MATH 2030 – MIDTERM II

Fall 2014

Name: SOLUTIONS Banner ID: ____________________

Signature: ______________________________

SECTION (circle one): 02 (Erey MWF 8:35-9:25)

01 (Brown MWF 9:35-10:25)

NOTES:

1. Please complete all problems.
2. Use the backs of pages for rough work if necessary.
3. Only what is written in non-erasable ink can be REGRADED.
4. If your work continues on another page (or the back of a page, please indicate this clearly).
5. Show your work.
6. Please place a box around your final answer where appropriate.

Question	**Out of**	**Mark**
1	10	
2	10	
3	15	
4	10	
5	5	
Total	50	

1. [10 marks] Fill in the blanks:

(a) A square matrix with all entries above the main diagonal being 0 is said to be in lower triangular form.

(b) The determinant of a square matrix that is not invertible is 0.

(c) The linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \mathbf{0}$ is said to be homogeneous.

(d) Suppose that A is a square matrix with det A = -2, and A' is formed from A by first interchanging two columns of A and then multiplying a row by 3. Then det A' = 6.

(e) For a square matrix A the characteristic polynomial of A is $\det(A-\lambda I)$.

(f) If every vector in a subspace S can be written as a linear combinations of vectors in a subset T of S, then we say that T spans S.

(g) If the only way to write $\mathbf{0}$ as a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is with all scalars being 0, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be linearly independent.

(h) A line in $\mathbf{R}^3$ is a subspace of $\mathbf{R}^3$ if and only if it passes through the point (0,0,0).

(i) Is the set $S = \{(a,b):$ at least one of a and b are $0\}$ a subspace of $\mathbf{R}^2$? Answer yes or no. no

(j) Is the set $S = \{(-a,a) : a \in \mathbf{R}\}$ a subspace of $\mathbf{R}^2$? Answer yes or no. yes

2. [10 marks]

Let $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$.

(a) Find the inverse of A.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & -2 & 0 & -1 \\ -2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

(b) Find the determinant of A.

$$\det A = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= -(2-1)$$

$$= -1$$

3. [15 marks] Find the eigenvalues and the eigenvectors of the following matrix:

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

$$\det(A-\lambda I)=0 \Leftrightarrow \begin{vmatrix} -1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (-3-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (-3-\lambda)\left((-1-\lambda)^2-4\right)=0$$

$$\Leftrightarrow (-3-\lambda)(-1-\lambda-2)(-1-\lambda+2)=0$$

$$\Leftrightarrow (-3-\lambda)(-3-\lambda)(1-\lambda)=0$$

$$\Leftrightarrow \lambda=-3 \text{ or } 1.$$

$\lambda=-3$:

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x+y=0$

Set $y=s, z=t$

$x=-s$

$y=s$

$z=t$

$\therefore$ The eigenvectors with eigenvalue -3 are $(-s,s,t)$ where $s,t\in\mathbb{R}$ and not both are 0.

$\lambda=1$

$$\left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$x-y=0$

$z=0$

Set $y=s$

$x=s$

$y=s$

$z=0$

$\therefore$ the eigenvectors with eigenvalue 1 are $(s,s,0)$ where $s\in\mathbb{R}$, $s\neq 0$.

4. [10 marks]

(a) Is $\{(1,0,2,-1),(1,3,-1,2),(2,-3,7,-5)\}$ linearly independent? Explain your answer.

$c_1(1,0,2,-1)+c_2(1,3,-1,2)+c_3(2,-3,7,-5)=(0,0,0,0)$

$\leftrightarrow (c_1+c_2+2c_3,\ 3c_2-3c_3, 2c_1-c_2+7c_3, -c_1+2c_2-5c_3)=(0,0,0,0)$

$c_1+c_2+2c_3=0$

$3c_2-3c_3=0$

$2c_1-c_2+7c_3=0$

$-c_1+2c_2-5c_3=0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 2 & -1 & 7 & 0 \\ -1 & 2 & -5 & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore\ c_1+3c_3=0$

$c_2-c_3=0$

If $c_3=1$, then $c_2=1, c_1=-3$.

Thus

$1(1,0,2,-1)+1(1,3,-1,2)+(-3)(2,-3,7,-5)$

$=(0,0,0,0)$

so the set is not linearly independent.

(b) Express $(2,-1,3)$ as a linear combination of $(1,-1,0)$, $(1,1,1)$ and $(1,0,1)$.

$(2,-1,3)=a(1,-1,0)+b(1,1,1)+c(1,0,1)$

$\leftrightarrow\ a+b+c=2$

$-a+b=-1$

$b+c=3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -5 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$\therefore\ a=-1, b=-2, c=5$

$\therefore (-1)(1,-1,0)+(-2)(1,1,1)+5(1,0,1)$

$=(2,-1,3)$.

5. [5 marks] Let A be an $n \times n$ matrix. Suppose that for some positive integer k, A^k is the $n \times n$ zero matrix. Show that A is not invertible.

$A^k = 0$

$\rightarrow |A^k| = |0| = 0$

$\rightarrow \underbrace{|A| \cdots |A|}_{k \text{ times}} = 0$

$\rightarrow |A|^k = 0$

$\rightarrow |A| = 0$

$\therefore$ As $\det A = 0$, A is not invertible.