Research Statement

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My primary field of research is commutative algebra. I am especially interested in problems in commutative algebra which can be translated into the language of combinatorics or algebraic geometry. One of the most useful techniques applied to connect commutative algebra to combinatorics is assigning a squarefree monomial ideal to a graph or a simplicial complex to make a dictionary between their algebraic and combinatorial properties.

A squarefree monomial is a product of distinct variables, for example $x_1x_3x_4$ in the polynomial ring $k[x_1, \ldots, x_4]$. A squarefree monomial ideal is an ideal generated by squarefree monomials. The central theme of my research is the study of squarefree monomial ideals from a combinatorial perspective.

Here are some specific areas which I am currently studying. A more detailed description for each follows after.

- **Path ideals:**

  path ideals are a generalization of edge ideals of graphs. I investigate the arithmetic rank of these ideals for cycles.

- **Rees algebra of squarefree monomial ideals:**

  The equations of Rees algebra of quadratic squarefree monomial ideals are known. The focus of my work is to find a combinatorial interpretation for equations of Rees algebras of squarefree monomial ideals of higher degrees.

- **j-multiplicity of edge ideals:**

  j-multiplicity is a generalization of Hilbert-Samuel multiplicity for ideals which are not primary. I am working on j-multiplicity of edge ideals of a graph. This is joint work with J. Validashti.

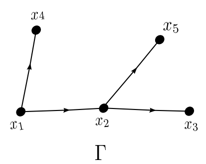
- **Regularity of powers of edge ideals:**

  In [3] with collaboration with A. Banerjee we investigated the asymptotic behavior of regularity of a class of bipartite graphs. Banerjee and Hà and I plan to do more with improving the asymptotic formula for the edge ideals of trees and cycles in [25] to a more general case.

Below I describe the principal themes and techniques for each of these projects.

1 Path Ideals

Path ideals were first introduced by Conca and De Negri in [12]. We define the **path ideal** of a graph G, denoted by $I_t(G)$ to be the ideal of R, generated by monomials of the form of $x_{i_1}x_{i_2}\dots x_{i_t}$ where $x_{i_1}, x_{i_2}, \dots, x_{i_t}$ is a path in G. The case $t = 2$ defines the edge ideal of a graph, first introduced by Villarreal in [27]. For the following tree we have



$$I_3(\Gamma) = (x_1x_2x_3, x_1x_2x_5, x_1x_2x_4).$$

Let I be an ideal in R, then the minimum number of elements of R that generate I up to radical is called the **arithmetic rank** of I. More precisely the arithmetic rank of I is

$$ara(I) = \min\left\{r : \exists a_1, \dots, a_r \in R; \sqrt{I} = \sqrt{(a_1, \dots, a_r)}\right\}.$$

Finding the arithmetic rank of squarefree monomial ideals is inspired by work of Lyubeznik [22] in the 1980's, who used Taylor's resolution of monomial ideals to study questions about cohomological dimensions.

The computation of the arithmetic rank of an ideal is a challenging problem. For this reason researchers are developing bounds for the arithmetic rank of classes of ideals. The simplest upper bound for the arithmetic rank of an ideal I is $\mu(I)$, the minimal number of generators of I. Also, by using Krull's Principal Ideal Theorem [29], one can find a lower bound; it is the height of the ideal I. Thus

$$ht(I) \le ara(I) \le \mu(I).$$

According to Lyubeznik's result in [22], one can find a less obvious lower bound for the arithmetic rank of squarefree monomial ideals. To state this lower bound we need to introduce the free resolution of an ideal.

For any homogeneous ideal I of the polynomial ring $R = K[x_1, \dots, x_n]$ there exists a **graded minimal finite free resolution**

$$0 \longrightarrow \bigoplus_d R(-d)^{\beta_{p,d}} \longrightarrow \cdots \longrightarrow \bigoplus_d R(-d)^{\beta_{1,d}} \to R \longrightarrow R/I \longrightarrow 0$$

of R/I in which $R(-d)$ denotes the graded free module obtained by shifting the degrees of elements in R by d.

The numbers $\beta_{i,d}$, which we shall refer to as the i-th $\mathbb{N}$-**graded Betti numbers** of degree d of R/I, are independent of the choice of graded minimal finite free resolution.

