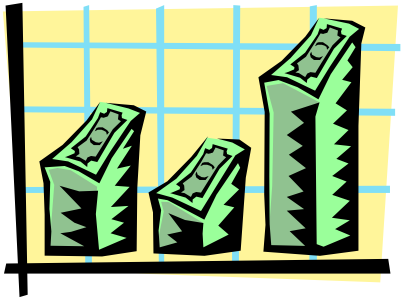

# Introduction to Probability and Statistics

## Slides 6 – Chapter 6

Ammar M. Sarhan,
asarhan@mathstat.dal.ca
Department of Mathematics and Statistics,
Dalhousie University
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# Chapter 6

# Point Estimation

# 6.1 General Concepts of Point Estimation

## Point Estimator

A ***point estimator*** of a parameter $\theta$ is a single number that can be regarded as a sensible value for $\theta$. A ***point estimator*** can be obtained by selecting a suitable statistic and computing its value from the given sample data.

**Notation:**

We use $\hat{\theta}$ to denote the point estimator of $\theta$.

**Examples:** 6.1 and 6.2 p. 229

# Unbiased Estimator

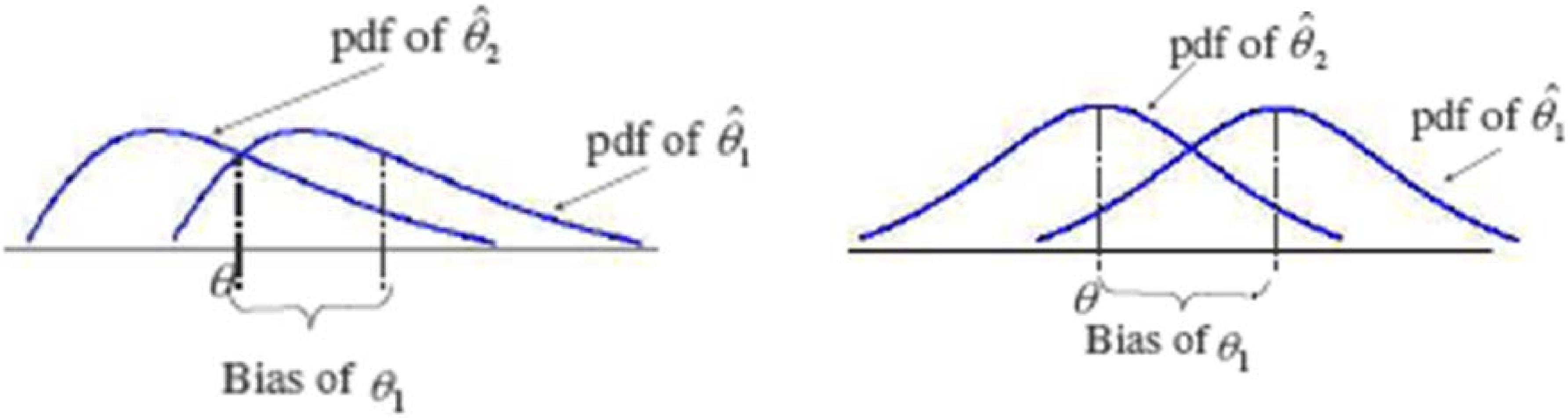
A point estimator $\hat{\theta}$ is said to be an *unbiased estimator* of $\theta$ if

$$E\left(\hat{\theta}\right) = \theta$$

for every possible value of $\theta$.

If $\hat{\theta}$ is biased, the difference $E\left(\hat{\theta}\right) - \theta$ is called the ***bias*** of $\hat{\theta}$ .

The pdf's of a biased estimator $\hat{\theta}_1$ and an unbiased estimator $\hat{\theta}_2$ for a parameter $\theta$.




## Principle of Unbiased Estimation

When choosing among several different estimators of $\theta$, select one that is unbiased.

# Examples of Unbiased Estimator

**(1)** When $X$ *is a binomial rv with* parameters $n$ and $p$, the sample proportion

$$\hat{p} = \frac{X}{n}$$

is an unbiased estimator of $p$. This is because:

$$E(\hat{p}) = E\left(\frac{X}{n}\right) = \frac{E(X)}{n} = \frac{n\,p}{n} = p$$

**(2)** Let $X_1, \ldots, X_n$ be a random sample from a population with an unknown mean μ. The sample mean is an unbiased estimator of μ.

$$E(\hat{\mu}) = E(\overline{X}) = \mu$$

**(3)** Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ and variance $\sigma^2$. The sample variance is an unbiased estimator of $\sigma^2$.

