

The category of quantum graphs is closed

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Quantum sets model “discrete” quantum systems.

Definition. A **quantum set** is an indexed family

$$X = (X_\alpha \mid \alpha \in I)$$

of nonzero finite-dimensional Hilbert spaces.

We refer to the indices $\alpha \in I$ as the **atoms** of X , and write $\text{At}(X) := I$.

The atoms of X model the superselection sectors of the quantum system.

Example. We identify each n.z.f.d. Hilbert space H with the quantum set

$$H = (H \mid \alpha \in \{*\}).$$

Example. We identify each set A with the quantum set $A = (\mathbb{C} \mid \alpha \in A)$.

Definition. The **Cartesian product** of quantum sets X and Y is

$$X \times Y = (X_\alpha \otimes Y_\beta \mid (\alpha, \beta) \in \text{At}(X) \times \text{At}(Y)).$$

Quantum sets can be used to obtain quantum generalizations:

1. **Stoch** $\hookrightarrow$ **qStoch**,
2. **Rel** $\hookrightarrow$ **qRel**,
3. **Gph** $\hookrightarrow$ **qGph**.

In each case, we have a full and faithful symmetric monoidal functor.

In each case, we think of the functor as an “inclusion functor.”

In each case, a set is mapped to the associated quantum set.

These quantum generalizations are from noncommutative geometry.

Definition. We define the symm. mon. categories **Stoch** and **Rel**:

1. in both categories, the objects are sets,
2. in both categories, the monoidal product is the Cartesian product,
- 3a. in **Stoch**, a morphism $\mathcal{W}: A \rightarrow B$ is a stochastic map,

$$\mathcal{W}(\beta|\alpha) \in [0, 1], \quad \sum_{\beta \in B} \mathcal{W}(\beta|\alpha) = 1,$$

- 3b. in **Rel**, a morphism $R: A \rightarrow B$ is a relation $R \subseteq A \times B$.

Stoch models discrete classical systems and total probabilistic channels.

Rel models discrete classical systems and partial possibilistic channels.

Proposition. We have the support functor **Stoch** $\rightarrow$ **Rel**.

Definition. We define **qStoch**:

1. objects are quantum sets,
2. the monoidal product is the Cartesian product,
3. a morphism $\mathcal{N}: X \rightarrow Y$ is a trace-preserving completely positive map

$$\mathcal{N}: \ell^1(X) \rightarrow \ell^1(Y), \quad \ell^1(X) = \bigoplus_{\alpha \in \text{At}(X)}^{\ell^1} L(X_\alpha).$$

qStoch models discrete quantum systems and total probabilistic channels.

Definition. The **inclusion functor** $\mathbf{Stoch} \hookrightarrow \mathbf{qStoch}$ is defined by

$$\mathcal{W}: A \rightarrow B \quad \mapsto \quad \mathcal{N}_{\mathcal{W}}: \ell^1(A) \rightarrow \ell^1(B), \quad \mathcal{N}_{\mathcal{W}}(\rho)(\beta) = \sum_{\alpha \in A} \mathcal{W}(\beta|\alpha) \rho(\alpha).$$

Definition (Weaver; K). We define **qRel**:

1. objects are quantum sets,
2. the monoidal product is the Cartesian product,
3. a morphism $K: X \rightarrow Y$ is an indexed family of subspaces

$$K = (K_{\alpha,\beta} \subseteq L(X_\alpha, Y_\beta) \mid \alpha \in \text{At}(X), \beta \in \text{At}(Y)).$$

qRel models discrete quantum systems and partial possibilistic channels.

Definition. The **inclusion functor Rel** $\hookrightarrow$ **qRel** is defined by

$$R: A \rightarrow B \quad \mapsto \quad K_R = ((K_R)_{\alpha,\beta}), \quad (K_R)_{\alpha,\beta} = \begin{cases} L(\mathbb{C}) & (\alpha, \beta) \in R, \\ \{0_{\mathbb{C}}\} & (\alpha, \beta) \notin R. \end{cases}$$

Proposition. We have the **support functor** $\mathbf{qStoch} \rightarrow \mathbf{qRel}$,

$$\mathcal{N} : \ell^1(X) \rightarrow \ell^1(Y) \quad \mapsto \quad K_{\mathcal{N}} = ((K_{\mathcal{N}})_{\alpha,\beta}),$$

where $(K_{\mathcal{N}})_{\alpha,\beta} \subseteq L(X_{\alpha}, Y_{\beta})$ is the span of the Kraus operators of

$$\mathcal{N}| : L(X_{\alpha}) \rightarrow L(Y_{\beta}).$$

Altogether, we have a commutative square:

$$\begin{array}{ccc} \mathbf{Stoch} & \xrightarrow{\text{Supp}} & \mathbf{Rel} \\ \downarrow & & \downarrow \\ \mathbf{qStoch} & \xrightarrow{\text{Supp}} & \mathbf{qRel} \end{array}$$

Theorem. **Rel** and **qRel** are, furthermore,

1. compact closed categories,
2. dagger categories,
3. enriched over complete ortholattices and $\vee$ -homomorphisms.