The length of the minimal free resolution is unique up to change of basis and it is defined as the **projective dimension** of R/I, and denoted by $pd(R/I)$. The following is a well known theorem by Lyubeznik.

Theorem 1.1 (Lyubeznik, [22])**.** *Let $I \subset R$ be a squarefree monomial ideal then $pd(R/I) \leq ara(I)$.*

So for a squarefree monomial ideal I we have the following

$$ht(I) \leq pd(R/I) \leq ara(I) \leq \mu(I).$$

In 2000, Zhao Yan [30] showed that the inequality in Theorem 1.1 is sharp in some cases. The following question arises immediately.

Question 1. For which classes of squarefree monomial ideals does $pd(R/I) = ara(I)$?

This is a broad question. Some classes of squarefree monomial ideals in which the equality holds have been investigated (see for example [7], [8], [9], [10], [11]).

The edge ideals of cycles have also been investigated. Barile, Kiani, Mohammadi and Yassemi [10] showed that

$$ara(I_2(C_n)) = pd(R/I_2(C_n)),$$

where C_n is a graph cycle with n vertices. Now we can ask what happens for $t \neq 2$? In my thesis [2] and in [4], I computed the projective dimension and all Betti numbers of the path ideals of a cycle.

Furthermore, in collaboration with S. Faridi, I proved the following result which can be found in [5].

Theorem 1.2 (**Projective dimension of path ideals of cycles**)**.** *Let n, t, p and d be integers such that $n \geq 2$, $2 \leq t \leq n$, $n = (t+1)p + d$, where $p \geq 0$, $0 \leq d \leq t$. Then the projective dimension of the path ideal of a graph cycle C_n is given by*

$$pd(R/I_t(C_n)) = \begin{cases} 2p+1 & d \neq 0 \\ 2p & d = 0. \end{cases}$$

1.1 Further Work

Regarding the arithmetic rank of path ideals of a cycle, Macchia [23] in his unpublished work showed the following.

Lemma 1.3. *Let C_n be a cycle graph over n vertices and let $2 \leq t \leq n$. Then we have*

$$ara(I_t(C_n)) \in \{pd(R/I_t(C_n)), pd(R/I_t(C_n)) + 1\}.$$

Since it is proved for $t = 2$ in [10] that $ara(I_2(C_n)) = pd(R/I_2(C_n))$, we have the following open question.

Question 2. Let C_n be a graph cycle over n vertices and $2 < t \leq n$. Is it true to say $ara(I_t(C_n)) = pd(R/I_t(C_n))$?

I have shown that for some values of t and n this equality holds. However, there are still lots of unexamined cases.

2 Rees Algebra of Squarefree Monomial Ideals

Let $I = (f_1, \ldots, f_q)$ be a monomial ideal in a polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k. We denote the **Rees algebra** of I by $R[It] = R[f_1t, \ldots, f_qt]$ and consider the homomorphism ψ of algebras

$$\psi : R[T_1, \ldots, T_q] \longrightarrow R[It], \ \ T_i \mapsto f_it$$

where T_i are indeterminate. If J is kernel of ψ, then $R[It] = S/J$ where $S = R[T_1, \ldots, T_q]$. The ideal J is called **defining ideal** of $R[It]$ and its minimal generators are called the **Rees equations** of I. The defining ideal J of $R[It]$ is graded. In other words, we have

$$J = J_1 \oplus J_2 \oplus \cdots$$

where J_s for $s \geq 1$ is R-modules. These equations carry a lot of information about $R[It]$; see for example [26], for more details.

An ideal I is said to be **of linear type** if $J = (J_1)$; in other words, the defining ideal of $R[It]$ is generated by linear combinations of variables $T_1, \ldots, T_q$.

Rees algebras and ideals of linear type have been investigated by many authors (see for example [12], [13], [16], [19],[20])

We can define $T_{\alpha,\beta}(I) \in J_s$ as a **redundant Rees equation of** I, if we have

$$T_{\alpha,\beta}(I) \in J_1S + \cdots + J_{s-1}S.$$

To identify ideals of linear type, it is enough to characterize redundant equations.

I am interested in giving a combinatorial interpretation of irredundant $T_{\alpha,\beta}(I)$ for a squarefree monomial ideal I. The simplest case of a squarefree monomial ideal is an edge ideal.