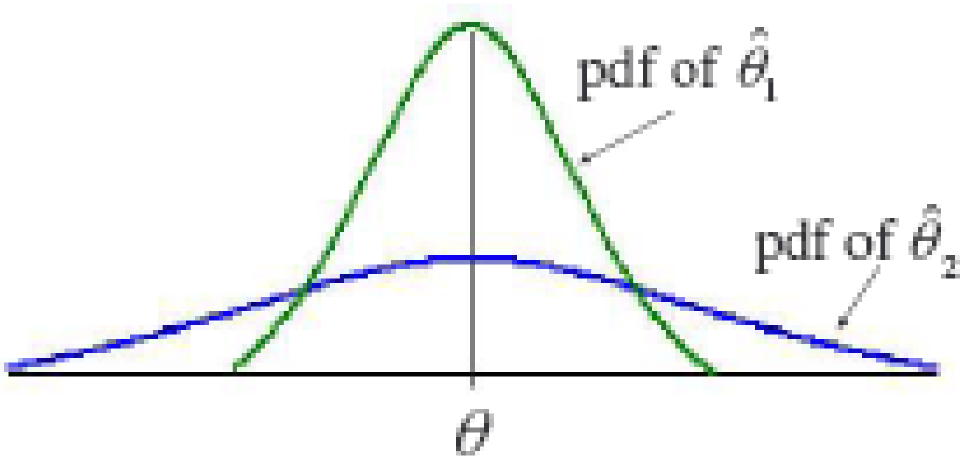
$$E\left(\hat{\sigma}^2\right) = E\left(S^2\right) = E\left(\frac{\sum\left(X_i - \overline{X}\right)^2}{n-1}\right) = \sigma^2$$

## Principle of Minimum Variance Unbiased Estimation

Among all estimators of $\theta$ that are unbiased, choose the one that has the minimum variance. The resulting $\hat{\theta}$ is called the ***minimum variance unbiased estimator*** *(MVUE) of $\theta$.*

Note: Sometimes, MVUE is called as *the best unbiased estimator*.

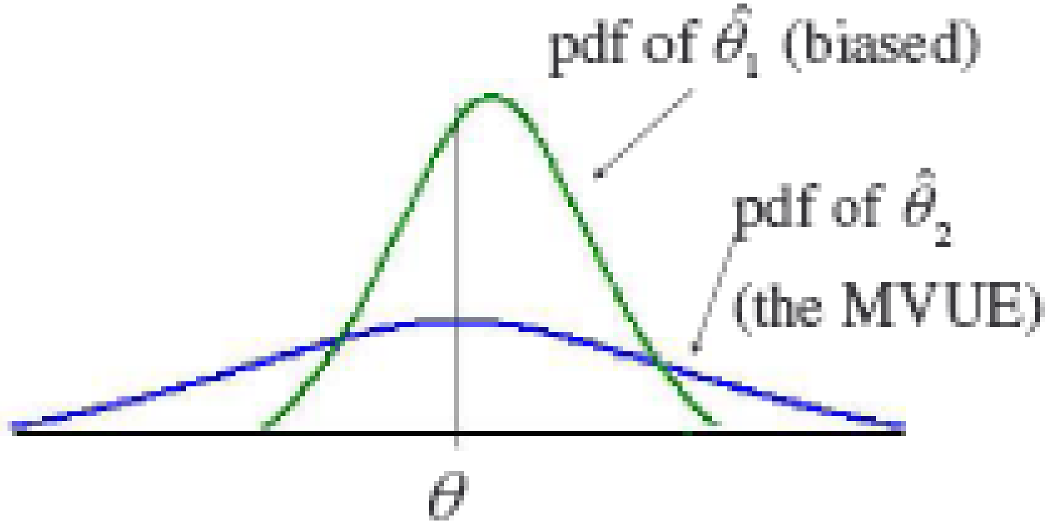
Graphs of the pdf's of two different unbiased estimators



# MVUE for a Normal Distribution

Let $X_1, \ldots, X_n$ be a random sample from normal distribution with mean $\mu$ and variance $\sigma^2$. Then the ***sample mean*** is the **MVUE** for $\mu$.

In some situation, it is possible to obtain an estimator with small bias that would be preferred to the beast unbiased estimator. This is illustrated in the following figure.




A bias estimator that is preferable to the MVUE.

