MATH 2030 Make-Up Midterm Test 1 Fall 2014

NAME: SOLUTIONS

ID #:

NOTE:

1. Show all your work.
2. No calculators are permitted.
3. No scratch paper is allowed. Use the back of a page if necessary, but indicate you are doing so with a →.
4. Put a BOX around your final answers where appropriate.
5. Only what is written in INK can be regraded.

#1.	/ 8
#2.	/12
#3.	/10
#4.	/10
#5.	/10
TOTAL	

[8] 1. Complete the following sentences.

(a) If **u** and **v** are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ such that $\mathbf{u} \cdot \mathbf{v} < 0$, then the angle between **u** and **v** is obtuse.

(b) If **u** is a nonzero vector, then a different vector that is parallel to **u** and has the same length as **u** is $-u$.

(c) Three forms of a plane in $\mathbb{R}^3$ are point-normal form, the standard form and parametric form.

(d) A line in $\mathbb{R}^3$ is determined by 2 points.

(e) In order to solve a linear system, you first form the associated augmented matrix, and then use elementary row operations to reduce the matrix into row-reduced echelon form. This procedure is called Gauss-Jordan elimination.

[12] 2. Let $\mathbf{u} = (-3, 0, 4)$, $\mathbf{v} = (1, 1, -1)$ and P be the point $(-2, 1, 0)$.

(a) Write down a vector of length 2 that is parallel to **u**.

$\|u\| = \sqrt{(-3)^2+0^2+4^2} = 5$, so $\frac{1}{\|u\|}u = \frac{1}{5}(-3,0,4)$ is a vector of length 1 parallel to u. The vector $2\frac{1}{\|u\|}u = \frac{2}{5}(-3,0,4) = \left(-\frac{6}{5}, 0, \frac{8}{5}\right)$ is a vector of length 2 parallel to u.

(b) Write down the parametric form of the equation of the line that contains P and is parallel to **v**.

$x = (-2,1,0) + t(1,1,-1) \Leftrightarrow (x,y,z) = (-2+t, 1+t, -t),\ t \in \mathbb{R}$

$\therefore$ Parametric form is $\begin{cases} x = -2+t \\ y = 1+t \\ z = -t \end{cases}$, $t \in \mathbb{R}$

2. (continued) Let $\mathbf{u} = (-3, 0, 4)$, $\mathbf{v} = (1, 1, -1)$ and P be the point $(-2, 1, 0)$.

(c) Find the point-normal form of the equation of the plane containing P that has $\mathbf{u}$ as a normal.

$$(x - (-2, 1, 0)) \cdot (-3, 0, 4) = 0$$

(d) Find the standard form of the equation of the plane that contains P and is parallel to both $\mathbf{u}$ and $\mathbf{v}$.

If the plane is parallel to both $\underline{u}$ and $\underline{v}$, then a normal is

$$n = \underline{u} \times \underline{v} = \left(\begin{vmatrix} 0 & 4 \\ 1 & -1 \end{vmatrix}, -\begin{vmatrix} -3 & 4 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} -3 & 0 \\ 1 & 1 \end{vmatrix} \right)$$

$$= (-4, 1, -3).$$

Thus the point-normal form of the plane is

$$(\underline{x} - (-2, 1, 0)) \cdot (-4, 1, -3) = 0$$

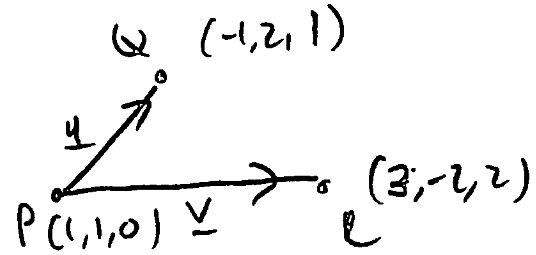
$$\Leftrightarrow ((x, y, z) - (-2, 1, 0)) \cdot (-4, 1, -3) = 0$$

$$\Leftrightarrow (x+2, y-1, z) \cdot (-4, 1, -3) = 0$$

$$\Leftrightarrow -4x - 8 + y - 1 - 3z = 0$$

$$\Leftrightarrow -4x + y - 3z = 9$$

[10] 3. Find the area of the triangle determined by the points $(1,1,0)$, $(-1,2,1)$ and $(3,-2,2)$.



Let $\underline{u} = \vec{PQ}$

$\underline{v} = \vec{PR}$

$$\underline{u} = (-1,2,1) - (1,1,0) = (-2,1,1)$$

$$\underline{v} = (3,-2,2) - (1,1,0) = (2,-3,2)$$

$$\underline{u} \times \underline{v} = \left(\begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix}, -\begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix}, \begin{vmatrix} -2 & 1 \\ 2 & -3 \end{vmatrix} \right)$$

$$= (5, 6, 4)$$

$\therefore$ Area of triangle $= \frac{1}{2} \| \underline{u} \times \underline{v} \|$

$$= \frac{1}{2} \sqrt{5^2 + 6^2 + 4^2}$$

$$= \frac{1}{2} \sqrt{25 + 36 + 16}$$

$$= \frac{\sqrt{77}}{2}$$

[10] 4. Find all of the the solutions to the following linear system:

$$
\begin{array}{rcrcrcrcr}
2x & + & 3y & - & z & - & 9w & = & -16 \\
-x & + & 2y & + & 3z & + & 4w & = & 8 \\
x & + & 2y & + & z & & & = & 0 \\
2x & & 7y & + & 3z & - & 5w & = & -8
\end{array}
$$

$$
\left[\begin{array}{cccc|c} 2 & 3 & -1 & -9 & -16 \\ -1 & 2 & 3 & 4 & 8 \\ 1 & 2 & 1 & 0 & 0 \\ 2 & 7 & 3 & -5 & -8 \end{array}\right] \leftrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 4 & 8 \\ 2 & 3 & -1 & -9 & -16 \\ 2 & 7 & 3 & -5 & -8 \end{array}\right]
$$

$$
\leftrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -1 & -3 & -9 & -16 \\ 0 & 3 & 1 & -5 & -8 \end{array}\right] \leftrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -9 & -16 \\ 0 & 3 & 1 & -5 & -8 \end{array}\right]
$$

$$
\leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -4 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -2 & -8 & -14 \\ 0 & 0 & -2 & -8 & -14 \end{array}\right] \leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -4 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & -2 & -8 & -14 \end{array}\right]
$$

$$
\leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 3 \\ 0 & 1 & 0 & -3 & -5 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

$\therefore$ Setting $w = s$, we find the solutions are
$$
\left\{ \begin{array}{l} x = 3 - 5s \\ y = -5 + 3s \\ z = 7 - 4s \\ w = s \end{array} \right. , \quad s \in \mathbb{R}
$$

[10] 5. For what conditions on a, b and c will the linear system have (i) no solutions, (ii) a unique solution, (iii) infinitely many solutions?

$$\begin{array}{rcrcrcl} x & + & 5y & + & z & = & 0 \\ x & + & 6y & - & z & = & 0 \\ 2x & + & ay & + & bz & = & c \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 1 & 6 & -1 & 0 \\ 2 & a & b & c \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & a-10 & b-2 & c \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 11 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & b-2+2(a-10) & c \end{array}\right]$$

There are (i) no solutions if $b-2+2(a-10)=0$ and $c\neq 0$, that is, if $2a+b=22$ and $c\neq 0$

(iii) infinitely many solutions if $b-2+2(a-10)=0$ and $c=0$, that is, if $2a+b=22$ and $c=0$

(ii) a unique solution if $b-2+2(a-10)\neq 0$, that is, if $2a+b\neq 22$.