In 1995 Villarreal gave a combinatorial characterization of irredundant generators for edge ideals of graphs [28] by attributing irredundant generators of J_s to closed even walks. Motivated by this work I defined **simplicial closed even walks** in my thesis for simplicial complexes to generalize graph case.

Faridi and I proved in [6] that if $T_{\alpha,\beta}(I)$ is an irredundant generator of J_s, then the generators of I involved in $T_{\alpha,\beta}(I)$ form a simplicial even walk. Our class of simplicial even walks includes *even special cycles* (see [17]) as known in hypergraph theory.

By using the concept of the simplicial closed even walks, we can give a necessary condition for a squarefree monomial ideals of linear type [6].

Theorem 2.1. *Let I be a squarefree monomial ideal and suppose the facet complex $\mathcal{F}(I)$ has no simplicial closed even walk. Then I is of linear type.*

Also in my thesis I showed that every simplicial closed even walk contains a simplicial cycle (not necessarily of the same length).

By using this new result we can conclude every simplicial tree is of linear type. (Simplicial trees are generalization of trees in graph theory. For more details see [15].) This fact can also be deduced by the concept of M-sequences [12].

2.1 Further Work

Generally the converse of Theorem 2.1 is not correct. My goal is to further investigate the structure of simplicial closed even walk so that I can give a more effective criterion to replace Theorem 2.1.

Question 3. Find a more effective criterion for squarefree monomial ideals of linear type.

Fouli and Lin [16] used the line graphs of squarefree monomial ideals to give a simple test for ideals of linear type. In my thesis I improved their test as demonstrated in the following theorem.

Theorem 2.2. *Let I be a squarefree monomial ideal and Δ be its facet complex. If $L(\Delta)$ (line graph of Δ) contains no even cycle, then I is of linear type.*

One can ask what happens for line graphs which contain an even cycle. In general we can say nothing about linearity of I by using these line graphs. I am currently working on improving Theorem 2.2.

3 j-Multiplicity of Edge Ideals

Hilbert-Samuel multiplicity is one of the most important invariants in homological algebra, commutative algebra and algebraic geometry. This notation was defined in 1954 by D. Northcott and D. Rees [24]. The Hilbert-Samuel multiplicity is defined in a local ring for $\mathfrak{m}$-primary ideals where $\mathfrak{m}$ is the maximal ideal. In 1993 j-multiplicity was defined by R. Achilles and M. Manaresi [1] to extend Hilbert-Samuel multiplicity notation for an ideal that is not necessarily $\mathfrak{m}$-primary.

We are ready to state the definition of j-multiplicity of an ideal.

Definition 3.1. Let $(R, \mathfrak{m}, k)$ be a Notherian local ring of dimension d and I be an ideal in R. We call the positive integer given by

$$j(I) = \lim_{n\to\infty} \frac{(d-1)!\lambda(H^0_{\mathfrak{m}}(I^n/I^{n+1}))}{n^{d-1}}.$$

the ***j*-multiplicity** of I in R.

We have the following properties for j-multiplicity (see [1])

- $j(I) \neq 0 \iff \ell(I) = dim(R)$ where $\ell(I)$ denoted the analytic spread of I (The Krull dimension of the fiber cone of I).
- If I is an m-primary ideal in R, then we have

  $$j(I) = e(I, R) = e(Gr_I(R))$$

  where $e(-)$ denotes the Hilbert-Samuel multiplicity.
- For a monomial ideal I in the polynomial ring $R = k[x_1, \ldots, x_d]$, we have

  $$j(I) = d!vol(pyr(I)).$$

  where $pyr(I)$ is a union of pyramid of bounded faces of Newton polyhedron of I. (see Theorem 3.2 in [21])

In general computing j-multiplicity of an ideal is not easy. In my work with J. Validashti, we have studied j-multiplicity of edge ideal of a graph. Our results are described below.

3.1 Edge Ideals

Let G be a simple graph and $P(G)$ be the Newton Polyhedron of the edge ideal $I(G)$. To compute j-multiplicity of the edge ideal of G from Theorem 3.2 in [21] we need to consider bounded faces of $P(G)$. The convex hull of the vertices of $P(G)$, called the **edge Polytope** $F(G)$ of G, is the only bounded face.

By using this fact I in collaboration with Validashti proved the following theorem to classify graphs with non-zero j-multiplicity.

