposition. In every case, we obtain a quantization of the starting class.

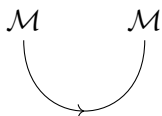
The graphical calculus for morphisms in a symmetric monoidal category:



The graphical representation of dual objects:



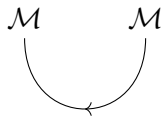
$$\delta_{\mathcal{M}}^{\circ}$$



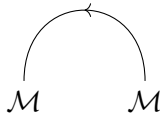
$$1 \otimes \delta_{\mathcal{M}}$$



$$\delta_{\mathcal{M}} \otimes 1$$



$$1 \otimes \delta_{\mathcal{M}}^{\circ}$$



$$\delta_{\mathcal{M}}^{\circ} \otimes 1$$

The graphical representation of maximum morphisms:



$$1 \otimes 1_{\mathcal{M}}$$

$$1 \otimes 1_{\mathcal{M}}^{\circ}$$

$$1_{\mathcal{M}}^{\circ} \otimes 1$$

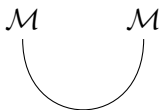
$$1 \otimes 1_{\mathcal{M}}$$

$$1_{\mathcal{M}}^{\circ} \otimes 1_{\mathcal{N}}$$

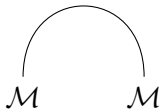
The graphical representation of self-dual objects ($\mathcal{M}^\circ = \mathcal{M}$):



$\delta_{\mathcal{M}}$



$1 \otimes \delta_{\mathcal{M}}$



$\delta_{\mathcal{M}} \otimes 1$

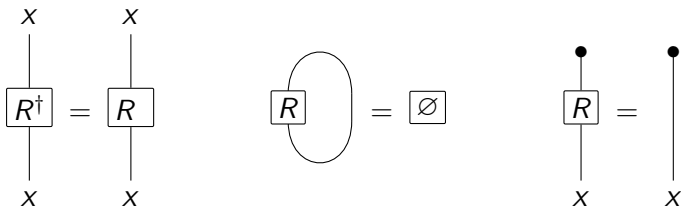


$1 \otimes 1_{\mathcal{M}}$

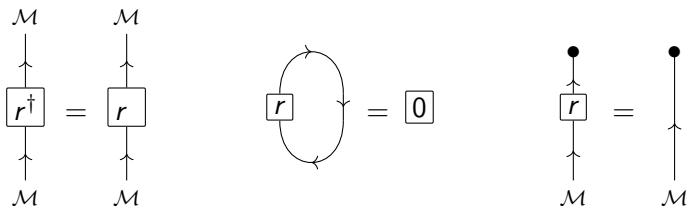


$1_{\mathcal{M}} \otimes 1$

Example. Consider the following conditions on a relation $R \subseteq X \times X$:



1. The left condition expresses that R is symmetric.
2. The middle condition expresses that R is irreflexive.
3. The right condition expresses that R is entire.

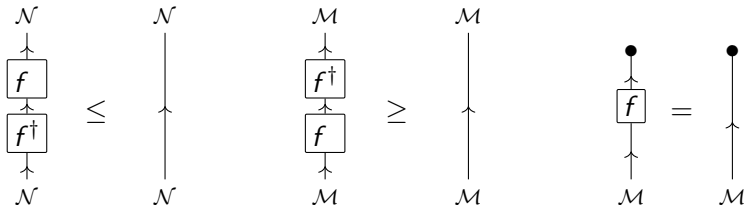


Definition. A **simple quantum graph** is

1. a hereditarily atomic von Neumann algebra \mathcal{M} ,
2. a projection $r \in \mathcal{M}^\circ \otimes \mathcal{M}$

such that the left and middle conditions hold.

Definition. It has **no isolated vertices** if the right condition holds.



Definition (Freyd, Scedrov). For a quantum relation $f \in \mathcal{M}^\circ \otimes \mathcal{N}$,

1. f is a **partial quantum map** if the left condition holds,
2. f is a **quantum map** if also the middle condition holds.

