MATH 2030 REVIEW PROBLEMS (BOLD) - MT2 FALL 2014

Section 3.1

37. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, so $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an example of an idempotent matrix.

38. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, so $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is an example of an idempotent matrix.

Section 3.2

32. Note that $(P^{-1}AP)(P^{-1}A^{-1}P) = P^{-1}A(PP^{-1})A^{-1}P = P^{-1}AIA^{-1}P = P^{-1}AA^{-1}P = P^{-1}IP = P^{-1}P = I$, and similarly $(P^{-1}AP)(P^{-1}AP) = I$, so $P^{-1}AP$ has an inverse, namely $P^{-1}A^{-1}P$.

33. If all a_i's are nonzero, then

$$\underbrace{\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}}_{D}\begin{bmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & 0 \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \frac{1}{a_n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I, \text{ and}$$

similarly, $\begin{bmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \frac{1}{a_n} \end{bmatrix}\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} = I$, so $D^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_n} \end{bmatrix}$.

Section 3.3

20. $A^TB^T = (BA)^T = I^T = I$, since $BA = I$ implies $AB = I$ by Theorem 2. By the same theorem it follows that $B^TA^T = I$ as well, so $(A^T)^{-1} = B^T$.

Chapter 3's Challenge Yourself:

34. $\det A = 0 \Leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & a \\ -1 & a & -1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & a+1 \\ 0 & a+1 & 0 \end{vmatrix} = 0$

$\Leftrightarrow (1) \begin{vmatrix} 2 & a+1 \\ a+1 & 0 \end{vmatrix} = 0$

$\Leftrightarrow \quad -(a+1)^2 = 0$

$\Leftrightarrow \quad a = -1$

Thus A is invertible $\Leftrightarrow \det A \neq 0 \Leftrightarrow a \neq -1$.

35. $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ iff

$AB = BA$

$\Leftrightarrow \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} a+c & b+d \\ -a & -b \end{bmatrix} = \begin{bmatrix} a-b & a \\ c-d & c \end{bmatrix}$

$$\Leftrightarrow \begin{aligned} a + c &= a - b \\ b + d &= a \\ -a &= c - d \\ -b &= c \end{aligned}$$

$$\Leftrightarrow \begin{aligned} b + c &= 0 \\ -a + b + d &= 0 \\ -a - c + d &= 0 \\ -b - c &= 0 \end{aligned}$$

$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{array}\right]$

$\Leftrightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{array}\right]$

$$\Leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} a \quad + c - d = 0 \\ \quad b + c \quad = 0 \end{cases}$$

$$\Leftrightarrow \begin{array}{l} a = d - c \\ b = -c \end{array} \qquad c, d \in \mathbb{R}.$$

$\therefore$ B commutes with A iff B is of the form $\begin{bmatrix} d-c & -c \\ c & d \end{bmatrix}$.

Section 4.3

17. Suppose A is invertible and symmetric. Then $\det A \neq 0$, so as $\det A^{-1} = \dfrac{1}{\det A} \neq 0$, A^{-1} is invertible. Moreover, A^{-1} is symmetric, as $(A^{-1})^T = (A^T)^{-1}$ ($(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$ and similarly $A^T (A^{-1})^T = I$), so

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}.$$

Section 4.4

18. 0 is an eigenvalue of A $\Leftrightarrow$ $A\underline{x} = \underline{0}$ has a nontrivial solution $\underline{x}$
$\Leftrightarrow$ A is not invertible (by Thm 5, Section 3.3)

20. Suppose A is nonsingular (i.e. A is invertible). Then from Exercise 18, 0 is not an eigenvalue of A. Now
λ is an eigenvalue of A $\Leftrightarrow$ there is a nonzero vector $\underline{x}$ s.t. $A\underline{x} = \lambda\underline{x}$

Now $A\underline{x} = \lambda\underline{x} \Leftrightarrow A^{-1}A\underline{x} = A^{-1}\lambda\underline{x} = \lambda A^{-1}\underline{x}$

$$\Leftrightarrow \underline{x} = \lambda A^{-1}\underline{x}$$

$$\Leftrightarrow \frac{1}{\lambda}\underline{x} = A^{-1}\underline{x}$$

so λ is an eigenvalue of A iff $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. $\square$

The argument shows that A and A^{-1} have the same eigenvectors.

Chapter 4's Challenge Yourself

30. See solutions to written assignment #2.

Section 5.1

30. Suppose $c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 = \underline{0}$. Then $c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 + 0\underline{v}_4 = \underline{0}$ so as $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ is linearly independent, all of the coefficients are 0, and so

$c_1 = 0 \quad c_2 = 0 \quad , c_3 = 0.$

It follows that $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is also linearly independent.