MATH 2030 – MIDTERM II.

Fall 2014

Name: SOLUTIONS Banner ID: ____________

Signature: ____________

SECTION (circle one): 02 (Erey MWF 8:35-9:25)

01 (Brown MWF 9:35-10:25)

NOTES:

1. Please complete all problems.
2. Use the backs of pages for rough work if necessary.
3. Only what is written in non-erasable ink can be REGRADED.
4. If your work continues on another page (or the back of a page, please indicate this clearly).
5. Show your work.
6. Please place a box around your final answer where appropriate.

Question	Out of	Mark
1	10	
2	10	
3	15	
4	10	
5	5	
Total	50	

1. [10 marks] Fill in the blanks:

(a) A square matrix with all entries below the main diagonal being 0 is said to be in

upper triangular form.

(b) The determinant of a square matrix that is invertible is nonzero.

(c) Suppose that A is a square matrix with $\det A = 4$, and A' is formed from A by first interchanging two columns of A and then multiplying a row by -2. Then

$\det A' =$ -8.

(d) The linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$ is said to be inhomogeneous.

(e) For a square matrix A the characteristic equation of A is $\det(A - \lambda I) = 0$.

(f) If every vector in a subspace S can be written as a linear combinations of vectors in a subset T of S, then we say that T spans S.

(g) If we can write $\mathbf{0}$ as a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with at least one of the scalars not equal to 0, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be

linearly dependent.

(h) A plane in $\mathbf{R}^3$ is a subspace of $\mathbf{R}^3$ if and only if it passes through the point

$(0,0,0)$.

(i) Is the set $S = \{(a,b) : a \text{ and } b \text{ are both integers}\}$ a subspace of $\mathbf{R}^2$? Answer yes or no. no

(j) Is the set $S = \{(2a,a) : a \in \mathbf{R}\}$ a subspace of $\mathbf{R}^2$? Answer yes or no. yes

2. [10 marks]

Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$.

(a) Find the inverse of A.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -2 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & -2 & 0 & -1 \\ -2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

(b) Find the determinant of A.

$$\det A = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= -(2-1)$$

$$= -1$$

3. [15 marks] Find the eigenvalues and the eigenvectors of the following matrix:

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

$\det(A-\lambda I)=0 \Leftrightarrow \begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$

$\Leftrightarrow (5-\lambda)((3-\lambda)^2-4)=0$

$\Leftrightarrow (5-\lambda)(3-\lambda-2)(3-\lambda+2)=0$

$\Leftrightarrow (5-\lambda)(1-\lambda)(5-\lambda)=0$

$\therefore \lambda = 5$ or 1

$\lambda=1$

$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$

$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore x-y=0$
$z=0$

Set $y=s$.
$x=s$
$y=s$
$z=0$

$\therefore$ The eigenvectors with eigenvalue 1 are
$(s,s,0)$
where $s\in\mathbb{R}, s\neq 0$

$\lambda=5$

$\left[\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x+y=0$

Set $y=s, z=t$.
$x=-s$
$y=s$
$z=t$

$\therefore$ The eigenvectors with eigenvalue 5 are
$(-s,s,t)$
where $s,t\in\mathbb{R}$, not both s and t are 0.

4. [10 marks]

(a) Is $\{(1,0,2,-1),(1,3,-1,2),(0,2,-2,2)\}$ linearly independent? Explain your answer.

$$c_1(1,0,2,-1)+c_2(1,3,-1,2)+c_3(0,2,-2,2)=(0,0,0,0)$$

$$\Leftrightarrow (c_1+c_2,\ 3c_2+2c_3,\ 2c_1-c_2-2c_3,\ -c_1+2c_2+2c_3)=(0,0,0,0)$$

$$\begin{aligned} c_1+c_2 \qquad &= 0 \\ 3c_2+2c_3 &= 0 \\ 2c_1-c_2-2c_3 &= 0 \\ -c_1+2c_2+2c_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 2 & -1 & -2 & 0 \\ -1 & 2 & 2 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & -3 & -2 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore\ c_1 - \frac{2}{3}c_3 = 0$

$c_2 + \frac{2}{3}c_3 = 0$

So $c_1 = \frac{2}{3}c_3$

$c_2 = -\frac{2}{3}c_3$

We can take $c_3 = 3$, $c_1 = 2$, $c_2 = -2$ so

$2(1,0,2,-1)+(-2)(1,3,-1,2)+3(0,2,-2,2)=(0,0,0)$. $\therefore$ Not linearly independent

(b) Express $(1,1,1)$ as a linear combination of $(1,-1,0)$, $(0,1,-1)$ and $(1,0,1)$.

$$(1,1,1) = a(1,-1,0)+b(0,1,-1)+c(0,1,-1)+d(1,0,1)$$

$$\begin{aligned} a \qquad + d &= 1 \\ -a+b+c \qquad &= 1 \\ -b-c+d &= 1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & 1 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 3 \end{array}\right] \Leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3/2 \end{array}\right] \Leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 3/2 \end{array}\right]$$

$\therefore\ a = -\frac{1}{2}$

$b+c = 1/2$

$d = 3/2$

So set $c=0$, and then $b = 1/2$.

$\therefore\ (1,1,1) = -\frac{1}{2}(1,-1,0)+\frac{1}{2}(0,1,-1)+0(0,1,-1)+\frac{3}{2}(1,0,1)$.

5. [5 marks] Let A be an $n \times n$ matrix. Suppose that for some positive integer k, A^k is the $n \times n$ identity matrix. Show that $\det A = 1$ or -1.

$$A^k = I$$

$$\rightarrow |A^k| = |I| = 1$$

$$\rightarrow \underbrace{|A| \cdots |A|}_{k \text{ times}} = |I| = 1$$

$$\rightarrow |A|^k = 1$$

So $|A| = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 1 \text{ or } -1 & \text{if } k \text{ is even.} \end{cases}$