Remark 3.2. $j(I(G)) \neq 0$ if and only if no connected component of G is bipartite.

To compute j-multiplicity we first consider connected unicycle graphs in which obviously the number of vertices and edges are the same. For a unicycle graph we have the following formula to compute j-multiplicity of its edge ideal. This formula has been computed by Jeffries and Montaño in [21] by using different method.

Corollary 3.3. *If G is a graph which its components are unicycle, then*

$$j(I(G)) = 2^c \alpha(G)$$

where c is the number of connected component of G and $\alpha(G) = 1$ if G has no bipartite connected component, and zero otherwise .

By using $K[G]$, the edge subrings of a graph G, Validashi and I showed the following theorem.

Theorem 3.4. *If G is a simple graph with $c(G)$ connected components, then*

$$j(I(G)) = 2^{c(G)} e(K[G]).$$

By using this theorem we have the following interesting formula for j-multiplicity of complete graphs.

Corollary 3.5. *If G is a complete graph with n vertices, we have*

$$j(I(G)) = 2(2^{n-1} - n) = 2^n - 2n.$$

3.2 Further Work

Question 4. We gave formulas to compute j-multiplicity for some classes of graphs (trees, unicycles, bipartite and complete). Is there any other classes of graphs which their j-multiplicity can be given by a formula?

Question 5. What is the interpretation of $2^n - 2n$ for complete graphs with n vertices?

Question 6. If $\ell(I(G)) \neq dimR$, can we say $e(K[G])$ is the next generalized Hilbert coefficient of $I(G)$?

4 Regularity of Powers of Edge Ideals

The **Castelnuovo-Mumford regularity** of R/I, written as $\text{reg}(R/I)$, is given by

$$\text{reg}(R/I) := \max\{j - i | \beta_{i,j}(R/I) \neq 0\}.$$

Asymptotic behavior of the regularity is one of the interesting ongoing research areas in commutative algebra. The following theorem by Cutkosky, Herzog and Trung is very motivating for many researchers.

Theorem 4.1 (Theorem 1.1, [14])**.** *Let I be a homogeneous ideal, the regularity I^n is a linear function for $n \gg 0$.*

Determining the exact linear function of $\text{reg}(I^n)$ and evaluating n for which $\text{reg}(I^n)$ becomes linear, have been studied by many researchers.

In my research I consider this question for edge ideal of a simple graph. More precisely if G is a simple graph whose edge ideal is denoted by $I(G)$, I am interested in finding values b and n_0 for which we have

$$\text{reg}(I(G)^n) = 2n + b \quad \text{for } n \geq n_0.$$

The values b and n_0 have been computed for a few classes of graphs. These classes are as follows

- **Herzog, Hibi and Zheng [18]:** If $\operatorname{reg}(I(G)) = 2$ ($I(G)$ has linear resolution), then $\operatorname{reg}(I(G)^n) = 2n$ for all $n \geq 1$ ($b = 0, n_0 = 1$).
- **Alilooee and Banerjee [3]:** If G is a bipartite graph and $\operatorname{reg}(I(G)) = 3$, then $\operatorname{reg}(I(G)^n) = 2n + 1$ for all $n \geq 1$ ($b = 1, n_0 = 1$).
- **Beyarslan, Hà and Trung [25]:** If G is a forest, then $\operatorname{reg}(I(G)^n) = 2n + \nu(G) - 1$ for all $n \geq 1$ ($b = \nu(G) - 1, n_0 = 1$) where $\nu(G)$ denotes the maximum induced matching number of G.
- **Beyarslan, Hà and Trung [25]:** $\operatorname{reg}(I(C_m)^n) = 2n + \nu(C_m) - 1$ for all $n \geq 2$ ($b = \nu(C_m) - 1, n_0 = 2$).

4.1 Further Work:

In [25] the authors raised the following question.

Question 7. If G is a simple graph whose edge ideal is $I = I(G)$, then is it true that $\operatorname{reg}(I(G)^n) = 2n + \nu(G) - 1$ for all $n \geq \operatorname{reg}(I(G)) - 1$.

In [25] Beyarslan, Hà and Trung showed that this question is true for forests and cycles. Banerjee, Hà and I are working to see if this question is also true for bipartite and unicycle graphs.

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