MATH 2030 Midterm Test 1 Fall 2014

NAME: SOLUTIONS

ID #:

NOTE:

1. Show all your work.

2. No calculators are permitted.

3. No scratch paper is allowed. Use the back of a page if necessary, but indicate you are doing so with a $\rightarrow$.

4. Put a BOX around your final answers where appropriate.

5. Only what is written in INK can be regraded.

#1.	/ 8
#2.	/12
#3.	/10
#4.	/10
#5.	/10
TOTAL	

[8] 1. Complete the following sentences.

(a) A vector that is perpendicular to a plane is said to be normal to the plane.

(b) For two non-collinear vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ the nonzero vector $\underline{u} \times \underline{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

(c) Two vectors are orthogonal if their dot product is equal to 0.

(d) A vector is said to be a unit vector if its length is equal to 1.

(e) In order to solve a linear system, you first form the associated augmented matrix, and then use elementary row operations to reduce the matrix into row-reduced echelon form. This procedure is called Gauss-Jordan elimination.

[12] 2. Let $\mathbf{u} = (1, -2, 2)$, $\mathbf{v} = (-1, 1, 0)$ and P be the point $(1, 3, -2)$.

(a) Write down a vector of length 4 that is parallel to $\mathbf{u}$.

$\|\underline{u}\| = \sqrt{1^2+(-2)^2+2^2} = \sqrt{9} = 3$

$\therefore$ $\frac{1}{3}\underline{u}$ is a unit vector in the direction of $\underline{u}$, so

$\frac{4}{3}\underline{u} = \left(\frac{4}{3}, -\frac{8}{3}, \frac{8}{3}\right)$ is a vector of length 4 that is parallel to $\underline{u}$

(b) Write down the parametric form of the equation of the line that contains P and is parallel to $\mathbf{u}$.

Point-parallel form: $\underline{x}(t) = (1,3,-2) + t(1,-2,2)$, $t \in \mathbb{R}$

$= (1+t, 3-2t, -2+2t)$

Parametric form: $\begin{cases} x = 1+t \\ y = 3-2t \\ z = -2+2t \end{cases}$, $t \in \mathbb{R}$.

2. (continued) Let $\mathbf{u} = (1, -2, 2)$, $\mathbf{v} = (-1, 1, 0)$ and P be the point $(1, 3, -2)$.

(c) Find the point-normal form of the equation of the plane containing P that has $\mathbf{v}$ as a normal.

$$(-1,1,0) \cdot \left(\underline{x} - (1,3,-2)\right) = 0$$

(d) Find the standard form of the equation of the plane that completely contains both $\mathbf{u}$ and $\mathbf{v}$.

A normal to the plane will be $\underline{u} \times \underline{v}$.

$$\begin{vmatrix} 1 & -2 & 2 \\ -1 & 1 & 0 \end{vmatrix}$$

$$\underline{u} \times \underline{v} = \left(\begin{vmatrix} -2 & 2 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} \right)$$

$$= (-2, -2, -1)$$

As the plane contains the origin $(0,0,0)$, the point-normal form is

$$(-2,-2,-1) \cdot \left(\underline{x} - (0,0,0)\right) = 0$$

$$\Leftrightarrow (-2,-2,-1) \cdot \left((x,y,z) - (0,0,0)\right) = 0$$

$$\Leftrightarrow -2x - 2y - z = 0$$

$$(\text{or } 2x + 2y + z = 0)$$

[10] 3. Find the point on the plane $x - 2y + 3z = 5$ closest to the point $(0, 3, 4)$.

A normal to the plane is $\underline{n} = (1,-2,3)$, and the point-parallel line ℓ through $P(0,3,4)$ with direction vector $\underline{n}$ has the equation

$$\underline{x} = (0,3,4) + t(1,-2,3), \quad t \in \mathbb{R}$$

$$\rightarrow (x,y,t) = (t, 3-2t, 4+3t), \quad t \in \mathbb{R}$$

As the point on the plane closest to $(0,3,4)$ must lie on this line, we find the correct value of t as below:

$$t - 2(3-2t) + 3(4+3t) = 5$$

$$\Leftrightarrow t - 6 + 4t + 12 + 9t = 5$$

$$\Leftrightarrow 14t = -1$$

$$\Leftrightarrow t = \frac{-1}{14}$$

So the point on the plane closest to $(0,3,4)$ is

$$\left(\frac{-1}{14}, 3 - 2\left(\frac{-1}{14}\right), 4 + 3\left(\frac{-1}{14}\right)\right) = \left(\frac{-1}{4}, \frac{22}{7}, \frac{53}{14}\right).$$

[10] 4. Find all of the the solutions to the following linear system:

$$\begin{array}{rcrcrcr} x & + & 2y & - & 3z & = & 4 \\ x & + & 3y & + & z & = & 11 \\ 2x & + & 5y & - & 4z & = & 13 \\ 2x & + & 6y & + & 2z & = & 22 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 1 & 3 & 1 & 11 \\ 2 & 5 & -4 & 13 \\ 2 & 6 & 2 & 22 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & 4 & 7 \\ 0 & 1 & 2 & 5 \\ 0 & 2 & 8 & 14 \end{array}\right] \begin{array}{l} \\ R_2+(-1)R_1 \\ R_3+(-2)R_1 \\ R_4 \end{array}$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -11 & -10 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1+(-2)R_2 \\ \\ R_3+(-1)R_2 \\ R_4+(-2)R_2 \end{array}$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -11 & -10 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ (-\tfrac{1}{2})R_3 \\ \\ \end{array}$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1+11R_3 \\ R_2+(-4)R_3 \\ \\ \\ \end{array}$$

$\therefore$ The solution is $(x,y,z)=(1,3,1)$, i.e. $x=1, y=3, z=1$.

[10] 5. For what values of k will the linear system have (i) no solutions, (ii) a unqiue solution, (iii) infinitely many solutions?

$$\begin{array}{rcrcrcr} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (k^2-14)z & = & k+2 \end{array}$$

Solving using augmented matrices:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & k^2-14 & k+2 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & k^2-2 & k-14 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & -7 & k^2-2 & k-14 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & k^2-16 & k-4 \end{array}\right]$$

So (i) there are no solutions if $\underbrace{k^2-16}_{(k-4)(k+4)}=0$ and $k-4\neq 0$, that is, if $k=-4$

(iii) there are infinitely many solutions if $k^2-16=0$ and $k-4=0$, that is, if $k=4$

(ii) there is a unique solution if $k^2-16\neq 0$, that is, if $k\neq -4, 4$.