Proposition. For a partial quantum map $f \in \mathcal{M}^\circ \otimes \mathcal{N}$, t.f.a.e.:

1. the middle condition,
2. the right condition.

Theorem (K). There is a one-to-one correspondence between

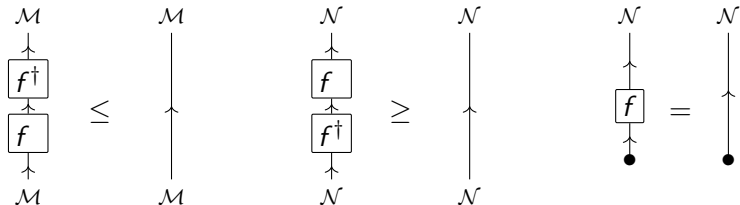
1. partial quantum maps $f \in \mathcal{M}^\circ \otimes \mathcal{N}$,
2. normal $*$ -homomorphisms $\phi: \mathcal{N} \rightarrow \mathcal{M}$

that restricts to a one-to-one correspondence between

1. quantum maps $f \in \mathcal{M}^\circ \otimes \mathcal{N}$,
2. unital normal $*$ -homomorphisms $\phi: \mathcal{N} \rightarrow \mathcal{M}$.

Furthermore, for all projections $p \in \mathcal{N} = \mathbb{C}^\circ \otimes \mathcal{N}$,

$$\begin{array}{c} \boxed{p^\dagger} \\ \uparrow \\ \boxed{f} \\ \uparrow \\ \mathcal{M} \end{array} = \begin{array}{c} \boxed{\phi(p)^\dagger} \\ \uparrow \\ \mathcal{M} \end{array}$$



Definition. For a quantum map $f \in \mathcal{M}^\circ \otimes \mathcal{N}$,

1. f is **injective** if the left condition holds,
2. f is **surjective** if the middle condition holds.

Proposition. For a quantum map $f \in \mathcal{M}^\circ \otimes \mathcal{N}$, t.f.a.e.:

1. the middle condition,
2. the right condition.

Theorem (K). The one-to-one correspondence between

1. quantum maps $f \in \mathcal{M}^\circ \otimes \mathcal{N}$,
2. unital normal $*$ -homomorphisms $\phi: \mathcal{N} \rightarrow \mathcal{M}$

restricts to a one-to-one correspondence between

1. injective quantum maps $f \in \mathcal{M}^\circ \otimes \mathcal{N}$,
2. surjective unital normal $*$ -homomorphisms $\phi: \mathcal{N} \rightarrow \mathcal{M}$

and to a one-to-one correspondence between

1. surjective quantum maps $f \in \mathcal{M}^\circ \otimes \mathcal{N}$,
2. injective unital normal $*$ -homomorphisms $\phi: \mathcal{N} \rightarrow \mathcal{M}$.

Let $r \in \mathcal{M}^\circ \otimes \mathcal{N}$ be an arbitrary quantum relation.

$$\begin{array}{c} \mathcal{M} \\ \uparrow \\ \boxed{r^\dagger} \\ \uparrow \\ \boxed{r} \\ \uparrow \\ \mathcal{M} \end{array} \geq \begin{array}{c} \mathcal{M} \\ \uparrow \\ \mathcal{M} \end{array} \qquad \begin{array}{c} \bullet \\ \uparrow \\ \boxed{r} \\ \uparrow \\ \mathcal{M} \end{array} = \begin{array}{c} \bullet \\ \uparrow \\ \mathcal{M} \end{array}$$

For commutative \mathcal{M} and \mathcal{N} , these two conditions are equivalent.

In general, these conditions are **not** equivalent.

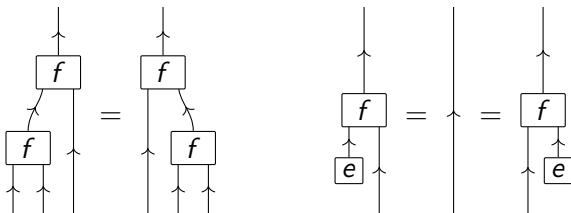
We will see that this phenomenon distinguishes

1. discrete quantum groups of Kac type,
2. discrete quantum groups.