This structure is all compatible, and **Rel** $\hookrightarrow$ **qRel** preserves it.

Definition. We define graphs and quantum graphs.

1. A **graph** is a pair (V_G, E_G) such that V_G is an object of **Rel** and $E_G: V_G \rightarrow V_G$ is a morphism of **Rel** with $E_G^\dagger = E_G$.
2. A **quantum graph** is a pair (V_G, E_G) such that V_G is an object of **qRel** and $E_G: V_G \rightarrow V_G$ is a morphism of **qRel** with $E_G^\dagger = E_G$.

Every graph is a quantum graph via the inclusion functor **Rel** $\hookrightarrow$ **qRel**.

Let $\mathcal{N}: X \rightarrow Y$ be a **quantum channel**, i.e., a morphism of **qStoch**.

Definition. The **confusability quantum graph** G of $\mathcal{N}$ has $V_G = X$ and

$$K_{\rho_1}^\dagger \circ E_G \circ K_{\rho_2} = 0 \quad \text{iff} \quad \text{tr}(\mathcal{N}(\rho_1)\mathcal{N}(\rho_2)) = 0$$

for all density operators $\rho \in \ell^1(X)$, i.e., all $\rho: \{*\} \rightarrow X$ in **qStoch**.

The possibilistic $K_{\mathcal{N}}: X \rightarrow Y$ is the corresponding morphism in **qRel**.

Proposition. The confusability quantum graph of $\mathcal{N}$ is $(X, K_{\mathcal{N}}^\dagger K_{\mathcal{N}})$.

Quantum graphs were introduced by Duan, Severini and Winter.

Theorem (Duan; Verdon). Every countable quantum graph G with $\text{id}_{V_G} \leq E_G$ is the confusability quantum graph of a quantum channel.

Let G and H be quantum graphs.

Definition (Musto, Reutter, Verdon; K, Lindenhovius). A **homomorphism** $\phi: G \rightarrow H$ is a morphism $V_G \rightarrow V_H$ in **qRel** such that

$$\phi^\dagger \circ \phi \geq \text{id}_{V_G}, \quad \phi \circ \phi^\dagger \leq \text{id}_{V_H}, \quad \phi \circ E_G \leq E_H \circ \phi.$$

When G and H are graphs, then $\phi: V_G \rightarrow V_H$ is a homomorphism iff

$$\phi \text{ is a map,} \quad (x_1, x_2) \in E_G \Rightarrow (\phi(x_1), \phi(x_2)) \in E_H.$$

Definition. The **box product** of G and H is the quantum graph

$$G \square H = (V_G \times V_H, E_G \times \text{id}_H \vee \text{id}_G \times E_H).$$

This defines the symm. mon. categories **Gph** $\hookrightarrow$ **qGph**.

Proposition. The symm. mon. category **Gph** is closed.

$$\begin{array}{ccc}
 G \square [G, H] & \xrightarrow{\varepsilon} & H \\
 \uparrow \text{id}_G \times ! & \nearrow \phi & \\
 G \square G' & &
 \end{array}$$

Theorem (K, Lindenhovius). The symm. mon. category **qGph** is closed.

When G and H are graphs, $[G, H]$ depends on the setting: **Gph** or **qGph**.

In **Gph**, $[G, H]$ is the graph of homomorphisms from G to H .

In **qGph**, $[G, H]$ is the **quantum graph of homomorphisms** from G to H .

This is comparable to classical vs. quantum automorphism groups.

The closure of **Gph** and **qGph** is related to graph homomorphism games. Graph homomorphism games demonstrate quantum nonlocality.

Definition (Mančinska, Roberson). Let G and H be finite simple graphs. The **(G, H) -homomorphism game** is played by two players who are cooperating against an adverse environment. The play proceeds as follows:

1. the players agree on a strategy,
2. the players are separated,
3. each player $i \in \{1, 2\}$ encounters a random vertex x_i of G ,
4. each player $i \in \{1, 2\}$ responds with a vertex y_i of H ,
5. the players win unless one of the following happens:
 - (a) $x_1 \sim x_2$ but $y_1 \not\sim y_2$,
 - (b) $x_1 = x_2$ but $y_1 \neq y_2$.

Proposition. For finite simple graphs G and H , t.f.a.e.:

1. the (G, H) -homomorphism game has a winning classical strategy,
2. $[G, H] \not\cong K_0$ in **Gph**.

Theorem (K, Lindenhovius). For finite simple graphs G and H , t.f.a.e.:

1. the (G, H) -homomorphism game has a winning quantum strategy,
2. $[G, H] \not\cong K_0$ in **qGph**.

Theorem (Mančinska, Roberson). There exist simple graphs G with 14 vertices and H with 4 vertices such that the (G, H) -homomorphism game

1. has no winning classical strategy,
2. has a winning quantum strategy.