# Standard Error

The ***standard error*** of an estimator $\hat{\theta}$ *is* its standard deviation

$$\sigma_{\hat{\theta}} = \sqrt{V(\hat{\theta})}$$

If the standard error itself involves unknown parameters whose values can be estimated, substitution into $\sigma_{\hat{\theta}}$ yields the *estimated standard error of the* estimator, denoted $\hat{\sigma}_{\hat{\theta}}$ or $s_{\hat{\theta}}$ .

**Examples:** 6.9 and 6.10 p. 238.

The sample mean of a normal distributed sample, with mean μ and unknown variance $\sigma^2$, is the best estimator of μ. The standard error is

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} \quad \Longrightarrow \quad \hat{\sigma}_{\overline{X}} = \frac{\hat{\sigma}}{\sqrt{n}}$$

## Moments

***The sample moments*:**

Let $X_1, X_2, \ldots, X_n$ be a random sample from a pmf or pdf $f(x)$. The ***kth sample moment*** *is*

$$\frac{1}{n}\sum_{i=1}^{n} X_i^k$$

***The population moments*:**

For $k = 1, 2, \ldots$ the ***kth population moment***, *or* ***kth moment*** of the distribution $f(x)$ is

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x)\,dx$$

***The 1st and 2nd population moments are*** $E(X) = \mu,\ E(X^2)$

***The 1st 1st and 2nd sample moments are*** $\frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X},\ \frac{1}{n}\sum_{i=1}^{n} X_i^2$

# Moment Estimators

Let $X_1, X_2,\ldots,X_n$ *be a random sample* from a distribution with pmf or pdf $f(x; \theta_1,\ldots, \theta_m)$, where $\theta_1,\ldots, \theta_m$ are parameters whose values are unknown. Then the ***moment estimators*** are obtained by **equating** the first *m sample moments to the* corresponding first *m population* moments and **solving** for, $\theta_1,\ldots, \theta_m$ .

If, for example, m=2, E(X) and E(X²) will be functions of $\theta_1$ and $\theta_2$. Setting:

$$E(X) = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X} \quad \text{and} \quad E(X^2) = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

gives two equations in $\theta_1$ and $\theta_2$.

The solution then defines the estimators.

For estimating the population mean, the method gives $\hat{\mu} = \overline{X}$

so, the estimator is the sample mean.

# Examples

**Example:** 6.12, p. 244

Assume $X_1, X_2,\ldots,X_n \sim Exp(\lambda)$, find the moment estimator for $\lambda$?

**Example:**

Assume $X_1, X_2,\ldots,X_n \sim Binom(m,p)$, find the moment estimator for $m,p$?

# Likelihood Function

Let $X_1$, $X_2$,…,$X_n$ have joint *pmf* or *pdf*

$$f(x_1,\ldots, x_n ; \theta_1,\ldots, \theta_m)$$

where parameters $\theta_1,\ldots, \theta_m$ have unknown values. When $x_1,\ldots,x_n$ *are* the observed sample values and $f$ is regarded as a function of $\theta_1,\ldots,\theta_m$, it is called the ***likelihood function***.

# Maximum Likelihood Estimators

The maximum likelihood estimates (mle's) are those values of the $\hat{\theta}_1, \cdots, \hat{\theta}_m$ are those values of the 's that maximize the likelihood function so that

$$f\left(x_1, \cdots, x_n; \hat{\theta}_1, \cdots, \hat{\theta}_m\right) \geq f\left(x_1, \cdots, x_n; \theta_1, \cdots, \theta_m\right)$$

for all $\theta_1, \ldots, \theta_m$

When the $X_i$'s are substituted in the place of the $x_i$'s, the ***maximum likelihood estimators*** result.

**Example:** 6.16, p. 246

Assume $X_1, X_2, \ldots, X_n \sim Exp(\lambda)$, find the mle for λ?

**Example:** 6.17, p. 247

Assume $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$, find the mle's for μ and $\sigma^2$?

## The Invariance Principle

Let $\hat{\theta}_1, \ldots, \hat{\theta}_m$ be the mle's of the parameters $\theta_1, \ldots, \theta_m$ Then the mle of any function $h(\theta_1, \ldots, \theta_m)$ of these parameters is the function $h(\hat{\theta}_1, \ldots, \hat{\theta}_m)$ of the mle's.

## Desirable Property of the Maximum Likelihood Estimate

Under very general conditions on the joint distribution of the sample, when the sample size $n$ is large, the maximum likelihood estimator of any parameter $\theta$ is approx.

unbiased $[E(\hat{\theta}) \approx \theta]$ and has variance that is nearly as small as can be achieved by any estimator.

$$\text{mle } \hat{\theta} \approx \text{MVUE of } \theta$$