Definition. A **discrete quantum monoid** is a triple (\mathcal{M}, f, e) with

$$f \in (\mathcal{M} \otimes \mathcal{M})^\circ \otimes \mathcal{M} \quad e \in \mathbb{C}^\circ \otimes \mathcal{M}$$

being quantum maps such that

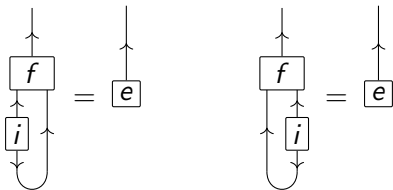


(The wires are implicitly labeled \mathcal{M} .)

Definition. A discrete quantum monoid **has inverses** if



Definition. A discrete quantum monoid **has an inversion map** i if



Theorem (Vaes). There is a one-to-one correspondence:

1. discrete quantum monoids with inverses,
2. discrete quantum groups.

Theorem (Chirvasitu, K). There is a one-to-one correspondence:

1. discrete quantum monoids with an inversion map,
2. discrete quantum groups of Kac type.

Definition. A discrete quantum group of **Kac type** is the Pontryagin dual of a compact quantum group of Kac type (tracial Haar state).

Definition. A **discrete quantum graph** is a pair (\mathcal{M}, r) with

$$r \in \mathcal{M}^\circ \otimes \mathcal{M}$$

being a quantum relation such that $r^\dagger = r$.

Definition. A **quantum homomorphism** $(\mathcal{M}, r) \rightarrow (\mathcal{N}, s)$ is a quantum map $f \in \mathcal{M}^\circ \otimes \mathcal{N}$ with

$$f \circ r \leq s \circ f.$$

Definition. The **box product** of discrete quantum graphs is defined by

$$(\mathcal{M}, r) \square (\mathcal{N}, s) = (\mathcal{M} \otimes \mathcal{N}, r \boxtimes \delta_{\mathcal{N}} \vee \delta_{\mathcal{M}} \boxtimes s).$$

Definition. Overall, we obtain a symmetric monoidal category **qGph**.

Proposition. **Gph** is symmetric monoidal closed:

$$\begin{array}{ccc}
 Z \square X & & \\
 \vdots \square \text{id}_X & \searrow f & \\
 [X, Y] \square X & \xrightarrow{\text{eval}} & Y
 \end{array}$$

Proposition. $[X, Y]$ is the graph of all homomorphisms $X \rightarrow Y$.

Proposition. For finite simple graphs X and Y , t.f.a.e.:

1. $[X, Y]$ is nonempty,
2. the (X, Y) -homomorphism game has a winning strategy.

Theorem (K, Lindenhovius). **qGph** is symmetric monoidal closed:

$$\begin{array}{ccc}
 \mathcal{O} \square \mathcal{M} & & \\
 \downarrow \text{!} \square \text{id}_{\mathcal{M}} & \searrow f & \\
 [\mathcal{M}, \mathcal{N}]_q \square \mathcal{M} & \xrightarrow{\text{eval}} & \mathcal{N}
 \end{array}$$

Intuition. $[\mathcal{M}, \mathcal{N}]_q$ consists of all quantum homomorphisms $\mathcal{M} \rightarrow \mathcal{N}$.

Corollary. For finite simple graphs X and Y , t.f.a.e.:

1. $[l^\infty(X), l^\infty(Y)]_q$ is nonempty, i.e., the algebra is nonzero,
2. the (X, Y) -homomorphism game has a quantum winning strategy.

Definition (Mančinska, Roberson). Let X and Y be finite simple graphs. The (X, Y) -**homomorphism game** is played by two players who are cooperating against an adverse environment. The play proceeds as follows:

1. the players agree on a strategy,
2. the players are separated,
3. each player encounters a random vertex of X ,
4. each player responds with a vertex of Y ,
5. the players win unless one of the following happens:
 - (a) $x_1 \sim x_2$ but $y_1 \not\sim y_2$,
 - (b) $x_1 = x_2$ but $y_1 \neq y_2$.

Theorem (Mančinska, Roberson). There exist finite simple graphs

1. X on 14 vertices,
2. Y on 4 vertices

such that the (X, Y) -homomorphism game

1. has no winning strategy,
2. has a quantum winning strategy.

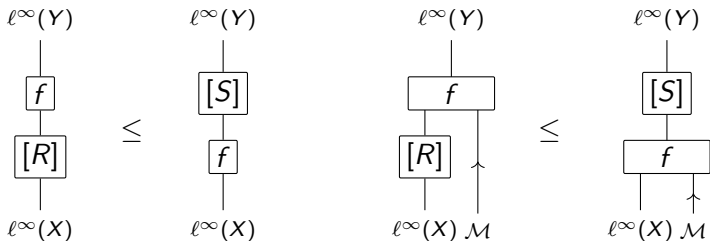
Proposition. A pair of finite simple graphs (X, Y) is such iff

1. $[X, Y]$ is empty,
2. $[\ell^\infty(X), \ell^\infty(Y)]_q$ is nonempty.

Definition (Atserias, Mančinska, Roberson, Šámal, Severini, Varvitsiotis).
The (X, Y) -**isomorphism game** is played by two players who are cooperating against an adverse environment. The play proceeds as follows:

1. the players agree on a strategy,
2. the players are separated,
3. each player encounters a random vertex of X or of Y ,
4. each player responds with a vertex of X or of Y ,
5. the players win unless their responses are *prima facie* contrary to playing according to an isomorphism $X \leftrightarrow Y$.

Let R and S be the adjacency relations of X and Y , respectively.



A homomorphism $X \rightarrow Y$ is the same thing as a quantum homomorphism $(l^\infty(X), [R]) \rightarrow (l^\infty(Y), [S])$, i.e., a quantum map $f \in l^\infty(X)^\circ \otimes l^\infty(Y)$ such that the left condition holds.

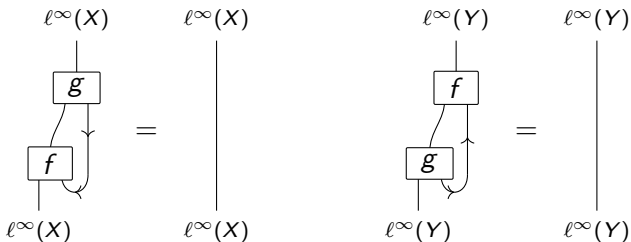
Definition. A **quantum family of homomorphisms** $X \rightarrow Y$ is a quantum map $f \in (l^\infty(X) \otimes \mathcal{M})^\circ \otimes l^\infty(Y)$ such that the right condition holds.

Theorem (Atserias, Mančinska, Roberson, Šámal, Severini, Varvitsiotis; Musto, Reutter, Verdon). For all finite simple graphs X and Y , t.f.a.e.:

1. there exists a quantum winning strategy for the (X, Y) -iso. game,
2. there exist quantum families of homomorphisms

$$f \in (\ell^\infty(X) \otimes \mathcal{M})^\circ \otimes \ell^\infty(Y) \quad g \in (\ell^\infty(Y) \otimes \mathcal{M}^\circ)^\circ \otimes \ell^\infty(X)$$

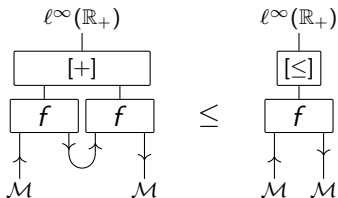
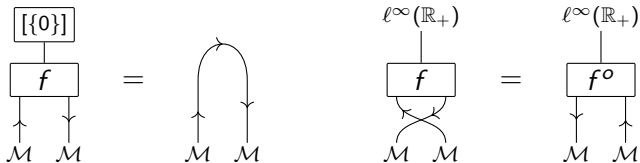
such that $\mathcal{M} \neq \{0\}$ and



Definition. A **quantum metric** on \mathcal{M} is a quantum map

$$f \in (\mathcal{M} \otimes \mathcal{M}^\circ)^\circ \otimes \ell^\infty(\mathbb{R}_+)$$

such that



Theorem (K). There is a one-to-one correspondence:

1. quantum metrics on \mathcal{M} ,
2. quantum metrics on \mathcal{M} in the sense of Kuperberg and Weaver.

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra.

Theorem (Kuperberg, Weaver). A quantum metric on \mathcal{M} is a family

$$(\mathcal{V}_t \subseteq \mathcal{B}(\mathcal{H}) \mid t \in [0, \infty))$$

of ultraweakly closed subspaces such that $\mathcal{V}_t = \bigcap_{s>t} \mathcal{V}_s$ and such that

1. $\mathcal{V}_0 = \mathcal{M}'$,
2. $\mathcal{V}_t^* = \mathcal{V}_t$,
3. $\mathcal{V}_s \mathcal{V}_t \subseteq \mathcal{V}_{s+t}$.

Definition. The symmetric monoidal category **qRel**:

1. an object is a hereditarily atomic von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$,
2. a morphism from $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ to $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ is a closed subspace

$$\mathcal{R} \subseteq \mathcal{B}_2(\mathcal{H}, \mathcal{K})$$

such that $\mathcal{N}'\mathcal{R}\mathcal{M}' = \mathcal{R}$,

3. the composition of \mathcal{R} and \mathcal{S} is $\mathcal{S}\mathcal{R} = \overline{\text{span}\{sr \mid r \in \mathcal{R}, s \in \mathcal{S}\}}$,
4. the identity morphism on $\mathcal{M} \subseteq \mathcal{B}_2(\mathcal{H})$ is $\mathcal{M}' \cap \mathcal{B}_2(\mathcal{H})$,
5. the monoidal product of \mathcal{M} and \mathcal{N} is $\mathcal{M} \otimes \mathcal{N}$ (ultraweakly closed),
6. the monoidal unit is $\mathcal{B}(\mathbb{C})$,
7. the monoidal product of \mathcal{R} and \mathcal{S} is $\mathcal{R} \otimes \mathcal{S}$ (norm closed).

Definition. The complete ortholattice enrichment of \mathbf{qRel} :

$$\mathcal{R} \perp \mathcal{S} \iff \text{tr}(\mathcal{R}^\dagger \mathcal{S}) = \{0\}.$$

Definition. The dagger structure on \mathbf{qRel} : for a morphism \mathcal{R} ,

$$\mathcal{R}^\dagger = \mathcal{R}^* = \{r^* \mid r \in \mathcal{R}\}.$$

Definition. Dual objects in \mathbf{qRel} :

1. the dual of an object $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is $\mathcal{M}^\circ \subseteq \mathcal{B}(\mathcal{H}^*)$,
2. unit $D_{\mathcal{M}}$ is the subspace of $\mathcal{B}_2(\mathbb{C}, \mathcal{H} \otimes \mathcal{H}^*)$ with range $\mathcal{M}' \cap \mathcal{B}_2(\mathcal{H})$,
3. the counit is $\mathcal{D}_{\mathcal{M}^\circ}^\dagger$.

(This definition reverses left and right.)

Theses. (Least controversial to most controversial.)

1. In the case of discrete structures, the noncommutative metaphor has been fully formalized.
2. This formalization is entirely canonical.
3. The noncommutative metaphor as a whole is a mathematical structure that has not yet been fully formalized or understood.
4. Formalizing and understanding the noncommutative metaphor is important for the long-term future of noncommutative geometry.
5. Noncommutative geometry is currently accumulating a conceptual debt that will need to be paid eventually.